Tatra Mt. Math. Publ. **28** (2004), 153–157

Tatra Mountains
Mathematical Publications

CONVERGENCE FIELDS OF REGULAR MATRIX TRANSFORMATIONS

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ABSTRACT. The limitation method performed by a regular matrix transformation is well-known and it is treated in monographs (see [R. G. Cooke: *Infinite Matrices and Sequence Spaces*], [G. M. Petersen: *Regular Matrix Transformations*]). In the present paper we investigate, convergence fields of regular matrix transformations. The notions of both the Baire category and the porosity of sets seem to be suitable. We show, in general, that the set of sequences which are convergent in this limitation method is "small". Note that paper by [T. Šalát: *On convergence fields of regular matrix transformations*, Czech. Math. J. **26** (1976), 613–627] is devoted to the problem.

1. Introduction

Let $A = (a_{n,k})$ $(n, k = 1, 2, \dots)$ be an infinite matrix of real numbers. A sequence (x_n) of reals is said to be A-limitable (limitable by a method (A)) to a number t, if $\lim\limits_{n\to\infty} t_n = t$, where

$$t_n = t_n(x) = \sum_{k=1}^{\infty} a_{n,k} x_k\,, \quad n = 1, 2, \dots .$$

If (x_k) is A-limitable to the number t, we write $A\text{-}\lim\limits_{k\to\infty} x_k = t$. The symbol $F(A)$ denotes the set of all A-limitable sequences. The set $F(A)$ is called convergence field of the method (A) or the convergence field of a matrix transformation A (see [Pet]). The method (A) defined by the matrix A is said to be regular provided $F(A)$ contains all convergent sequences and $\lim\limits_{k\to\infty} x_k = t$ implies $A\text{-}\lim\limits_{k\to\infty} x_k = t$. If the method (A) is regular then the matrix A is called a regular matrix.

2000 Mathematics Subject Classification: 40D09.
Keywords: convergence field, Baire category, porosity.

Supported by the Grant VEGA 1/7173/20.

It is well-known that the method (A) is regular if and only if the matrix A fulfils the following three conditions ([Coo], p. 79; [Pet], p. 8):

(i) There exists $M>0$ such that for each $n=1,2,\dots$ we have $\sum\limits_{k=1}^{\infty}|a_{n,k}| \le M$;

(ii) $\lim\limits_{n\to\infty} a_{n,k}=0$ holds for each fixed positive integer k;

(iii) $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{\infty} a_{n,k}=1$.

H. Steinhaus has shown that for each regular method (A) there exists a sequence of 0's and 1's which is not A-limitable (see [Con]; [Coo], p. 93–94; [Šal]).

Further we shall deal with the set s of all real sequences and with some of its subsets. We shall consider them a metric linear space or a normed linear space, respectively. The symbol l_∞ ($l_\infty \subset s$) stands for the set of all bounded sequences. Operations of addition and multiplication by a real number are defined in the natural way.

The notion of porosity has been introduced by L. Zajíček [Zaj]. It is a suitable tool to describe small sets in a metric space. Let (Y,ρ) be a metric space, $Z\subset Y$. Let $y\in Y$, $\delta>0$ and $B(y,\delta)=\{x\in Y:\ \rho(x,y)<\delta\}$. We put $\gamma(y,\delta,Z)=\sup\{t>0:\ \exists z\in B(y,\delta):\ B(z,t)\subset B(y,\delta) \text{ and } B(z,t)\cap Z=\emptyset\}$. If such a $t>0$ does not exist, we put $\gamma(y,\delta,Z)=0$. The numbers

$$\underline{p}(y,Z)=\liminf_{\delta\to 0+}\gamma(y,\delta,Z)/\delta \quad \text{and} \quad \overline{p}(y,Z)=\limsup_{\delta\to 0+}\gamma(y,\delta,Z)/\delta$$

are called lower and upper porosity of the set Z at y. If we have $\overline{p}(y,Z)>0$ for all $y\in Y$, then Z is said to be porous in Y. Obviously every set porous in Y is nowhere dense in Y. If $\underline{p}(y,Z)=\overline{p}(y,Z)$ $(=p(y,Z))$ then the number $p(y,Z)$ is called porosity of Z at y. If $p(y,Z)=1$, then Z is said to be strongly porous at y. If for all $y\in Y$, we have $p(y,Z)=1$ for $y\notin Z$, and $p(y,Z)=1/2$ for $y\in Z$, then Z is said to be strongly porous at Y. The set W is said to be σ-porous (σ-strongly porous) at Y, if $W=\bigcup\limits_{n=1}^{\infty} Z_n$ and each Z_n is porous (strongly porous) at Y.

