Tatra Mt. Math. Publ. **30** (2005), 71–85

TATRA MOUNTAINS
Mathematical Publications

CONVEXITIES OF RIESZ GROUPS

Judita Lihová

ABSTRACT. In this paper, the ordered class of all convexities of Riesz groups is investigated. Further, some principal convexities are dealt with.

Riesz groups are special partially ordered groups. Their importance consists in the fact that they include lattice ordered groups and occur in different fields of mathematics. The foundations of the theory of Riesz groups were laid by L. Fuchs ([1]). The research was continued by further authors, e.g., J. R. Teller ([12]), M. Jakubíková ([8]), K. R. Goodearl ([3]).

The notion of a convexity of lattices has been introduced by E. Fried at the Problems Session of the Conference on General Algebra at Krems in 1988 [11]. E. Fried formulated a question concerning the "number" of convexities of lattices. This problem was solved in [4]. Convexities can be defined also for other types of ordered algebraic structures. J. Jakubík defined and studied convexities of d-groups ([5]) and l-groups ([6], [7]).

In the present paper we investigate the ordered class $\mathcal{C}(\mathcal{R})$ of all convexities of Riesz groups. We prove that $\mathcal{C}(\mathcal{R})$ is a distributive lattice (omitting the fact that $\mathcal{C}(\mathcal{R})$ is a proper class). Further we deal with some principal convexities.

1. Preliminaries

We will use the standard terminology and notations for partially ordered groups (see, e.g., [2]). We start with the key-definition.

Definition 1.1. *Riesz group* is a directed partially ordered group $(G, +, \leq)$ satisfying the following interpolation property:

(IP) to all $a_i, b_j \in G$ with $a_i \leq b_j$, $i, j \in \{1, 2\}$, there exists $c \in G$ such that $a_i \leq c \leq b_j$.

2000 Mathematics Subject Classification: 06F20.
Keywords: Riesz group, convexity.

Supported by grant VEGA 1/0423/03.

This definition is due to L. Fuchs (see [1] or [2]). Definitions of Riesz groups introduced in [12] and [3] are slightly different.

Lattice ordered groups (l-groups, for short) can serve as examples of Riesz groups. Riesz groups which fail to be l-groups can be found in [1].

By an *antilattice* such a partially ordered group will be meant, in which only pairs of comparable elements may have a greatest lower and a least upper bound. Linearly ordered groups are just those which are both l-groups and antilattices. A partially ordered group G will be said to be *dense* if, whenever $a, b \in G$, $a < b$, there exists $c \in G$ with $a < c < b$.

2. Convexities of Riesz groups

Let $\mathcal{R}$ denote the class of all Riesz groups. If $\{G_i : i \in I\}$ is a nonempty subsystem of $\mathcal{R}$, the symbol $\prod\limits_{i\in I} G_i$ will be used for the direct product of G_i $(i \in I)$; π_j will be the projection of $\prod\limits_{i\in I} G_i$ into G_j. If $f \in \prod\limits_{i\in I} G_i$, by the support of f, the set $\operatorname{supp} f = \{i \in I : \pi_i(f) = f(i) \neq 0\}$ will be meant. If $G_i = G$ for each $i \in I$, we will write G^I instead of $\prod\limits_{i\in I} G_i$. The direct product of two Riesz groups G_1, G_2 will be denoted by $G_1 \times G_2$. It is easy to see that the direct product of Riesz groups is also a Riesz group.

In what follows we will deal with convex directed subgroups and o-ideals of Riesz groups. By an *o-ideal* of a Riesz group G, a convex directed normal subgroup of G will be meant. It is easy to verify that each convex directed subgroup of a Riesz group is a Riesz group, too. If A is an o-ideal of $G \in \mathcal{R}$, then the factor group G/A is also a Riesz group, if we define

$$g_1 + A \le g_2 + A \iff g_1' \le g_2' \text{ for some } g_1' \in g_1 + A,\ g_2' \in g_2 + A$$
$$\text{(or, equivalently, } g_1 \le g_2 + a \text{ for some } a \in A)$$

(cf. [1]). Now let G_1, G_2 be partially ordered groups, $\varphi\colon G_1 \to G_2$. In [10], φ was defined to be a homomorphism if it satisfies the following conditions:

(i) φ is an order-preserving group homomorphism (shortly an o-homomorphism);

(ii) the kernel of φ is a directed set;

(iii) if $a \in G_1$, $\varphi(a) \ge 0$, then there exists $b \in G_1$, $b \ge 0$ with $\varphi(a) = \varphi(b)$.

Under this definition, homomorphic images of a Riesz group G are just (up to isomorphism) factor groups of G by all possible o-ideals of G (cf. [10], (3.5)).

We will use the following lemmas (cf. [10], (3.6), (3.7)):

Lemma 2.1. *Let G be a Riesz group, H a partially ordered group, φ a homomorphism of G into H and let A be a convex directed subgroup of G. Then*

(i) *the restriction $\varphi \restriction A$ of φ to A is a homomorphism of A into H;*

(ii) *$\varphi(A)$ is a convex directed subgroup of $\varphi(G)$.*

Lemma 2.2. *Let $G, H \in \mathcal{R}$ and let φ be a homomorphism of G onto H. If B is a convex directed subgroup of H, then $\varphi^{-1}(B)$ is a convex directed subgroup of G.*

The following lemma is easy to prove.

Lemma 2.3. *Let $G \in \mathcal{R}$ and let H, A be o-ideals of G such that $A \subseteq H$. Then A is an o-ideal of H, H/A is an o-ideal of G/A and $G/A/H/A$ is isomorphic to G/H.*

L. Fuchs proved the following theorem (cf. [1]), which will be also needed.

Theorem 2.4. *The partially ordered system of all o-ideals of a Riesz group G is a complete lattice satisfying the following infinite distributive law*

$$H \wedge \Big(\bigvee_{i\in I} A_i \Big) = \bigvee_{i\in I} (H \wedge A_i) .$$

Let us mention that the infimum in the above theorem is the set-theoretical intersection only in the case of finitely many o-ideals.

For a subclass $\mathcal{X}$ of the class $\mathcal{R}$ we denote by

$H\mathcal{X}$ the class of all homomorphic images of elements of $\mathcal{X}$;

$C\mathcal{X}$ the class of all convex directed subgroups of elements of $\mathcal{X}$ and their isomorphic copies;

$P\mathcal{X}$ the class of all direct products of elements of $\mathcal{X}$ and their isomorphic qcopies.

Definition 2.5. A nonempty subclass of $\mathcal{R}$ will be said to be a *convexity* of *Riesz groups* (or shortly a *convexity*) if it is closed under the operators H, C and P.

