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A CANTOR CUBE AS A SPACE OF HIGHER LEVEL ORDERINGS

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ABSTRACT. Let K be an ordered field. The set $X(K)$ of its orderings can be topologized to make a Boolean space. Moreover, it has been shown by Craven that for any Boolean space X there exists a field K such that $X(K)$ is homeomorphic to X. Especially, the Cantor cube D_m of infinite weight m is homeomorphic to the space of orderings of some field K. Becker's higher level ordering is a generalization of the usual concept of ordering. We prove that there exists a field, which space of orderings of fixed level n is homeomorphic to the Cantor cube D_m.

1. Notation and terminology

Let K be a formally real field. A subset P of K is called *an ordering of higher level* if $\dot{P} := P \setminus \{0\}$ is a subgroup of the multiplicative group $\dot{K}$, $P + P \subset P$ and $\dot{K}/\dot{P}$ is a finite, cyclic group of even exponent. If $\#\dot{K}/\dot{P} = 2n$ then P is called an *ordering of exact level* n. In this terminology an ordering (a total order) is just an ordering of exact level 1.

We use the notation introduced by Becker, Harman and Rosenberg [2]. Let $\mu(n)$ be the group of all complex nth roots of unity and let

$$\mu = \bigcup_{n \in \mathbb{N}} \mu(n).$$

A character $\chi\colon \dot{K} \to \mu$, with $\ker(\chi)$ additively closed is called a *signature* of the field K. The *exact level* $s(\chi)$ of a signature χ, if finite, is defined as $\#\operatorname{Im}(\chi)/2$. A subset P of K is an *ordering of exact level* n if and only if $\dot{P}$ is the kernel of some signature χ of K of exact level n. Therefore, it is useful to think about an ordering P as a set without 0 and we will do it from now on. In general, several signatures have the same kernel. If χ_1 and χ_2 are signatures of K of exact level n, then $\ker(\chi_1) = \ker(\chi_2)$ if and only if there exists an automorphism κ of $\mu(2n)$ such that $\chi_1 = \kappa \circ \chi_2$ (cf. [2, Prop. 3.21, p. 71]).

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Let χ be a signature of the field K. By [2, Th. 2.7, p. 60] the set

$$A(\chi) := \{a \in K \colon \exists_{q\in\mathbb{Q}^+} : q \pm a \in \ker(\chi)\}$$

is a valuation ring of K with the maximal ideal

$$I(\chi) := \{a \in K \colon \forall_{q\in\mathbb{Q}^+} : q \pm a \in \ker(\chi)\}$$

and the residue field $k(\chi)$ such that the map

$$\bar{\chi} \colon \dot{k}(\chi) \to \mu\,, \quad \bar{\chi}\big(a + I(\chi)\big) = \chi(a)\,, \qquad \text{for } a \in \dot{A}(\chi) = A(\chi)\backslash I(\chi)\,,$$

is a signature of archimedean total order of $k(\chi)$.

By [2, Th. 1.12(iii), p. 56 and Prop. 2.5, p. 59], for every signature χ of K there exists a signature χ_0 of exact level 1 such that $A(\chi) = A(\chi_0)$ and $\bar{\chi} = \bar{\chi}_0$.

Note, that for any signature χ of K of exact level n and $\kappa \in \operatorname{Aut}\big(\mu(2n)\big)$ we have $A(\kappa \circ \chi) = A(\chi)$. Therefore in the terminology of orderings we define

$$A(P) := A(\chi)\,, \quad I(P) := I(\chi)\,,$$

where χ is any signature with $P = \ker(\chi)$. The set $\bar{P} := \big(P \cap \dot{A}(P)\big) + I(P)$ is an archimedean total order of $k(P)$. For every higher level ordering P there exists a total order P_0 such that $A(P) = A(P_0)$ and $\bar{P} = \bar{P}_0$ on $k(P)$.

In [2] the authors described the connection between the signature χ of the ordering $P = \ker(\chi)$ of exact level n and the signature χ_0 such that $A(\chi) = A(\chi_0)$ and $\bar{\chi} = \bar{\chi}_0$. If v is the valuation with the valuation ring $A(\chi)$ then by [2, Prop. 1.14, p. 58 and Cor. 2.6, p. 59], there exists unique character τ of value group of v such that

$$\chi = \chi_0 \cdot \tau \circ v\,.$$

Then we have

$$s(\chi) = \frac{1}{2}\ \operatorname{lcm}\big(2, \#\operatorname{Im}(\tau)\big)\,.$$

Therefore

$$\#\operatorname{Im}(\tau) = \begin{cases} 2n & \text{if } n \text{ is even}\,, \\ n \text{ or } 2n & \text{if } n \text{ is odd}\,. \end{cases}$$

For a field K denote by $X_n(K)$ and $eX_n(K)$ the set of all orderings of level m dividing n and the set of all orderings of exact level n, respectively. Similarly, denote by $\operatorname{Sgn}_n(K)$ and $\operatorname{eSgn}_n(K)$ the set of all signatures of level m dividing n and the set of all signatures of exact level n, respectively. We also write $\operatorname{Sgn}(K)$ for the set of all signatures (i.e., of finite and infinite level).