2. Results

A complete metric linear space (S,ρ) of sequences, i.e., $S\subset s$, is said to be FK-space provided the functions $f_k\colon S\to\mathbb{R}$, $f_k(x)=x_k$ ($x=(x_k)$), $k=1,2,\dots$, are continuous ([BK], p. 24). The following simple proposition can be formulated omitting a proof.

Proposition. *Let (S,ρ) be a complete metric linear space of sequences. Then (S,ρ) is an FK-space if and only if the convergence in the sense of the metric ρ is the pointwise convergence, i.e., if $x=(x_k)$, $x^{(r)}=\big(x_k^{(r)}\big)\in S$ and $\rho\big(x^{(r)},x\big)\to 0$ as $r\to\infty$, then $x_k^{(r)}\to x_k$ as $r\to\infty$ for each $k=1,2,\dots$.*

In the proof of Theorem 1 we shall use the following form of Banach's Subgroup Theorem ([KN], p. 93): If G is any proper subgroup of linear topological space E, then either G is of the first category in E, or G fails to satisfy the condition of Baire.

Theorem 1. *In any FK-space (S,ρ), $l_\infty\subset S$, the set $F(A)$ is of the first Baire category in S for every regular matrix A.*

P r o o f. Put $t_{n,s}(x)=\sum\limits_{k=1}^{s}a_{n,k}x_k$ $(n,s=1,2,\dots)$. From above assumptions it follows that each of the real functions $t_{n,s}\colon S\to\mathbb{R}$ is continuous on S and each of the functions $t_n(x)=\lim\limits_{s\to\infty}t_{n,s}(x)$ $(n=1,2,\dots)$ is the Baire one function on the set $D_n=\big\{x\in S\colon \lim\limits_{s\to\infty}t_{n,s}(x)\text{ exists}\big\}$. Obviously,

$$\begin{aligned}D_n&=\Big\{x\in S\colon \mathop{\forall}_{p=1,2,\dots}\ \mathop{\exists}_{s_0}\ \mathop{\forall}_{s>s_0}\ \mathop{\forall}_{q>s_0}\big|t_{n,s}(x)-t_{n,q}(x)\big|\le 1/p\Big\}\\&=\bigcap_{p=1}^{\infty}\ \bigcup_{s_0=1}^{\infty}\ \bigcap_{s>s_0}\ \bigcap_{q>s_0}\big\{x\in S\colon -1/p\le t_{n,s}(x)-t_{n,q}(x)\le 1/p\big\}.\end{aligned}$$

Since every function $t_{n,s}(x)-t_{n,q}(x)$ is continuous, every set D_n is an $F_{\sigma\delta}$ set. The common domain of all the functions t_n—the set $\bigcap\limits_{n=1}^{\infty}D_n$—is also an $F_{\sigma\delta}$ set. Put $S(A)=\bigcap\limits_{n=1}^{\infty}D_n$. Then $F(A)\subset S(A)$ and

$$\begin{aligned}F(A)&=\big\{x\in S(A)\colon \lim_{n\to\infty}t_n(x)\ \text{ exists}\big\}\\&=\Big\{x\in S(A)\colon \mathop{\forall}_{p=1,2,\dots}\ \mathop{\exists}_{n_0}\ \mathop{\forall}_{n>n_0}\ \mathop{\exists}_{m>n_0}\big|t_n(x)-t_m(x)\big|\le 1/p\Big\}\\&=\bigcap_{p=1}^{\infty}\ \bigcup_{n_0=1}^{\infty}\ \bigcap_{n>n_0}\ \bigcap_{m>n_0}\big\{x\in S(A)\colon -1/p\le t_n(x)-t_m(x)\le 1/p\big\}.\end{aligned}$$

Each of the functions t_n-t_m is a Baire one function, hence the set $F(A)$ is a $G_{\delta\sigma\delta}$ set. $F(A)$ satisfies the condition of Baire. Since $F(A)$ is a proper subgroup of S, Theorem 1 is a consequence of Banach's Subgroup Theorem. □

Further we shall use the following assertion.