A convexity of l-groups was defined by J. Jakubík (see [6]) as a subclass of the class $\mathcal{M}$ of all l-groups closed under homomorphic images, convex l-subgroups and direct products. As evidently each convex directed subgroup and o-ideal of an l-group G is a convex l-subgroup and l-ideal of G, respectively, it is easy to see that each convexity of l-groups is also a convexity of Riesz groups. It is obvious that the class of all convexities of l-groups includes all varieties of l-groups. But, while varieties of l-groups form evidently a set, convexities of l-groups form a proper class. The latter was proved by J. Jakubík (see [6]). Hence the convexities of Riesz groups form a proper class, too.

Proceeding standardly and using (2.1), (2.2) we obtain

Lemma 2.6. *Let $\emptyset \neq \mathcal{X} \subseteq \mathcal{R}$. Then*

$$HH\mathcal{X} = H\mathcal{X}, \quad CC\mathcal{X} = C\mathcal{X}, \quad PP\mathcal{X} = P\mathcal{X},$$
$$CH\mathcal{X} \subseteq HC\mathcal{X}, \quad PH\mathcal{X} \subseteq HP\mathcal{X}, \quad PC\mathcal{X} \subseteq CP\mathcal{X}.$$

Then we have:

Theorem 2.7. *Let $\emptyset \neq \mathcal{X} \subseteq \mathcal{R}$. Then $HCP\mathcal{X}$ is the least convexity containing $\mathcal{X}$.*

The convexity $HCP\mathcal{X}$ will be said to be *generated by* $\mathcal{X}$. If $\mathcal{X}$ is a one-element class, then the convexity $HCP\mathcal{X}$ will be called *principal.*

3. Subdirect product of Riesz groups

Definition 3.1. Let G be a Riesz group, $\{H_i : i \in I\}$ a system of non-zero o-ideals of G such that $\bigcap\limits_{i \in I} H_i = \{0\}$ and for any $x \in G$, there holds $x \geq 0$ whenever $x + H_i \geq H_i$ for all $i \in I$. Then the mapping $\varphi \colon G \to \prod\limits_{i \in I} G/H_i$ defined by $\pi_i\bigl(\varphi(x)\bigr) = x + H_i$ for all $i \in I$, will be referred to as a *subdirect decomposition* of G.

Lemma 3.2. *If $\varphi \colon G \to \prod\limits_{i \in I} G/H_i$ is a subdirect decomposition of G, then φ is an injective homomorphism.*

P r o o f. Evidently φ is an o-homomorphism, Ker $\varphi = \{0\}$, so that Ker φ is a directed set and φ is injective. If $\varphi(x) \geq 0$, then $x + H_i \geq H_i$ for all $i \in I$ and hence $x \geq 0$. ☐

Let us notice that if I is a finite set, then the assumption

$$x + H_i \geq H_i \text{ for all } i \in I \implies x \geq 0$$

in the Definition (3.1) can be omitted, as the following lemma shows.

Lemma 3.3. *Let G be a Riesz group, H_1, H_2 non-zero o-ideals of G with $H_1 \cap H_2 = \{0\}$. Then the mapping $\varphi \colon G \to G/H_1 \times G/H_2$ defined by $\varphi(x) = (x + H_1,\ x + H_2)$ satisfies the condition*

$$x + H_i \geq H_i \ (i = 1, 2) \implies x \geq 0 .$$

P r o o f. Let $x + H_1 \geq H_1$, $x + H_2 \geq H_2$. Then $x + h_1 \geq 0$, $x + h_2 \geq 0$ for some $h_1 \in H_1$, $h_2 \in H_2$. In view of the directedness of H_1, H_2, we can suppose that $h_1, h_2 \geq 0$. We show that $0 = \inf\{h_1, h_2\}$. Let $p \in G$, $p \leq h_1, h_2$. By (IP) there exists $p' \in G$ such that $0, p \leq p' \leq h_1, h_2$. Since H_1, H_2 are convex, we have $p' \in H_1 \cap H_2 = \{0\}$. Therefore $0 = p' \geq p$. Now $0 \leq x + h_1$, $x + h_2$ and $x \leq x + h_1$, $x + h_2$, so that there exists $t \in G$ such that $0, x \leq t \leq x + h_1$, $x + h_2$. But then $-x + t \leq h_1, h_2$, which implies $-x + t \leq 0$. Hence $0 \leq t \leq x$. □

Definition 3.4. A non-zero Riesz group G will be said to be *finitely subdirectly irreducible* (*f-subdirectly irreducible*, for short) if the intersection of any two non-zero o-ideals of G is also non-zero.

The following two theorems are proved in [1].

Theorem 3.5. *An Abelian Riesz group is f-subdirectly irreducible if and only if it is a non-zero antilattice.*

Let us remark that the part "if" holds also without the assumption of the commutativity.

Theorem 3.6. *Each non-zero Abelian Riesz group G has a subdirect decomposition $\varphi\colon G \to \prod_{i\in I} G/H_i$ with f-subdirectly irreducible G/H_i $(i \in I)$.*

Let $\mathcal{K}$ be a convexity of Riesz groups. We will show that f-subdirectly irreducible elements of $\mathcal{K}$ can be described by means of the ultraproduct.

Definition 3.7. Let I be a nonempty set, $\mathcal{F}$ a system of subsets of I. Then $\mathcal{F}$ is said to be a *filter* over I, if it satisfies the following conditions:

(1) $I \in \mathcal{F}$, $\emptyset \notin \mathcal{F}$;

(2) $J \subseteq K$, $J \in \mathcal{F} \Rightarrow K \in \mathcal{F}$;

(3) $J, K \in \mathcal{F} \Rightarrow J \cap K \in \mathcal{F}$.

A filter $\mathcal{F}$ is called an *ultrafilter* if it satisfies:

(4) $J \subseteq I \Rightarrow J \in \mathcal{F}$ or $I - J \in \mathcal{F}$.

We will use the following lemma due to B. Jònsson (cf. [9]).

Lemma 3.8. *Let I be a nonempty set, $\mathcal{W}$ a family of subsets of I such that*

(i) $I \in \mathcal{W}$;

(ii) $J \subseteq K$, $J \in \mathcal{W} \Rightarrow K \in \mathcal{W}$;

(iii) $J \cup K \in \mathcal{W} \Rightarrow J \in \mathcal{W}$ *or* $K \in \mathcal{W}$.

Then there exists an ultrafilter $\mathcal{F}$ over I with $\mathcal{F} \subseteq \mathcal{W}$.