It is known that the set $X_1(K) = eX_1(K)$ can be made into a Boolean space (i.e., compact, Hausdorff and totally disconnected) by using as a subbasis the family of Harrison sets

$$H(a) := \{P \in X_1(K) \colon a \in P\}\,.$$

In the similar way one can define topology on $X_n(K)$ by the subbasis consisting of all sets of the form

$$H_n(a) = \{P \in X_n(K) \colon a \in P\} \quad \text{and} \quad H_n^c(a) = \{P \in X_n(K) \colon a \notin P\}\,.$$

It is clear that the above sets are clopen and $X_n(K)$ is a Hausdorff space. We show that it is also Boolean. Becker and Rosenberg have shown in [3, Prop 1.4, p. 479] that for a fixed natural number n the set $\operatorname{Sgn}_n(K)$ is a Boolean space with the family of sets

$$U(\omega, a) = \{\chi \in \operatorname{Sgn}_n(K) \colon \chi(a) = \omega(a)\}, \qquad \omega \in \operatorname{Sgn}_n(K)\,,$$

as a subbasis.

Define an equivalence relation ρ on $\operatorname{Sgn}_n(K)$, by

$$\chi_1 \rho \chi_2 \iff \ker(\chi_1) = \ker(\chi_2)\,.$$

Thus we get the quotient topological space $\operatorname{Sgn}_n(K)/\rho$ and a bijective map

$$\Phi\colon \operatorname{Sgn}_n(K)/\rho \longrightarrow X_n(K)\,, \quad \Phi([\chi]_\rho) := \ker \chi\,.$$

Proposition 1. *The spaces $\operatorname{Sgn}_n(K)/\rho$ and $X_n(K)$ are Boolean. Moreover the map Φ is a homeomorphism.*

Proof. First of all we shall show that $\operatorname{Sgn}_n(K)/\rho$ is a compact space. Since $\operatorname{Sgn}_n(K)$ is Boolean and the canonical map $\rho\colon \operatorname{Sgn}_n(K) \longrightarrow \operatorname{Sgn}_n(K)/\rho$ is continuous it suffices to show that $\operatorname{Sgn}_n(K)/\rho$ is a Hausdorff space. Take $[\chi_0]_\rho,\ [\eta_0]_\rho \in \operatorname{Sgn}_n(K)/\rho,\ [\chi_0]_\rho \neq [\eta_0]_\rho$. Thus there exists $a \in \dot{K}$ such that $\chi_0(a) = 1,\ \eta_0(a) \neq 1$. Consider

$$U_1 = \{[\chi]_\rho \colon \chi(a) = 1\}\,, \quad U_2 = \{[\chi]_\rho \colon \exists_{\eta \in [\chi]_\rho} \eta(a) = \eta_0(a)\}\,.$$

Of course, $U_1 \cap U_2 = \emptyset$ and $[\chi_0]_\rho \in U_1$, $[\eta_0]_\rho \in U_2$. Moreover both sets are open since

$$\rho^{-1}(U_1) = U(\chi_0, a)\,, \quad \rho^{-1}(U_2) = \bigcup\{U((\tau^{-1} \circ \eta_0), a) \colon \tau \in \operatorname{Aut}(\mu(2n))\}\,.$$

It means that $\operatorname{Sgn}_n(K)/\rho$ is a Hausdorff space. Therefore it is compact, as a continuous image of a compact space.

The map Φ is continuous, because the set

$$\rho^{-1}(\Phi^{-1}(H_n(a))) = \{\chi \in \operatorname{Sgn}_n(K) \colon \chi(a) = 1\}$$

is clopen, for each $a \in \dot{K}$. A continuous bijection of compact space onto a Hausdorff space is a homeomorphism. □

Remark 2. Notice that

$$eX_n(K) = X_n(K) \setminus \bigcup\{X_m(K) \colon m \mid n,\, m < n\}\,.$$

Thus $eX_n(K)$ is an open subset of the Boolean space $X_n(K)$. It has not to be clopen. However, in the case of the field K which we construct in our main theorem (Theorem 12) the subspace $eX_n(K)$ in fact is clopen.

In the second section we recall some results on extensions of signatures obtained by Becker, Harman and Rosenberg [2]. We are mainly interested in the case of Galois extensions. Among other results we show that if L/K is a Galois extension of fields and a total order P of K extends to L not changing the level, then there is a homeomorphism between the Galois group $G(L/K)$ and the subspace of $eX_n(L)$ consisting of all orderings P^L such that $P^L \cap K = P$.

