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# ON DIFFERENTIABLE AND LIPSCHITZ SELECTIONS IN NORMED SPACES

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ABSTRACT. Given a separable normed space we construct differentiable selections from differentiable multifunctions provided that some special hyperspace metrics, instead of the Hausdorff metric, are used. Our construction also gives Lipschitz selections from Lipschitz multifunctions.

## 1. Introduction

The existence of differentiable selections of set-valued functions in finite dimensional normed spaces were studied in papers: [BaJa, StZe, Farb, Hot, BGMO, Asee, Domm, GaMo, Albr, LuDi, KnWo, Dent]. For Lipschitz selections see [Posi, Yo, PrYo, Ziva, GaMo, Prz1, Prz2, Prze3, Shv] and references therein.

Our aim is to construct, in arbitrary (also infinite dimensional) separable normed spaces, differentiable (Lipschitz) selections from differentiable (Lipschitz) multifunctions. To do this we use total sequences of continuous linear functionals.

Let $X$ be a normed space, $U \subset Y$ an open subset, $Y$ a separable normed space, $\mathrm{Kv}(Y)$ the cone of all nonempty compact convex subsets of $Y$, $d_H$ the Hausdorff metric on $\mathrm{Kv}(Y)$, i.e., $d_H(A,B) = \inf\{\varepsilon > 0\colon A \subset B + K(\varepsilon),\ B \subset A + K(\varepsilon)\}$, where $K(\varepsilon)$ is an open ball in $Y$ with the center $0$ and the radius $\varepsilon$. Given a total sequence of functionals, i.e., a sequence of continuous linear functionals separating the points of $Y$, we construct a sequence of Minkowski additive and positively homogeneous set-valued function on $\mathrm{Kv}(Y)$ and then a translation invariant and positively homogeneous metric $d_H^* \geq d_H$ on $\mathrm{Kv}(Y)$ such that $(\mathrm{Kv}(Y), d_H^*)$ can be embedded, in a Rådström manner, as a convex metric cone into a normed space, say $(\widehat{Y}, \|\cdot\|_H^*)$. Also another semimetric $d^*$,

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not comparable with $d_H$, and the corresponding normed space $(\widehat{Y}, \|\cdot\|^*)$, will be constructed in the same manner.

Now, for a given function $F: U \to \mathrm{Kv}(Y)$, which is $d_H^*$-differentiable, i.e., (Fréchet or Gateaux) differentiable as a map $\widehat{F}$ from $U$ to $(\widehat{Y}, \|\cdot\|_H^*)$, we construct a differentiable selection of $F$. The same result holds for $d^*$-differentiability.

Our construction is based on the construction idea which can be found in [StZe], and on properties of continuous linear functionals. Namely, assuming that $F$ is $d_H^*$-differentiable we obtain a differentiable selection of $F$ as the composition $\widehat{m} \circ \widehat{F}$ of the $d_H^*$-differentiable map $\widehat{F}: U \to (\widehat{Y}, \|\cdot\|_H^*)$ and a continuous linear map $\widehat{m}: (\widehat{Y}, \|\cdot\|_H^*) \to Y$.

For $d_H$-differentiable set-valued functions and selections in finite dimensional spaces see [BaJa] and [StZe]. Recall that if $Y$ is a Banach space then there exists a $d_H$-Lipschitz selection on $\mathrm{Kv}(Y)$ if and only if $Y$ is finite dimensional (see [Posi, Prz1, Yost]), and that there is no $d_H$-uniformly continuous selection on $\mathrm{Kv}(Y)$ whenever $Y$ is infinite dimensional ([PrYo]). However, the question whether every $d_H$-differentiable multifunction has a differentiable selection is still open for infinite dimensional spaces.

## 2. Additive set-valued functions on $\mathrm{Kv}(Y)$

Let $Y$ be a real topological vector space and $Y^*$ its dual space, i.e., the space of all continuous linear functionals on $Y$. Given a functional $\varphi \in Y^*$ we can construct an additive and positively homogeneous function from $\mathrm{Kv}(Y)$ into $\mathrm{Kv}(Y)$.

**Lemma 2.1.** *For every $\varphi \in Y^*$, the function $M_\varphi: \mathrm{Kv}(Y) \to \mathrm{Kv}(Y)$ defined by the formula*

$$M_\varphi(A) = \{x \in A: \varphi(x) = \inf\{\varphi(y): y \in A\}\},$$

*is (Minkowski) additive and positively homogeneous, i.e., $M_\varphi(A+B) = M_\varphi(A) + M_\varphi(B)$ and $M_\varphi(\lambda A) = \lambda M_\varphi(A)$ for $A, B \in \mathrm{Kv}(Y)$ and $\lambda \geq 0$, where $A+B = \{a+b: a \in A,\ b \in B\}$ and $\lambda A = \{\lambda a: a \in A\}$. Moreover, $M_\varphi(A) \subset A$ for every $A \in \mathrm{Kv}(Y)$.*