Let $\emptyset \neq \{G_i\colon i \in I\} \subseteq \mathcal{R}$, $G = \prod_{i\in I} G_i$ and let $\mathcal{F}$ be a filter over I. Define a binary relation on G by

$$f \sim_{\mathcal{F}} g \iff \bigl\{i \in I\colon f(i) = g(i)\bigr\} \in \mathcal{F}.$$

Lemma 3.9. *The above defined relation $\sim_{\mathcal{F}}$ is an equivalence relation. The set $H_{\mathcal{F}} = \{a \in G : a \sim_{\mathcal{F}} 0\}$ is an o-ideal of G and elements of the quotient set $G/H_{\mathcal{F}}$ are just the equivalence classes of $\sim_{\mathcal{F}}$.*

Proof. It suffices to prove that $H_{\mathcal{F}}$ is an o-ideal; the remainder is evident. Let us remark that if $a \in G$, then $a \in H_{\mathcal{F}}$ if and only if $I - \operatorname{supp} a \in \mathcal{F}$. Since there holds $\operatorname{supp}(a-b) \subseteq \operatorname{supp} a \cup \operatorname{supp} b$, $\operatorname{supp}(-x+a+x) = \operatorname{supp} a$ for any $a, b, x \in G$, we can easily show that $H_{\mathcal{F}}$ is a normal subgroup of G. To verify that $H_{\mathcal{F}}$ is convex, let $0 \le x \le h$, $h \in H_{\mathcal{F}}$. Then $I - \operatorname{supp} x \supseteq I - \operatorname{supp} h \in \mathcal{F}$ so that $x \in H_{\mathcal{F}}$, too. It remains to show that $H_{\mathcal{F}}$ is directed. Let $h \in H_{\mathcal{F}}$; we are going to show the existence of $t \in H_{\mathcal{F}}$ with $t \ge h, 0$. For $i \in \operatorname{supp} h$ let t_i be any element of G_i with $t_i \ge h(i), 0$. Now define

$$t(i) = \begin{cases} t_i & \text{if } \ i \in \operatorname{supp} h\,, \\ 0 & \text{if } \ i \notin \operatorname{supp} h\,. \end{cases}$$

Then t is as we need. □

Using the above notation, the factor group $G/H_{\mathcal{F}}$ is said to be a reduced product of $G = \prod\limits_{i \in G} G_i$ by the filter $\mathcal{F}$. If $\mathcal{F}$ is an ultrafilter over I, then the reduced product is called an ultraproduct. If all G_i are equal to H, then the ultraproduct $G/H_{\mathcal{F}}$ is referred to as an ultrapower of H by $\mathcal{F}$.

As to the denotation, we will write $G/\mathcal{F}$ instead of $G/H_{\mathcal{F}}$, the symbol $[g]\mathcal{F}$ will be used for $g + H_{\mathcal{F}} \in G/\mathcal{F}$.

The following lemmas concern the partial order in $\left(\prod\limits_{i \in I} G_i\right)/\mathcal{F}$.

Lemma 3.10. *Let $\emptyset \ne \{G_i : i \in I\} \subseteq \mathcal{R}$, $G = \prod\limits_{i \in I} G_i$ and let $\mathcal{F}$ be a filter over I. If $f \in G$, then $[f]\mathcal{F} \ge [0]\mathcal{F}$ if and only if $\{i \in I : f(i) \ge 0\} \in \mathcal{F}$.*

Proof. Let $f \in G$. If $[f]\mathcal{F} \ge [0]\mathcal{F} = H_{\mathcal{F}}$, then $f \ge g$ for some $g \in H_{\mathcal{F}}$. Consequently $\{i \in I : f(i) \ge 0\} \supseteq I - \operatorname{supp} g$ and since $I - \operatorname{supp} g \in \mathcal{F}$, we have also $\{i \in I : f(i) \ge 0\} \in \mathcal{F}$. Conversely, let $J = \{i \in I : f(i) \ge 0\} \in \mathcal{F}$. Define $g \in G$ by

$$g(i) = \begin{cases} 0 & \text{if } \ i \in J\,, \\ f(i) & \text{if } \ i \notin J\,. \end{cases}$$

Evidently $f \ge g$ and $g \in [0]\mathcal{F}$, so that $[f]\mathcal{F} \ge [0]\mathcal{F}$. □

Lemma 3.11. *Let $\emptyset \ne \{G_i : i \in I\} \subseteq \mathcal{R}$, $G = \prod\limits_{i \in I} G_i$ and let $\mathcal{F}$ be an ultrafilter over I. If $f \in G$, then $[f]\mathcal{F} > [0]\mathcal{F}$ is equivalent to $\{i \in I : f(i) > 0\} \in \mathcal{F}$.*

Proof. Let $f \in G$. Then $[f]\mathcal{F} > [0]\mathcal{F}$ is equivalent to $[f]\mathcal{F} \ge [0]\mathcal{F}$, $[f]\mathcal{F} \ne$

$[0]\mathcal{F}$, which holds if and only if $\{i \in I \colon f(i) \geq 0\} \in \mathcal{F}$, $\{i \in I \colon f(i) = 0\} \notin \mathcal{F}$. Now these two relations imply $\{i \in I \colon f(i) > 0\} \in \mathcal{F}$, since $\{i \in I \colon f(i) \geq 0\} = \{i \in I \colon f(i) > 0\} \cup \{i \in I \colon f(i) = 0\}$. Conversely, if $\{i \in I \colon f(i) > 0\} \in \mathcal{F}$, then also $\{i \in I \colon f(i) \geq 0\} \in \mathcal{F}$, because $\{i \in I \colon f(i) \geq 0\} \supseteq \{i \in I \colon f(i) > 0\}$ and $\{i \in I \colon f(i) = 0\} \notin \mathcal{F}$, since otherwise we would have $\emptyset = \{i \in I \colon f(i) > 0\} \cap \{i \in I \colon f(i) = 0\} \in \mathcal{F}$, a contradiction. $\square$

Evidently also dual statements to (3.10) and (3.11) are valid. Hence we obtain as a consequence, that if all G_i are linearly ordered, so is their ultraproduct.

If $\mathcal{X} \subseteq \mathcal{R}$, we denote by

$P_{\mathcal{U}}\mathcal{X}$ the class of all ultraproducts of elements of $\mathcal{X}$ and their isomorphic copies.

The following theorem is a generalization of that proved by Jònsson for l-groups (cf. [9]).

Theorem 3.12. *Let $\emptyset \neq \mathcal{X} \subseteq \mathcal{R}$, $\mathcal{K} = HCP\mathcal{X}$. If $A \in \mathcal{K}$ and A is f-subdirectly irreducible, then $A \in HCP_{\mathcal{U}}\mathcal{X}$.*

Proof. Let A be an f-subdirectly irreducible Riesz group, $A \in \mathcal{K}$. Then there exist $\{G_i \colon i \in I\} \subseteq \mathcal{X}$, a convex directed subgroup B of $G = \prod_{i\in I} G_i$ and a homomorphism φ of B onto A. Let $T = \operatorname{Ker}\varphi$ and for $J \subseteq I$ let $T_J = \{a \in G \colon J \subseteq I - \operatorname{supp} a\}$. It is easy to see that T_J is an o-ideal in G. Now let $T_{JB} = T_J \cap B$ and define $\mathcal{W} = \{J(\subseteq I) \colon T_{JB} \subseteq T\}$. Evidently T_{JB} is an o-ideal in B. We are going to show that $\mathcal{W}$ fulfils the hypothesis of (3.8).