For each infinite cardinal number $\mathfrak{m}$, let $D_{\mathfrak{m}}$ denote the Cantor cube of weight $\mathfrak{m}$, i.e., the product of $\mathfrak{m}$ copies of discrete two-point space $\{1, -1\}$ with the standard product topology. In Section 3 we use our previous results in construction of a field K with $eX_n(K)$ (and thus also $X_n(K)$) homeomorphic to a given Cantor cube $D_{\mathfrak{m}}$.

2. Extension theory

The third section of [2] develops the general extension theory of signatures. For convenience of the reader we recall some main results that we use in this paper.

Let L/K denote a field extension. If χ^L is a signature of L and χ its restriction to K, then χ is a signature of K. Thus χ^L is *an extension* of χ and we shall write $\chi^L \mid \chi$. For $\chi^L \mid \chi$ we have $s(\chi) \mid s(\chi^L)$. If $s(\chi) = s(\chi^L)$, then χ^L is called *a faithful extension* of χ.

By analogy, if $P^L \in X_n(L)$ then $P = P^L \cap K \in X_n(K)$. We say that P^L is *an extension* of P and we write $P^L \mid P$. If exact levels of P^L and P coincide, then we say that P^L is *a faithful extension* of P.

The next theorem is a particular version of [2, Th. 3.4, p. 65].

Theorem 3. *Let L/K be a field extension, $\chi \in \mathrm{Sgn}(K)$, $\chi^L \in \mathrm{Sgn}(L)$, $\chi^L \mid \chi$.*

(1) *We have $A(\chi^L) \cap K = A(\chi)$ and the signature $\overline{\chi^L}$ of $k(\chi^L)$ is an extension of the signature $\bar{\chi}$ of $k(\chi)$.*
(2) *If $\eta \in \mathrm{Sgn}(K)$ is such that $A(\eta) = A(\chi)$ and $\bar{\eta} = \bar{\chi}$, then there exists $\eta^L \mid \eta$ such that $A(\eta^L) = A(\chi^L)$ and $\overline{\eta^L} = \overline{\chi^L}$.*

Corollary 4. *Let L/K be a field extension. Then $\chi \in \mathrm{Sgn}(K)$ has an extension χ^L to L if and only if there exists $\chi_0 \in \mathrm{Sgn}_1(K)$ such that $A(\chi) = A(\chi_0)$, $\bar{\chi} = \bar{\chi}_0$ and χ_0 has a faithful extension to L.*

P r o o f . According to previous theorem it suffices to show the existence of χ_0.

Let χ_0^L be a signature of L of exact level 1 such that $A(\chi_0^L) = A(\chi^L)$ and $\overline{\chi_0^L} = \overline{\chi^L}$. We define χ_0 as the restriction $\chi_0^L|_K$. □

We have to notice that the problem of extension of signatures differs from the problem of extension of orderings. It is possible that ordering P does not extend, whereas its signature does. For example, let K be a field with a signature χ of higher level and let χ_0 be a signature of a total order such that $A(\chi) = A(\chi_0)$ and $\bar{\chi} = \bar{\chi}_0$. Consider a real closure F of $\big(K, \ker(\chi_0)\big)$. By Theorem 3, the signature χ has an extension to F, since χ_0 extends to F. Thus an extension of χ must be of infinite level, because F does not admit an ordering of level $n > 1$ [2, Prop. 2.9. iii, p. 61].

In the fourth section of [2] the authors show that in the case of Galois extensions the extension theory of signatures takes on a particularly simple form. The following theorem describes possible levels of extensions of $\chi \in \mathrm{Sgn}(K)$ to Galois extension L/K.

Theorem 5. [2, Th. 4.4, p. 73] *Let L/K be a Galois extension of fields and $\chi \in \mathrm{Sgn}(K)$. If χ extends to L, then either all extensions are faithful or all have level $2s(\chi)$.*

For a Galois extension L/K we shall consider $G(L/K)$ as a topological group with subbasis consisting of all sets of the form

$$V(b, a) = \{\sigma \in G(L/K)\colon \sigma(b) = a\}, \qquad a, b \in L.$$

Then $G(L/K)$ is compact and totally disconnected [5, Chap. V, §10, Sec. 3].

Theorem 6. *Let L/K be a Galois extension of fields, $\eta \in \mathrm{Sgn}_n(L)$, $\chi = \eta|_K$. Then the map*

$$G(L/K) \longrightarrow \{\chi^L \in \mathrm{Sgn}_n(L)\colon \chi^L \mid \chi\}, \qquad \sigma \longmapsto \sigma\eta,$$

where $\sigma\eta(a) = \eta\big(\sigma^{-1}(a)\big)$, is a homeomorphism.