Lemma. ([Ole]) *Let Z be a convex nowhere dense set in a Banach space X. Then Z is strongly porous on X.*

THEOREM 2. *In the Banach space* $(l_\infty, \|\cdot\|)$ $(\|x\| = \sup\{|x_k|\})$ *the set* $F(A)$ *is strongly porous in* l_∞ *for any regular matrix* A.

P r o o f. Note that each of functions $t_n\colon l_\infty \to \mathbb{R}$, $t_n(x) = \sum\limits_{k=1}^{\infty} a_{n,k} x_k$ is Lipschitz. It follows from the property (i) of the regular matrix $A\bigl(x = (x_k),\ y = (y_k)\bigr)$ that

$$\bigl|t_n(x) - t_n(y)\bigr| = \left|\sum_{k=1}^{\infty} a_{n,k}(x_k - y_k)\right| \le \|x-y\| \sum_{k=1}^{\infty} |a_{n,k}| \le M\|x-y\|. \tag{1}$$

The set $F(A) = \bigl\{x \in l_\infty\colon \lim\limits_{n\to\infty} t_n(x) \text{ exists}\bigr\}$ is a subspace of l_∞ and is convex. If we put $\alpha(x) = \liminf\limits_{n\to\infty} t_n(x)$ and $\beta(x) = \limsup\limits_{n\to\infty} t_n(x)$, then

$$l_\infty \setminus F(A) = \bigl\{x \in l_\infty\colon \beta(x) - \alpha(x) > 0\bigr\}. \tag{2}$$

We show that the set $l_\infty \setminus F(A)$ is open. Let $x \in l_\infty$ and $\beta(x) - \alpha(x) > \varepsilon > 0$. Then

$$\alpha(x) + \varepsilon/2 < \beta(x) - \varepsilon/2. \tag{3}$$

There are increasing sequences of natural numbers $(n_r)_r$ and $(m_s)_s$ such that $\sum\limits_{k=1}^{\infty} a_{n_r,k} x_k \to \alpha(x)$ as $r \to \infty$ and $\sum\limits_{k=1}^{\infty} a_{m_s,k} x_k \to \beta(x)$ as $s \to \infty$. Hence we have

$$t_{n_r}(x) < \alpha(x) + \varepsilon/4 \quad \text{and} \quad t_{m_s}(x) > \beta(x) - \varepsilon/4 \tag{4}$$

for all sufficiently large r and s. Choose $\eta > 0$ such that $M\eta < \varepsilon/4$ and take $y \in B(x,\eta) = \bigl\{z \in l_\infty\colon \|z - x\| < \eta\bigr\}$. Then (1) implies $\bigl|t_{n_r}(y) - t_{n_r}(x)\bigr| < M\eta$ or equivalently $t_{n_r}(x) - M\eta < t_{n_r}(y) < t_{n_r}(x) + M\eta$ and $t_{n_r}(y) < t_{n_r}(x) + \varepsilon/4$. From (4) it follows $t_{n_r}(y) < \alpha(x) + \varepsilon/2$. Analogously we can show $t_{m_s}(y) > \beta(x) - \varepsilon/2$. Consequently (3) and (2) imply $y \in l_\infty \setminus F(A)$. The set $F(A)$ is a closed subspace of l_∞, hence it is a convex and nowhere dense subset of l_∞ and Theorem 2 follows from the lemma. □

Remark. Note that there are Banach spaces included in s such that the convergence field $F(A)$ of any regular matrix A is not porous. E.g., the Banach space $S_1 = \bigl\{x \in s\colon \|x\|_1 < \infty\bigr\}$ endowed with the norm $\|x\|_1 = \sum\limits_{k=1}^{\infty} |x_k| 2^{-k}$. It is easy to verify that both the sets $F(A)$ and $S_1 \setminus F(A)$ are dense in S_1 and consequently $F(A)$ is not porous in S_1. It follows from Theorem 1 that $F(A)$ is the first Baire category in S_1.

Problem. Is it true that the set $F(A)$ introduced in the above remark is σ-porous (σ-strongly porous) in S_1?

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Received October 28, 2002

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