Obviously $I \in \mathcal{W}$ and $J \subseteq K(\subseteq I)$, $J \in \mathcal{W}$ imply $K \in \mathcal{W}$. Now let $J \cup K \in \mathcal{W}$. We will show that then $J \in \mathcal{W}$ or $K \in \mathcal{W}$. We have $T_{J\cup KB} = T_{J\cup K} \cap B = (T_J \cap T_K) \cap B = (T_J \cap B) \cap (T_K \cap B) = T_{JB} \wedge T_{KB}$ in the lattice of o-ideals of B. The hypothesis $J \cup K \in \mathcal{W}$ implies $T_{J\cup KB} \subseteq T$, so that $T = T_{J\cup KB} \vee T = (T_{JB} \wedge T_{KB}) \vee T = (T_{JB} \vee T) \wedge (T_{KB} \vee T)$, because of the distributivity of the lattice of o-ideals of B. Now $T_{JB} \vee T/T$, $T_{KB} \vee T/T$ are o-ideals in B/T such that their intersection is $\{T\}$. But B/T is isomorphic to A and A is f-subdirectly irreducible, therefore either $T_{JB} \vee T = T$ or $T_{KB} \vee T = T$. In the first case we have $J \in \mathcal{W}$, in the latter case $K \in \mathcal{W}$.

By (3.8) there exists an ultrafilter $\mathcal{F}$ over I with $\mathcal{F} \subseteq \mathcal{W}$. Consider $G/\mathcal{F}$ and let ψ be the restriction of the natural homomorphism of G onto $G/\mathcal{F}$ to B. By (2.1) ψ is a homomorphism of B onto a convex directed subgroup of $G/\mathcal{F}$. It is easy to verify that $H_{\mathcal{F}} = \bigcup_{J\in\mathcal{F}} T_J$, so that $H_{\mathcal{F}} = \bigvee_{J\in\mathcal{F}} T_J$ in the lattice of o-ideals of G. Consequently $\operatorname{Ker}\psi = H_{\mathcal{F}} \cap B = \Big(\bigvee_{J\in\mathcal{F}} T_J\Big) \wedge B = \bigvee_{J\in\mathcal{F}} (T_J \wedge B) = \bigvee_{J\in\mathcal{F}} T_{JB} \subseteq \bigvee_{J\in\mathcal{W}} T_{JB} \subseteq T$ by (2.4). Now using (2.3) we obtain $A \cong B/T \cong (B/\operatorname{Ker}\psi)/(T/\operatorname{Ker}\psi)$, hence $A \in H\{B/\operatorname{Ker}\psi\}$. Further

$B/\operatorname{Ker}\psi \cong \psi(B)$, $\psi(B)$ being a convex directed subgroup of $G/\mathcal{F}$. We conclude that $A \in HCP_{\mathcal{U}}\mathcal{X}$. $\square$

4. The lattice of convexities

Let $\mathcal{C}(\mathcal{R})$ be the class of all convexities of Riesz groups. It is partially ordered by the class-theoretical inclusion. We can easily prove:

LEMMA 4.1. *Let $\{\mathcal{K}_\lambda : \lambda \in \Lambda\}$ be a nonempty subclass of the class $\mathcal{C}(\mathcal{R})$. Then $\bigcap\limits_{\lambda\in\Lambda} \mathcal{K}_\lambda \in \mathcal{C}(\mathcal{R})$.*

The class of all one-element Riesz groups is evidently the least element of $\mathcal{C}(\mathcal{R})$ and $\mathcal{R}$ is the greatest one. This and the above lemma provide the following lemma.

LEMMA 4.2. *Each nonempty subclass of $\mathcal{C}(\mathcal{R})$ has a least upper bound and a greatest lower bound.*

As we have already remarked, $\mathcal{C}(\mathcal{R})$ is a proper class. Nevertheless, in view of the above theorem we will apply for $\mathcal{C}(\mathcal{R})$ the usual lattice-theoretical terminology and notation. So we will use the symbol $\bigvee\limits_{\lambda\in\Lambda} \mathcal{K}_\lambda$ and $\bigwedge\limits_{\lambda\in\Lambda} \mathcal{K}_\lambda$ for the least upper bound and the greatest lower bound of $\{\mathcal{K}_\lambda : \lambda \in \Lambda\}(\subseteq \mathcal{C}(\mathcal{R}))$, respectively, and $\mathcal{C}(\mathcal{R})$ will be referred to as a lattice. The following assertion is evident.

LEMMA 4.3. *Let $\{\mathcal{K}_\lambda : \lambda \in \Lambda\}$ be a nonempty subclass of $\mathcal{C}(\mathcal{R})$. Then*

$$\bigwedge_{\lambda\in\Lambda} \mathcal{K}_\lambda = \bigcap_{\lambda\in\Lambda} \mathcal{K}_\lambda, \qquad \bigvee_{\lambda\in\Lambda} \mathcal{K}_\lambda = HCP\Big(\bigcup_{\lambda\in\Lambda} \mathcal{K}_\lambda\Big).$$

Evidently the class $\mathcal{C}(\mathcal{M})$ of all convexities of l-groups is a complete sublattice of the lattice $\mathcal{C}(\mathcal{R})$. It is known that the lattice $\mathcal{C}(\mathcal{M})$ is distributive (cf. [7]). We are going to prove that the lattice $\mathcal{C}(\mathcal{R})$ is also distributive.

DEFINITION 4.4. Let G be a Riesz group, $A, B \subseteq G$. We will say that G is the *internal product* of A, B and we will write $G \overset{i}{=} A \times B$, if the following conditions are satisfied:

(i) A, B are o-ideals of G;

(ii) for each $g \in G$ there exists a unique couple $(a, b) \in A \times B$ with $g = a + b$;

(iii) $a \in A$, $b \in B \Rightarrow a + b = b + a$;

(iv) $a \in A$, $b \in B$, $a + b \geq 0 \Rightarrow a \geq 0$, $b \geq 0$.

It is easy to verify that if $G \overset{i}{=} A \times B$, then $G \cong A \times B$ and conversely, if $G \cong A \times B$, then $G \overset{i}{=} A_1 \times B_1$ for some A_1 and B_1 isomorphic to A and B, respectively. Using this fact we can easily prove the following two lemmas which generalize those concerning l-groups.