P r o o f . By [2, Th. 4.2, p. 73] the map defined above is a bijection. Since $G(L/K)$ is compact it suffices to show that this bijection is continuous. Really, for $a \in L$ and $\omega \in \mathrm{Sgn}_n(L)$ the inverse image of the set $U(\omega, a) \cap \{\chi^L \in \mathrm{Sgn}_n(L)\colon \chi^L \mid \chi\}$ is of the form

$$\{\sigma \in G(L/K)\colon \sigma\eta(a) = \omega(a)\} = \{\sigma \in G(L/K)\colon \exists_{b\in B}\, \sigma^{-1}(a) = b\} = \bigcup_{b\in B} V(b, a),$$

where $B = \{b \in L\colon \eta(b) = \omega(a)\}$. □

Theorem 7. *Let L/K be a Galois extension of fields, $P \in X_n(K)$ and suppose that $Q \in X_n(L)$ is a faithful extension of P. Then the map*

$$G(L/K) \longrightarrow \{P^L \in X_n(L) \colon P^L \mid P\}, \quad \sigma \mapsto \sigma(Q)$$

is a homeomorphism.

Proof. By [2, Th. 4.5, p. 74] we have that:

(i) the set of all extensions of P is $\{\sigma(Q) \colon \sigma \in G(L/K)\}$,

(ii) all extensions of P are faithful, since Q is a faithful extension of P,

(iii) for any $P^L \mid P$ the set

$$Z(P^L) = \{\sigma \in G(L/K) \colon \sigma(P^L) = P^L\} = \{\mathrm{Id}\}.$$

Therefore the map is a bijection. It suffices to show that it is continuous.

Let $H_n(a)$ be a subbasis set of $X_n(L)$. Then the inverse image of

$$H_n(a) \cap \{P^L \in X_n(L) \colon P^L \mid P\}$$

is of the form

$$\{\sigma \in G(L/K) \colon a \in \sigma(Q)\} = \{\sigma \in G(L/K) \colon \sigma^{-1}(a) \in Q\} = \bigcup_{b \in Q} V(b, a),$$

so it is open. Similarly, the inverse image of

$$H_n^c(a) \cap \{P^L \in X_n(L) \colon P^L \mid P\}$$

is of the form

$$\bigcup_{b \in L \setminus Q} V(b, a),$$

and it is open, as well. □

It seems natural to ask when, for the fixed Galois extension L/K, the fixed signature χ of K extends faithfully to L.

Let $\chi_n \in \mathrm{eSgn}_n(K)$ and let χ_0 be a signature of a total order of K such that $A(\chi_0) = A(\chi_n)$ and $\bar{\chi}_0 = \bar{\chi}_n$. As we have pointed out in Section 1, there exists a unique character τ of the value group of a valuation v associated to $A(\chi_n)$, such that

$$\chi_n = \chi_0 \cdot \tau \circ v,$$

and

$$\#\,\mathrm{Im}(\tau) = \begin{cases} 2n & \text{if } n \text{ is even}, \\ n \text{ or } 2n & \text{if } n \text{ is odd}. \end{cases}$$

Let

$$\chi_1 = \chi_0 \cdot \tau^n \circ v.$$

According to [2, Th. 1.12, p. 56 and Th. 2.5, p. 59], we have that χ_1 is a signature of K of level 1 with $A(\chi_1) = A(\chi_n)$ and $\bar{\chi}_1 = \bar{\chi}_n$.

If $\#\,\mathrm{Im}(\tau) = 2n$ then $\chi_1 \neq \chi_0$. But if $\#\,\mathrm{Im}(\tau) = n$, where n is odd, then $\chi_1 = \chi_0$ and $\ker(\chi_n) \subset \ker(\chi_0)$.

Definition 8. Under the notation fixed above the pair (χ_0, χ_1) is called a *pair of signatures associated with* χ_n. If (χ_0, χ_1) is a pair of signatures associated with χ_n then the pair $\big(\ker(\chi_0), \ker(\chi_1)\big)$ is called a *pair of total orders associated with* $\ker(\chi_n)$.

Lemma 9. *Let L/K be a Galois extension of fields and $\chi_n \in \operatorname{eSgn}_n(K)$. If there exists a pair (χ_0, χ_1) of signatures associated with χ_n such that χ_0 and χ_1 have faithful extensions to L then χ_n also has a faithful extension to L.*

Proof. Let (χ_0, χ_1) be a pair of signatures associated with χ_n, such that χ_0 and χ_1 extend faithfully to L. Let v be the valuation determined by $A(\chi_n)$ and Γ its value group. Then there exists a unique character τ of Γ such that

$$\chi_n = \chi_0 \cdot \tau \circ v ,$$

and

$$\chi_1 = \chi_0 \cdot \tau^n \circ v .$$

Observe that:

(i) all extensions of χ_0 and all extensions of χ_1 are faithful (Theorem 5),

(ii) there exists an extension χ^L of χ_n (Corollary 4),

(iii) $s(\chi^L) = s(\chi_n)$ or $s(\chi^L) = 2s(\chi_n)$ (Theorem 5).

Suppose that $s(\chi^L) = 2s(\chi_n) = 2n$. By Theorem 3 there exists $\chi_0^L \mid \chi_0$ such that $A(\chi_0^L) = A(\chi^L)$ and $\overline{\chi_0^L} = \overline{\chi^L}$ on the residue field $k(\chi^L)$. Let v^L be the valuation determined by $A(\chi^L)$ with the value group Γ^L. Then there exists a unique character τ^L of Γ^L such that

$$\chi^L = \chi_0^L \cdot \tau^L \circ v^L .$$

We have $\#\operatorname{Im}(\tau^L) = 4n$, since $s(\chi^L) = 2n$. Notice that

$$\chi = \chi^L|_K = \chi_0^L|_K \cdot \tau^L \circ v^L|_K = \chi_0 \cdot \tau^L|_\Gamma .$$

Since τ is uniquely determined, we have that $\tau^L|_\Gamma = \tau$. Let

$$\chi_1^L = \chi_0^L \cdot (\tau^L)^n \circ v^L .$$

Then $s(\chi_1^L) = \frac{1}{2}\operatorname{lcm}(2, \#\operatorname{Im}\big((\tau^L)^n\big) = 2$.

But $\chi_1^L|_K = \chi_0^L|_K \cdot (\tau^L)^n \circ v^L|_K = \chi_0 \cdot \tau^n \circ v = \chi_1$. A contradiction with (i). □

Now suppose that n is odd and $\chi_n \in \operatorname{eSgn}_n(K)$ has a representation

$$\chi_n = \chi_0 \cdot \tau \circ v ,$$

for some $\chi_0 \in \mathrm{eSgn}_1(K)$ and some character τ of the value group of the valuation v determined by $A(\chi_n)$. As we have pointed out above, if $\#\,\mathrm{Im}(\tau) = n$, then $\ker(\chi_n) \subset \ker(\chi_0)$. If $\#\,\mathrm{Im}(\tau) = 2n$ and

$$\chi_1 = \chi_0 \cdot \tau^n \circ v\,,$$

then direct computation shows that $\ker(\chi_n) \subset \ker(\chi_1)$. Thus for every ordering P_n of odd exact level n there exists a total order P_0 such that $P_n \subset P_0$. By [4, Lem 1.6, p. 516] the total order P_0 is uniquely determined for each $P_n \in eX_n(K)$.

Lemma 10. *Let L/K be a Galois extension of fields. Let $\chi_n \in \mathrm{eSgn}_n(K)$, where n is odd and let $\chi_0 \in \mathrm{Sgn}_1(K)$ be such that $\ker(\chi_n) \subset \ker(\chi_0)$. Then χ_n has a faithful extension to L if and only if χ_0 has a faithful extension to L.*

P r o o f. We have $\chi_n = \chi_0 \cdot \tau \circ v$, where v is a valuation determined by $A(\chi_n)$ and τ is a character of the value group of v with $\#\,\mathrm{Im}(\tau) = n$. Then (χ_0, χ_0) is a pair of signatures associated with χ_n. If χ_0 has a faithful extension to L then, by Lemma 9, χ_n also has a faithful extension to L.

Now suppose that χ^L is a faithful extension of χ_n. By [4, Lem. 1.6, p. 516] there exists a unique $\chi_0^L \in \mathrm{Sgn}_1(L)$ such that $\ker(\chi^L) \subset \ker(\chi_0^L)$. Then $\ker(\chi_n) = \ker(\chi^L\,|_K) \subset \ker(\chi_0^L\,|_K)$. From the uniqueness of χ_0 it follows that $\chi_0^L\,|_K = \chi_0$. □

From Lemma 9 and Lemma 10 it immediately follows

Corollary 11. *Let L/K be a Galois extension of fields and let P be an ordering of K.*

(1) *If P is an ordering of even exact level and there exists a pair (P_0, P_1) of total orders associated with P such that P_0 and P_1 extend faithfully to L, then P also has a faithful extension to L.*

(2) *If P is an ordering of odd exact level and P_0 is a total order of K such that $P \subset P_0$, then P has a faithful extension to L if and only if P_0 has a faithful extension to L.*

3. The Cantor cube as a space of orderings of exact level n

Our aim in this chapter is to construct a field K with $eX_n(K)$ homeomorphic to the Cantor cube $D_{\mathfrak{m}}$ ($\mathfrak{m}$—any infinite cardinal number). It is known that for any real closed field F the space $X_1(K)$ of total orders of the field

$$K := F(X)\Big(\big\{\sqrt{X-a}\colon a \in \dot{F}\big\}\Big)$$

is homeomorphic to the Cantor cube $\{-1,1\}^{\dot{F}}$ (cf. [6, Remark, p. 230]). We show that similar result holds for orderings of exact level $n>1$. When n is odd we use the same field K, but in the case of even n we have to modify a little the construction of the field K.