LEMMA 4.5. *If $G = \mathcal{R}$, $G \overset{i}{=} A \times B$ and D is a convex directed subgroup of G, then $D \overset{i}{=} (A \cap D) \times (B \cap D)$.*

LEMMA 4.6. *If $G \in \mathcal{R}$, $G \overset{i}{=} A \times B$ and D is an o-ideal of G, then $A \cap D$ and $B \cap D$ is an o-ideal of A and B, respectively, and G/D is isomorphic to $A/A \cap D \times B/B \cap D$.*

LEMMA 4.7. *Let $\mathcal{K}_1, \mathcal{K}_2 \in \mathcal{C}(\mathcal{R})$ and let $\mathcal{K}$ be the class of all $G \in \mathcal{R}$ such that G is isomorphic to $A \times B$ for some $A \in \mathcal{K}_1$, $B \in \mathcal{K}_2$. Then $\mathcal{K} \in \mathcal{C}(\mathcal{R})$ and $\mathcal{K} = \mathcal{K}_1 \vee \mathcal{K}_2$.*

Proof. The inclusion $\mathcal{K} \subseteq \mathcal{K}_1 \vee \mathcal{K}_2$ is evident. Conversely, let $G \in \mathcal{K}_1 \vee \mathcal{K}_2$. In view of (4.3) we have $G \in HCP\ (\mathcal{K}_1 \cup \mathcal{K}_2)$, so that $G \in H\{B\}$, $B \in C\left\{\prod\limits_{i \in I} G_i\right\}$ for some $B \in \mathcal{R}$, $\{G_i : i \in I\} \subseteq \mathcal{K}_1 \cup \mathcal{K}_2$. Let $I_1 = \{i \in I : G_i \in \mathcal{K}_1\}$, $I_2 = I - I_1$, $U = \prod\limits_{i \in I_1} G_i$, $V = \prod\limits_{i \in I_2} G_i$. (If some of the sets I_1, I_2 is empty, the corresponding direct product is regarded as $\{0\}$). Then $U \in \mathcal{K}_1$, $V \in \mathcal{K}_2$. We can assume that $B \in C\{D\}$, $D \overset{i}{=} U \times V$. By (4.5) we have $B \overset{i}{=} (U \cap B) \times (V \cap B)$, where $U \cap B \in \mathcal{K}_1$, $V \cap B \in \mathcal{K}_2$, because $U \cap B \in C\{U\}$, $V \cap B \in C\{V\}$. Now using (4.6) we obtain $G \cong U_1 \times V_1$ for some $U_1 \in H\{U \cap B\}$, $V_1 \in H\{V \cap B\}$. Consequently $U_1 \in \mathcal{K}_1$, $V_1 \in \mathcal{K}_2$ and $G \in \mathcal{K}$. ☐

THEOREM 4.8. *The lattice $\mathcal{C}(\mathcal{R})$ is distributive.*

Proof. Let $\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3 \in \mathcal{C}(\mathcal{R})$. We are going to verify

$$\mathcal{K}_1 \wedge (\mathcal{K}_2 \vee \mathcal{K}_3) = (\mathcal{K}_1 \wedge \mathcal{K}_2) \vee (\mathcal{K}_1 \wedge \mathcal{K}_3).$$

Clearly $(\mathcal{K}_1 \wedge \mathcal{K}_2) \vee (\mathcal{K}_1 \wedge \mathcal{K}_3) \subseteq \mathcal{K}_1 \wedge (\mathcal{K}_2 \vee \mathcal{K}_3)$.

Now let $G \in \mathcal{K}_1 \wedge (\mathcal{K}_2 \vee \mathcal{K}_3)$. Then $G \in \mathcal{K}_1$ and $G \cong B \times D$ for some $B \in \mathcal{K}_2$, $D \in \mathcal{K}_3$ by (4.7). We can suppose that B, D are o-ideals of G, so that $B, D \in C\mathcal{K}_1 = \mathcal{K}_1$. Hence we have $G \in (\mathcal{K}_1 \wedge \mathcal{K}_2) \vee (\mathcal{K}_1 \wedge \mathcal{K}_3)$. ☐

5. Some principal convexities

Throughout this section, $L (\in \mathcal{R})$ will be a non-zero dense antilattice, G a non-zero finite trivially ordered group. By $L \circ G$, the lexicographic product of these groups will be meant. I.e., $L \circ G$ is the group $L \times G$ with the partial ordering defined by

$$(x,a) \le (y,b) \iff x < y \text{ or } (x = y \text{ and } a \le b).$$

In view of the assumption that G is trivially ordered, it is

$$(x,a) \le (y,b) \iff x < y \text{ or } (x,a) = (y,b),$$

so that

$$(x,a) < (y,b) \iff x < y.$$

In [10] it was shown that $L \circ G$ is a Riesz group. We will be interested in the convexity generated by $L \circ G$.

We want to describe f-subdirectly irreducible elements of $HCP\{L \circ G\}$. First we are going to study $P_{\mathcal{U}}\{L \circ G\}$.

Lemma 5.1. *Let I be a nonempty set, $\mathcal{F}$ an ultrafilter over I. Then the ultrapower $(L \circ G)^I/\mathcal{F}$ is isomorphic to $(L^I/\mathcal{F}) \circ G$.*

Proof. We are going to define a mapping ϕ of $(L \circ G)^I/\mathcal{F}$ to $(L^I/\mathcal{F}) \circ G$. For any $\varphi \in (L \circ G)^I$ with $\varphi(i) = (x_i, a_i) \in L \circ G$ set

$$\phi([\varphi]\mathcal{F}) = ([x]\mathcal{F}, g),$$

where $x \in L^I$ is defined by $x(i) = x_i$ for all $i \in I$ and g is the (unique) element of G such that $\{i \in I \colon a_i = g\} \in \mathcal{F}$. Now let also $\psi \in (L \circ G)^I$, $\psi(i) = (y_i, b_i)$,

$$\phi([\psi]\mathcal{F}) = ([y]\mathcal{F}, g').$$

Evidently $[\varphi]\mathcal{F} = [\psi]\mathcal{F}$ is equivalent to $\{i \in I \colon x_i = y_i\} \in \mathcal{F}$, $\{i \in I \colon a_i = b_i\} \in \mathcal{F}$. So if $[\varphi]\mathcal{F} = [\psi]\mathcal{F}$, then $[x]\mathcal{F} = [y]\mathcal{F}$ and the set $\{i \in I \colon a_i = g\} \cap \{i \in I \colon b_i = g'\} \cap \{i \in I \colon a_i = b_i\}$ is nonempty, because it belongs to $\mathcal{F}$, which implies $g = g'$. Conversely, if $[x]\mathcal{F} = [y]\mathcal{F}$ and $g = g'$, then evidently $\{i \in I \colon x_i = y_i\} \in \mathcal{F}$ and also $\{i \in I \colon a_i = b_i\} \in \mathcal{F}$, since $\{i \in I \colon a_i = b_i\} \supseteq \{i \in I \colon a_i = g\} \cap \{i \in I \colon b_i = g'\} \in \mathcal{F}$. We have proved that the definition of ϕ is correct and ϕ is one-to-one. To show that ϕ is onto, let $x \in L^I$, $g \in G$. Define $\varphi \in (L \circ G)^I$ by $\varphi(i) = (x(i), g)$ for all $i \in I$. Then evidently $\phi([\varphi]\mathcal{F}) = ([x]\mathcal{F}, g)$. Next we prove that ϕ preserves the group operation. Let φ, ψ be as above. We have $\phi([\varphi]\mathcal{F} + [\psi]\mathcal{F}) = \phi([\varphi + \psi]\mathcal{F}) = ([x+y]\mathcal{F}, t)$, where $\{i \in I \colon a_i + b_i = t\} \in \mathcal{F}$. On the other hand $\phi([\varphi]\mathcal{F}) + \phi([\psi]\mathcal{F}) = ([x]\mathcal{F}, g) + ([y]\mathcal{F}, g') = ([x+y]\mathcal{F}, g + g')$. Now the set $\{i \in I \colon a_i + b_i = t\} \cap \{i \in I \colon a_i = g\} \cap \{i \in I \colon b_i = g'\}$ belongs to $\mathcal{F}$, so it is nonempty. Hence there exists $i_0 \in I$ with $t = a_{i_0} + b_{i_0} = g + g'$. We have proved that $\phi([\varphi]\mathcal{F} + [\psi]\mathcal{F}) = \psi([\varphi]\mathcal{F}) + \phi([\psi]\mathcal{F})$. Finally we are going to show that

$$[\varphi]\mathcal{F} > 0 \iff \phi([\varphi]\mathcal{F}) > 0.$$

Using the notation as above we can write:

$$\phi\big([\varphi]\mathcal{F}\big) > 0 \Leftrightarrow \big([x]\mathcal{F}, g\big) > 0 \Leftrightarrow [x]\mathcal{F} > 0 \Leftrightarrow \{i \in I\colon x_i > 0\} \in \mathcal{F} \Leftrightarrow$$
$$\{i \in I\colon (x_i, a_i) > 0\} \in \mathcal{F} \Leftrightarrow \{i \in I\colon \varphi(i) > 0\} \in \mathcal{F} \Leftrightarrow [\varphi]\mathcal{F} > 0 .$$

The proof is complete. □

Let us remark that $L^I/\mathcal{F}$ is also a non-zero dense antilattice.

The following lemma can be proved easily.

Lemma 5.2. *The class $C\{L \circ G\}$ consists of $L_1 \circ G$ for all possible non-zero convex directed subgroups L_1 of L, together with $\{(0,0)\}$. Further, non-zero o-ideals of $L \circ G$ are just $L_1 \circ G$ for all possible non-zero o-ideals L_1 of L.*

By this lemma, $L \circ G$ is always f-subdirectly irreducible without respect to whether L is or is not f-subdirectly irreducible. In fact, the intersection of any two non-zero o-ideals of $L \circ G$ contains the set $\{0\} \times G$.

Notice that if $L_1 \circ G \in C\{L \circ G\}$, then L_1 is a dense antilattice, too.

Lemma 5.3. *Let $L_1 \circ G$ be a non-zero o-ideal of $L \circ G$. Then $L \circ G/L_1 \circ G$ is isomorphic to L/L_1.*

P r o o f . Define $\phi\colon L \circ G/L_1 \circ G \to L/L_1$ by

$$\phi\big((x,a) + L_1 \circ G\big) = x + L_1 .$$

It is easy to verify that this definition is correct and ϕ is an isomorphism. □

Using (3.12) and (5.1) – (5.3) we obtain:

Theorem 5.4. *Let $L \in \mathcal{R}$ be a non-zero dense antilattice, G be a non-zero finite trivially ordered group. If $A \in HCP\{L \circ G\}$ is f-subdirectly irreducible, then there exists $L_1 \in CP_{\mathcal{U}}\{L\}$, $L_1 \neq \{0\}$ such that either $A \cong L_1 \circ G$ or $A \cong L_1/L_2$ for some non-zero o-ideal L_2 of L_1, $L_2 \neq L_1$.*

As we have remarked above, all $L_1 \circ G$ for $L_1 \in CP_{\mathcal{U}}\{L\}$, $L_1 \neq \{0\}$ are f-subdirectly irreducible. But if $L_1 \in CP_{\mathcal{U}}\{L\}$, $L_1 \neq \{0\}$ and L_2 is a non-zero o-ideal of L_1, $L_2 \neq L_1$, then L_1/L_2 is not f-subdirectly irreducible in general, as the following example shows.

Example 5.5. Let $\mathbb{R}$ be the additive group of all real numbers with the natural linear order and $L = (\mathbb{R} \times \mathbb{R}) \circ \mathbb{R}$. It is easy to see that $L \in \mathcal{R}$, L is a dense antilattice. Put $L_1 = L$, $L_2 = \{(0,0)\} \circ \mathbb{R}$. Then evidently L_2 is an o-ideal of L_1. As to L_1/L_2, it is not f-subdirectly irreducible. To see this, let $G_1 = \big\{\big((a,0),c\big)\colon a,c \in \mathbb{R}\big\}$, $G_2 = \big\{\big((0,b),c\big)\colon b,c \in \mathbb{R}\big\}$. Then G_1, G_2 are o-ideals of L_1 and $G_1 \cap G_2 = L_2$. Thus G_1/L_2 and G_2/L_2 are non-zero o-ideals of L_1/L_2 with $G_1/L_2 \cap G_2/L_2 = \{L_2\}$.

A full description of f-subdirectly irreducible members of the convexity $HCP\{L \circ G\}$ can be given, e.g., in the case that L is linearly ordered.

THEOREM 5.6. *Let L be a non-zero dense linearly ordered group, G a non-zero finite trivially ordered group. Then f-subdirectly irreducible members of the convexity $HCP\{L \circ G\}$ are just (up to isomorphism) $L_1 \circ G$ and L_1/L_2 for all possible $L_1 \in CP_{\mathcal{U}}\{L\}$, $L_1 \neq \{0\}$ and o-ideals L_2 of L_1 with $\{0\} \neq L_2 \neq L_1$.*

Proof. We have only to prove that L_1/L_2 is f-subdirectly irreducible whenever $L_1 \in CP_{\mathcal{U}}\{L\}$, $L_1 \neq \{0\}$, L_2 is a non-zero o-ideal of L_1, $L_2 \neq L_1$, So let L_1, L_2 satisfy the above assumptions. Since L is linearly ordered, so are L_1 and L_1/L_2. Thus L_1/L_2 is a non-zero antilattice, which implies that it is f-subdirectly irreducible. □