Suppose that F is a real closed field. Let $A = A(\dot{F}^2)$ be the valuation ring determined by the unique order of F and I its maximal ideal. Let v be the valuation associated with A and let Γ and k be its value group and its residue field, respectively. By [2, Cor. 2.6, p. 59] we have that $\overline{\dot{F}^2}$ is the unique archimedean total order on k, since $\dot{F}^2$ is the unique total order on F. It is possible that Γ is not a trivial group, since the order $\dot{F}^2$ can be nonarchimedean. However, F does not admit orderings of level $n>1$, so Γ does not admit nontrivial characters of finite level. Direct computation shows that the set

$$P_0 = \left\{ \frac{f}{g} \in F(X)\colon \frac{a_s}{b_t} \in \dot{F}^2 \right\},$$

where $s=\deg(f)$, $t=\deg(g)$ and a_s, b_t are the leading coefficients of the polynomials f and g, respectively, is a total order of $F(X)$. One can easily check that

$$A(P_0) = \left\{ \frac{f}{g} \in F(X)\colon t>s \right\} \cup \left\{ \frac{f}{g} \in F(X)\colon t=s \wedge \frac{a_s}{b_t} \in A \right\},$$
$$I(P_0) = \left\{ \frac{f}{g} \in F(X)\colon t>s \right\} \cup \left\{ \frac{f}{g} \in F(X)\colon t=s \wedge \frac{a_s}{b_t} \in I \right\},$$
$$\dot{A}(P_0) = \left\{ \frac{f}{g} \in F(X)\colon t=s \wedge \frac{a_s}{b_t} \in \dot{A} \right\}.$$

Therefore $k(P_0) \cong k$. The map

$$v_0\colon \dot{F}(X) \longrightarrow \mathbb{Z}\times\Gamma, \quad v_0\left(\frac{f}{g}\right) = \left(t-s, v\left(\frac{a_s}{b_t}\right)\right),$$

is a group epimorphism with the kernel $\dot{A}(P_0)$. Thus $\Gamma_0 = \dot{F}(X)/\dot{A}(P_0)$ is isomorphic to the group $\mathbb{Z}\times\Gamma$ equipped with the lexicographic order.

Let χ_0 be a signature of P_0. We shall describe all the signatures of finite level of $F(X)$ which determine the same valuation ring as χ_0. By [2, Cor. 2.6, p. 59] it suffices to determine all characters (of finite level) of Γ_0, since $k(P_0)$ admits the only one archimedean total order.

Let τ be any character of finite level of Γ_0. Then the restriction of τ to Γ is a trivial character, since Γ does not admit nontrivial characters of finite level. Thus if $\#\operatorname{Im}(\chi)=m$ then $\tau(v_0(X))$ is a generator of $\mu(m)$, thus a primitive root of unity of level m and $\tau\left(v_0\left(\frac{f}{g}\right)\right) = \left(\tau(v_0(X))\right)^{t-s}$ (t, s are degrees of g, f, respectively).

Every signature of exact level n of $F(X)$ with valuation ring $A(P_0)$ is of the form

$$\chi_0 \cdot \tau \circ v_0 \,,$$

where τ is a character of Γ_0 such that:

$$\#\operatorname{Im}(\tau) = \begin{cases} 2n\,, & \text{for even } n\,,\\ n \text{ or } 2n\,, & \text{for odd } n\,. \end{cases}$$

Suppose that n is odd. Let ε_{2n} be a fixed primitive root of unity of level $2n$. Then $(\varepsilon_{2n})^2$ is a primitive root of unity of level n. Any signature of exact level n of $F(X)$ with valuation ring $A(P_0)$ has one of the form: $\kappa \circ \chi_n$ or $\kappa \circ \hat{\chi}_n$, where κ is running through $\operatorname{Aut}\big(\mu(2n)\big)$ and

$$\chi_n\left(\frac{f}{g}\right) = \chi_0\left(\frac{f}{g}\right) \cdot \varepsilon_{2n}^{t-s}\,,$$

$$\hat{\chi}_n\left(\frac{f}{g}\right) = \chi_0\left(\frac{f}{g}\right) \cdot (\varepsilon_{2n})^{2(t-s)}\,,$$

where t, s are degrees of g, f, respectively.

Note, that for $n=1$ we have $\hat{\chi}_1 = \chi_0$ and

$$\ker(\chi_1) = \left\{ \frac{f}{g} \in F(X) \colon (-1)^{t-s}\frac{a_s}{b_t} \in \dot{F}^2 \right\} =: P_1\,.$$

Moreover, $\ker(\chi_n)$ and $\ker(\hat{\chi}_n)$ are the unique orderings of exact level n of $F(X)$ with valuation ring $A(P_0)$ and we have

$$\ker(\chi_n) \neq \ker(\hat{\chi}_n)\,, \quad \ker(\hat{\chi}_n) \subset P_0\,, \quad \ker(\chi_n) \subset P_1\,.$$

Now suppose that n is even. Like previous let ε_{2n} be a fixed primitive root of unity of level $2n$. Then any signature of exact level n of $F(X)$ with valuation ring $A(P_0)$ has one of the form: $\kappa \circ \chi_n$, where κ is running through $\operatorname{Aut}\big(\mu(2n)\big)$ and

$$\chi_n\left(\frac{f}{g}\right) = \chi_0\left(\frac{f}{g}\right) \cdot \varepsilon_{2n}^{t-s}\,.$$

Thus $\ker(\chi_n)$ is the unique ordering of exact level n of $F(X)$ with valuation ring $A(P_0)$. Note, that

$$\chi_0\left(\frac{f}{g}\right) \cdot \big((\varepsilon_{2n})^{t-s}\big)^n = (-1)^{t-s}\chi_0\left(\frac{f}{g}\right) = \chi_1\left(\frac{f}{g}\right).$$

Therefore (χ_0, χ_1) is a pair of signatures associated with χ_n.