Let us remark that a convexity of Riesz groups need not be uniquely determined by its f-subdirectly irreducible elements. Suppose that $\mathcal{K}$ is a convexity of Riesz groups, $\mathcal{X}$ the subclass of all f-subdirectly irreducible elements of $\mathcal{K}$. Evidently $HCP\mathcal{X} \subseteq \mathcal{K}$, but the case that $\mathcal{K}$ properly contains $HCP\mathcal{X}$ cannot be excluded, not even in the case that $\mathcal{K}$ contains only Abelian groups. To make clear the substance of the problem, let $\mathcal{K}$ be a convexity of Abelian Riesz groups, $A \in \mathcal{K}$. By (3.6), A has a subdirect decomposition $\varphi\colon A \to \prod\limits_{i \in I} A/H_i$ with f-subdirectly irreducible factors A/H_i. Evidently all A/H_i belong to $\mathcal{K}$, so that $\prod\limits_{i \in I} A/H_i \in P\mathcal{X} \subseteq HCP\mathcal{X}$. But as $\varphi(A)$ needn't be convex in $\prod\limits_{i \in I} A/H_i$, we have not guaranteed that $A \in HCP\mathcal{X}$.

In spite of this fact, some relations between convexities of Riesz groups can be derived by means of f-subdirectly irreducible members of these convexities, as we will see later.

LEMMA 5.7. *Let L_1, L_2 be two non-zero dense linearly ordered groups, G_1, G_2 two non-zero finite trivially ordered groups. Then $L_1 \circ G_1$ is isomorphic to $L_2 \circ G_2$ if and only if L_1 is isomorphic to L_2 and G_1 is isomorphic to G_2.*

Proof. The part "if" is trivial. Now let ϕ be an isomorphism of $L_1 \circ G_1$ onto $L_2 \circ G_2$. Evidently $\phi\bigl(\{0\} \times G_1\bigr) \subseteq \{0\} \times G_2$, since $\{0\} \times G_1$ is a trivially ordered subgroup of $L_1 \circ G_1$. Analogously $\phi^{-1}\bigl(\{0\} \times G_2\bigr) \subseteq \{0\} \times G_1$. Therefore $\phi(\{0\} \times G_1) = \{0\} \times G_2$ and ϕ determines an isomorphism of G_1 onto G_2. Now consider $\phi\bigl(L_1 \times \{0\}\bigr)$. Since $L_1 \times \{0\}$ is a linearly ordered subgroup of $L_1 \circ G_1$, first components of elements of $\phi\bigl(L_1 \times \{0\}\bigr)$ are mutually different. So if we define $\varphi\colon L_1 \to L_2$ by $\varphi(x) = x'$ whenever $\phi\bigl((x,0)\bigr) = (x', a)$, the mapping φ is one-to-one, it preserves the group operation and both φ and φ^{-1} are order-preserving. To show that φ is onto, let $y \in L_2$. There exists $(x,a) \in L_1 \times G_1$ with $\phi\bigl((x,a)\bigr) = (y,0)$. Then $\phi\bigl((x,0)\bigr) = \phi\bigl((x,a) - (0,a)\bigr) = \phi\bigl((x,a)\bigr) - \phi\bigl((0,a)\bigr) = (y,0) - (0,b) = (y,-b)$ for some $b \in G_2$. So we have $\varphi(x) = y$. □

Now we are ready to make clear the relation between convexities generated by $L \circ G$ for various non-zero dense linearly ordered groups L and non-zero finite trivially ordered groups G.

Theorem 5.8. *Let L_1, L_2 be any non-zero dense linearly ordered groups, G_1, G_2 any non-zero finite trivially ordered groups. Then $HCP\{L_1 \circ G_1\} \subseteq HCP\{L_2 \circ G_2\}$ if and only if $G_1 \cong G_2$ and $L_1 \in CP_{\mathcal{U}}\{L_2\}$.*

P r o o f. Let $HCP\{L_1 \circ G_1\} \subseteq HCP\{L_2 \circ G_2\}$. Then $L_1 \circ G_1$ is isomorphic to $L_2' \circ G_2$ for some $L_2' \in CP_{\mathcal{U}}\{L_2\}$, $L_2' \neq \{0\}$ by (5.6), since $L_1 \circ G_1$ is f-subdirectly irreducible, but non-linearly ordered group. Then $L_1 \cong L_2'$, $G_1 \cong G_2$, by (5.7). Since $L_2' \in CP_{\mathcal{U}}\{L_2\}$, we have also $L_1 \in CP_{\mathcal{U}}\{L_2\}$. Conversely, let $G_1 \cong G_2$ and $L_1 \in CP_{\mathcal{U}}\{L_2\}$. Using (5.1) and (5.2) we obtain $L_1 \circ G_1 \cong L_1 \circ G_2 \in CP_{\mathcal{U}}\{L_2 \circ G_2\} \subseteq HCP\{L_2 \circ G_2\}$. Hence $HCP\{L_1 \circ G_1\} \subseteq HCP\{L_2 \circ G_2\}$. ☐

Corollary 5.9. *Let L_1, L_2 be any non-zero dense linearly ordered groups, G_1, G_2 non-isomorphic non-zero finite trivially ordered groups. Then the convexities $HCP\{L_1 \circ G_1\}$, $HCP\{L_2 \circ G_2\}$ are non-comparable.*

J. J a k u b í k [7] proved that the convexity $HCP\{\mathbb{R}\}$ generated by the additive group of real numbers (with the natural linear ordering) is contained in each convexity $\mathcal{K}$ of normal valued lattice ordered groups, except the least one. A natural question arises, which convexities of Riesz groups contain $HCP\{\mathbb{R}\}$. We do not know the answer. But we will show that $HCP\{\mathbb{R}\}$ is properly contained in $HCP\{L \circ G\}$ for each non-zero dense linearly ordered group L and non-zero finite trivially ordered group G.

Lemma 5.10. *Let $\mathcal{F}$ be an ultrafilter over the set $\mathbb{N}$ of all positive integers containing complements of all finite subsets of $\mathbb{N}$. For each non-zero linearly ordered group L, the ultrapower $L^{\mathbb{N}}/\mathcal{F}$ has a non-trivial convex subgroup.*

P r o o f. We will distinguish two cases:

1. L is archimedean,
2. L is non-archimedean.

First, let L be archimedean. Let us call $f \in L^{\mathbb{N}}$ almost bounded if there exists $r \in L$, $r > 0$ and $X \in \mathcal{F}$ with $-r \leq f(i) \leq r$ for all $i \in X$. Now set $\mathcal{H} = \big\{[f]\mathcal{F}\colon f \text{ is almost bounded}\big\}$. It is easy to see that $\mathcal{H}$ is a non-zero convex subgroup of $L^{\mathbb{N}}/\mathcal{F}$. Suppose that $\mathcal{H} = L^{\mathbb{N}}/\mathcal{F}$. Take any $a \in L$, $a > 0$, and let us define $g \in L^{\mathbb{N}}$ by $g(i) = ia$. The assumption $\mathcal{H} = L^{\mathbb{N}}/\mathcal{F}$ yields $[g]\mathcal{F} = [f]\mathcal{F}$ for an almost bounded $f \in L^{\mathbb{N}}$. So there exists $Z \in \mathcal{F}$ and $r \in L$, $r > 0$, with $-r \leq g(i) = f(i) \leq r$ for all $i \in Z$. But Z is infinite set, therefore $na = g(n) \leq r$ for all $n \in \mathbb{N}$, which contradicts the assumption that L is archimedean. Now let L be non-archimedean. Then there exist $a, b \in L$, $a > 0$,

$b > 0$, such that $na < b$ for all $n \in \mathbb{N}$. Now let $\mathcal{H}$ be the least convex subgroup of $L^I/\mathcal{F}$ containing $[\mathbf{a}]\mathcal{F}$, $\mathbf{a} \in L^{\mathbb{N}}$ being defined by $\mathbf{a}(i) = a$ for all $i \in \mathbb{N}$. Evidently $\mathcal{H}$ is non-zero and it is $\mathcal{H} = \{[f]\mathcal{F}\colon -n[\mathbf{a}]\mathcal{F} \le [f]\mathcal{F} \le n[\mathbf{a}]\mathcal{F}$ for some $n \in \mathbb{N}\} = \{[f]\mathcal{F}\colon$ there exist $n \in \mathbb{N}$ and $Z \in \mathcal{F}$ with $-na \le f(i) \le na$ for all $i \in Z\}$, so that $[\mathbf{b}]\mathcal{F} \notin \mathcal{H}$. □

THEOREM 5.11. *Let L be a non-zero dense linearly ordered group, G a non-zero finite trivially ordered group. Then the convexity $HCP\{\mathbb{R}\}$ is properly contained in $HCP\{L \circ G\}$.*

P r o o f. Let $\mathcal{F}$ be an ultrafilter over $\mathbb{N}$ as in (5.10) and let L' be any non-trivial convex subgroup of $L^{\mathbb{N}}/\mathcal{F}$. Further, let L_1, L_2 be such convex subgroups of $L^{\mathbb{N}}/\mathcal{F}$, that $L' \subseteq L_2 \subseteq L_1 \subseteq L^{\mathbb{N}}/\mathcal{F}$ and $L_2 \prec L_1$ is a jump (see, e.g., [2]). It is well known that L_2 is a normal subgroup of L_1 and that L_1/L_2 is o-isomorphic to a non-zero subgroup $\mathbb{R}_1$ of $\mathbb{R}$. So we have $\mathbb{R}_1 \cong L_1/L_2$, $L_1 \in CP_{\mathcal{U}}\{L\}$, L_2 is an o-ideal of L_1, $L_1 \neq \{0\}$, $L_2 \neq \{0\}$, $L_2 \neq L_1$, hence $\mathbb{R}_1 \in HCP\{L \circ G\}$ by (5.6). J. J a k u b í k proved (cf. [7], 2.11) that $HCP\{\mathbb{R}\} \subseteq HCP\{G_1\}$ for each non-zero l-subgroup G_1 of $\mathbb{R}$. So we have $HCP\{\mathbb{R}\} \subseteq HCP\{\mathbb{R}_1\} \subseteq HCP\{L \circ G\}$. Finally, as $HCP\{\mathbb{R}\}$ contains only l-groups, we have $HCP\{\mathbb{R}\} \subsetneq HCP\{L \circ G\}$. □

Some open questions:

1. Which convexities are below $HCP\{\mathbb{R} \circ G\}$?
2. Which convexities are below $HCP\{L \circ G\}$ (L, G are as in (5.11))?
3. If $\mathbb{R}_1$ is a non-trivial l-subgroup of $\mathbb{R}$, then $HCP\{\mathbb{R}_1 \circ G\} \nsubseteq HCP\{\mathbb{R} \circ G\}$. Namely, in the opposite case, we would have $\mathbb{R}_1 \in CP_{\mathcal{U}}\{\mathbb{R}\}$, which would imply $HCP\{\mathbb{R}_1\} \subseteq HCP\{\mathbb{R}\}$, a contradiction, because $HCP\{\mathbb{R}\} \subsetneq HCP\{\mathbb{R}_1\}$ by the above mentioned result of J. J a k u b í k. Can it be $HCP\{\mathbb{R} \circ G\} \subseteq HCP\{\mathbb{R}_1 \circ G\}$? This question can be reformulated as: can it be $\mathbb{R} \in CP_{\mathcal{U}}\{\mathbb{R}_1\}$?

REFERENCES

[1] FUCHS, L.: *Riesz groups*, Ann. Scuola Norm. Sup. Pisa **19** (1965), 1–34.
[2] FUCHS, L.: *Partially Ordered Algebraic Systems*, Moskva, 1965.
[3] GOODEARL, K. R.: *Partially Ordered Abelian Groups with Interpolation*, Amer. Math. Soc., New York, 1986.
[4] JAKUBÍK, J.: *On convexities of lattices*, Czechoslovak Math. J. **42** (1992), 325–330.
[5] JAKUBÍK, J.: *On convexities of d-groups*, Czechoslovak Math. J. **44** (1994), 305–314.
[6] JAKUBÍK, J.: *Convexities of lattice ordered groups*, Math. Bohemica **121** (1996), 59–67.
[7] JAKUBÍK, J.: *Convexities of normal valued lattice ordered groups*, Czechoslovak Math. J. **50** (2000), 641–649.

[8] JAKUBÍKOVÁ, J.: *Konvexe gerichtete Untergruppen der Rieszschen Gruppen*, Mat. Čas. **21** (1971), 3–8.

[9] KOPYTOV, V. M.—MEDVEDEV, N. YA.: *The Theory of Lattice–Ordered Groups*, Kluwer Academic Publishers, Dordrecht, 1994.

[10] LIHOVÁ, J.: *On Riesz groups*, Tatra Mt. Math. Publ. **27** (2003), 163–176.

[11] *General algebra 1988* (R. Mlitz, ed.), Proc. Internat. Conf. Krems, Austria, August 21–27, 1988, North Holland, Amsterdam, 1990, pp. 265.

[12] TELLER, J. R.: *On partially ordered groups satisfying the Riesz interpolation property*, Proc. Amer. Math. Soc. **16** (1965), 1392–1400.

Received November 14, 2003

Institute of Mathematics
Faculty of Science
P. J. Šafárik University
Jesenná 5
SK–041 54 Košice
SLOVAKIA

Mathematical Institute
Slovak Academy of Sciences
Grešákova 6
SK–040 01 Košice
SLOVAKIA

E-mail: lihova@science.upjs.sk