Theorem 12. *The Cantor cube $D_{\mathfrak{m}}$ is homeomorphic to the space $eX_n(K)$ for some formally real field K.*

P r o o f . Let F be a real closed field of (infinite) cardinality $\mathfrak{m}$. A possibility of such a choice was remarked in [6, Remark, p. 230].

Suppose that n is odd. Let

$$K := F(X)\Big(\big\{\sqrt{X-a}\colon a \in \dot{F}\big\}\Big).$$

Note, that K is a compositum of quadratic extensions of $F(X)$, so by [5, Chap. V, §10, Sec. 1, Prop. 1] it is a Galois extension of $F(X)$.

By Theorem 5, every ordering of exact level n (odd) of K is a faithful extension of some ordering of $F(X)$. Let P_0 be a total order of $F(X)$ which we have described above. By [6, Remark, p. 230], every total order of K is an extension of P_0. Therefore by Corollary 11, every ordering of exact level n of K is a faithful extension of ordering, which is contained in P_0. As we have already seen, $\ker(\hat{\chi}_n)$ is the unique ordering of exact level n contained in P_0, so we have

$$eX_n(K) = \big\{P^K \in X_n(K)\colon P^K \mid \ker(\hat{\chi}_n)\big\}.$$

By Theorem 7, the space $eX_n(K)$ is homeomorphic to $G\big(K/F(X)\big)$. Now it suffices to show that $G\big(K/F(X)\big)$ is homeomorphic to $D_{\mathfrak{m}} \cong \{-1,1\}^{\dot{F}}$.

Let $\mathcal{S}$ be a family of all finite subsets of $\dot{F}$. For any $S \in \mathcal{S}$ consider

$$K_S := F(X)\Big(\big\{\sqrt{X-a}\colon a \in S\,\big\}\Big).$$

Routine induction shows that the map

$$\varphi_S\colon G\big(K_S/F(X)\big) \longrightarrow \{-1,1\}^S,$$

$$\varphi_S(\sigma)(a) = \begin{cases} 1 & \text{if } \sigma\big(\sqrt{X-a}\,\big) = \sqrt{X-a}\,, \\ -1 & \text{if } \sigma\big(\sqrt{X-a}\,\big) = -\sqrt{X-a}\,. \end{cases}$$

for $a \in S$, is a bijection.

We have that $G\big(K/F(X)\big)$ is the inverse limit of $\big\{G\big(K_S/F(X)\big)\colon S \in \mathcal{S}\big\}$ (cf. [5, Chap. V, §10, Sec. 3]). Therefore the map

$$\varphi\colon G\big(K/F(X)\big) \longrightarrow \{-1,1\}^{\dot{F}},$$

$$\varphi(\sigma)(a) = \begin{cases} 1 & \text{if } \sigma\big(\sqrt{X-a}\,\big) = \sqrt{X-a}\,, \\ -1 & \text{if } \sigma\big(\sqrt{X-a}\,\big) = -\sqrt{X-a}\,. \end{cases}$$

is a bijection. Let $U(a,\varepsilon)$ be any subbasis subset of the product topology of $\{-1,1\}^{\dot{F}}$, i.e.,

$$U(a,\varepsilon) = \big\{\delta \in \{-1,1\}^{\dot{F}}\colon \delta(a) = \varepsilon\big\}, \qquad a \in \dot{F},\ \varepsilon \in \{1,-1\}.$$

Then $\varphi^{-1}(U(a,\varepsilon)) = \{\sigma \in G(K/F(X)) : \sigma(\sqrt{X-a}) = \varepsilon\sqrt{X-a}\} = V(\sqrt{X-a}, \varepsilon\sqrt{X-a})$. Therefore φ is continuous. Since $G(K/F(X))$ is compact, we have that φ is a homeomorphism.

Now suppose that n is even. Let

$$K := F(X)\left(\left\{\sqrt{\frac{X-a}{X}} : a \in \dot{F}\right\}\right).$$

Using the same arguments like above one can prove that $K/F(X)$ is a Galois extension and the map

$$\varphi\colon G(K/F(X)) \longrightarrow \{-1,1\}^{\dot{F}},$$

$$\varphi(\sigma)(a) = \begin{cases} 1 & \text{if } \sigma\left(\sqrt{\frac{X-a}{X}}\right) = \sqrt{\frac{X-a}{X}}, \\ -1 & \text{if } \sigma\left(\sqrt{\frac{X-a}{X}}\right) = -\sqrt{\frac{X-a}{X}}, \end{cases}$$

is a continuous bijection, so $G(K/F(X))$ is homeomorphic to $D_{\mathfrak{m}} = \{-1,1\}^{\dot{F}}$.

We have

$$X_1(K) = H(X) \,\dot{\cup}\, H(-X),$$

where $H(X)$ and $H(-X)$ are the sets of Harrison subbasis of $X_1(K)$. From the identity

$$X - a = X\left(\frac{X-a}{X}\right)^2$$

it follows that $H(X) = H(X-a)$ and $H(-X) = H(-X+a)$ for every $a \in F$. We shall show that

$$P \cap F(X) = \begin{cases} P_0 & \text{if } P \in H(X), \\ P_1 & \text{if } P \in H(-X), \end{cases}$$

where P_0, P_1 are total orders of $F(X)$ which we have described above.

Let $f \in F[X] \cap P_0$. Then f can be written as:

$$a^2(X-a_1)\dots(X-a_k)\cdot\{\text{product of irred. polynomials of deg. 2 with the leading coeff. 1}\}.$$

Every irreducible polynomial of degree 2 with the leading coefficient 1 is a sum of squares in K, so $f \in P$ for every $P \in H(X)$. Notice, that if $\frac{f}{g} \in P_0$ then f, $g \in P_0$ or $-f$, $-g \in P_0$. Thus $\frac{f}{g} \in P$ and $P_0 = F(X) \cap P_0 \subseteq F(X) \cap P$. By [8, Rem. 1.2.2, p. 13] we have that $P_0 = F(X) \cap P$, for every $P \in H(X)$.

Let $f \in F[X] \cap P_1$. Then f can be written as:

$$a^2(-X+a_1)\dots(-X+a_k)\cdot\{\text{product of irred. polyn. of deg. 2 with the leading coeff. 1}\},$$

where k is odd iff $\deg(f)$ is odd. Thus $f \in P$ for every $P \in H(-X)$. Like previously we have $P_1 = F(X) \cap P$, for every $P \in H(-X)$.

By Corollary 4, every ordering of K is an extension of some ordering P of $F(X)$ such that $A(P) = A(P_0) = A(P_1)$ and $\bar{P} = \bar{P}_0 = \bar{P}_1$ on $k(P_0)$. Since P_0 and P_1 are the unique total orders with the valuation ring $A(P)$ we have that P has a faithful extension to K (cf. Corollary 11). As we have already seen, $\ker(\chi_n)$ is the unique ordering of exact level n with valuation ring $A(P_0)$. Thus

$$eX_n(K) = \{P^K \in X_n(K)\colon P^K \mid \ker(\chi_n)\}.$$

By Theorem 7, the space $eX_n(K)$ is homeomorphic to $G(K/F(X))$, hence to the Cantor cube $D_{\mathfrak{m}}$. □

Remark 13. In fact, the space $eX_n(K)$ for

$$K = F(X)\left(\left\{\sqrt{\frac{X-a}{X}}\colon a \in \dot{F}\right\}\right)$$

is homeomorphic to the Cantor cube $D_{\mathfrak{m}}$ for any $n \in \mathbb{N}$. Let n be an odd natural number. All total orders of K are extensions of P_0 or P_1, thus all orderings of exact level n (cf. Corollary 11) are faithful extensions of $\ker(\hat{\chi}_n)$ or $\ker(\chi_n)$, where $\ker(\hat{\chi}_n) \subseteq P_0$ and $\ker(\chi_n) \subseteq P_1$, i.e.,

$$eX_n(K) = \{P \in X_n(K)\colon P \mid \ker(\chi_n)\} \,\dot{\cup}\, \{P \in X_n(K)\colon P \mid \ker(\hat{\chi}_n)\}.$$

By Theorem 7, each of those subsets is homeomorphic to $G(K/F(X))$, hence to $D_{\mathfrak{m}}$.

Corollary 14. *The Cantor cube $D_{\mathfrak{m}}$ is homeomorphic to the space $X_n(K)$ for some formally real field K.*

Proof. Since

$$X_n(K) = \dot{\bigcup}\{eX_m(K)\colon m \mid n\}$$

it follows from Theorem 12 and Remark 13 that the space $X_n(K)$, for

$$K = F(X)\left(\left\{\sqrt{\frac{X-a}{X}}\colon a \in \dot{F}\right\}\right),$$

is homeomorphic to the Cantor cube $D_{\mathfrak{m}}$. □

Remark 15. It is routine matter to observe that for

$$K = F(X)\left(\left\{\sqrt{\frac{X-a}{X}}\colon a \in \dot{F}\right\}\right)$$

the space $\mathrm{Sgn}_n(K)$ is a disjoint sum of finite number subspaces each of which is homeomorphic to $G(K/F(X)) \cong D_{\mathfrak{m}}$. Thus $\mathrm{Sgn}_n(K) \cong D_{\mathfrak{m}}$.

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