P r o o f. The proof uses standard arguments. By the compactness of $A$ and continuity of $\varphi$ the set $M_\varphi(A)$ is nonempty and compact. The convexity of $M_\varphi(A)$ is also clear because for every $x_1, x_2 \in M_\varphi(A)$ and every $\lambda \in [0,1]$ we have: $\varphi(\lambda x_1 + (1-\lambda)x_2) = \lambda\varphi(x_1) + (1-\lambda)\varphi(x_2) = \inf\{\varphi(y): y \in A\}$. First, we prove the additivity of $M_\varphi$. We have $M_\varphi(A+B) = \{x \in A+B: \varphi(x) =$

$\inf\{\varphi(a)+\varphi(b)\colon a\in A,\ b\in B\}\big\} = \big\{x_1+x_2\colon x_1\in A,\ x_2\in B,\ \varphi(x_1)+\varphi(x_2)=\inf\{\varphi(a)\colon a\in A\}+\inf\{\varphi(b)\colon b\in B\}\big\} = \big\{x_1+x_2\colon x_1\in A,\ x_2\in B,\ \varphi(x_1)=\inf\{\varphi(a)\colon a\in A\},\ \varphi(x_2)=\inf\{\varphi(b)\colon b\in B\}\big\} = \big\{x_1\in A\colon \varphi(x_1)=\inf\{\varphi(a)\colon a\in A\}\big\}+\big\{x_2\in B\colon \varphi(x_2)=\inf\{\varphi(b)\colon b\in B\}\big\} = M_\varphi(A)+M_\varphi(B)$ because $\varphi$ is additive, $\varphi(x_1)\ge\inf\big\{\varphi(a)\colon a\in A\big\}$ and $\varphi(x_2)\ge\inf\big\{\varphi(b)\colon b\in B\big\}$. To show the positive homogeneity observe that $M_\varphi(\lambda A)=\big\{\lambda x\colon x\in A,\ \varphi(\lambda x)=\inf\{\varphi(\lambda y)\colon y\in A\}\big\}=\lambda\big\{x\in A\colon \varphi(x)=\inf\{\varphi(y)\colon y\in A\}\big\}=\lambda M_\varphi(A)$. □

**Remarks.** 1. Recall that an additive function $M\colon \mathrm{Kv}(Y)\to Y$ is positively homogeneous provided $M(A)\in A$ holds for all $A\in\mathrm{Kv}(Y)$. Indeed, for every positive and rational $r$ we can easily obtain that $M(rA)=rM(A)$. Now, if $\lambda>0$ then $\lambda=\lim_n r_n$ for some positive rationals $r_n<\lambda$ and $M(\lambda A)=M\big((\lambda-r_n)A\big)+r_nM(A)$. The continuity of the scalar multiplication and the boundedness of $A$ completes the proof.

2. For every $\lambda\ge 0$ the formula $M(A)=\lambda A$, $A\in\mathrm{Kv}(Y)$, also defines an additive and positively homogeneous set-valued function on $\mathrm{Kv}(Y)$. However, $M_\varphi(A)\not\subset A$.

3. See [Prz2] and [Ziva] for additive set-valued functions on $\mathrm{Kv}(\mathbb{R}^n)$.

4. Instead of the infimum one can use the supremum and obtain another additive and positively homogeneous set-valued function on $\mathrm{Kv}(Y)$.

5. Instead of $\mathrm{Kv}(Y)$ one can consider the family $\mathrm{Kv}_\sigma(Y)$ of all nonempty convex and weakly compact subsets of $Y$.

**Lemma 2.2.** *Let $\varphi_n\in Y^*$, $n\in\mathbb{N}$, be an arbitrary sequence of continuous linear functionals. We define the following nonincreasing sequence of functions $M_n\colon \mathrm{Kv}(Y)\to\mathrm{Kv}(Y)$ inductively:*

$$M_0(A)=A\,,$$
$$M_1(A)=\big\{x\in A\colon \varphi_1(x)=\inf\{\varphi_1(y)\colon y\in A\}\big\}\,,$$
$$M_{n+1}(A)=\big\{x\in M_n(A)\colon \varphi_{n+1}(x)=\inf\{\varphi_{n+1}(y)\colon y\in M_n(A)\}\big\}\,.$$

*Then the function $M\colon \mathrm{Kv}(Y)\to\mathrm{Kv}(Y)$ defined by the formula*

$$M(A)=\bigcap_{n=1}^{\infty}M_n(A)=\bigcap_{n=0}^{\infty}M_n(A)\,,$$

*is additive and positively homogenous.*

P r o o f . By the compactness assumption the sets $M_n(A)$ are nonempty. It is obvious that $M_0$ is additive and positively homogeneous. In virtue of the Lemma 2.1 the functions $M_n$ are additive and positively homogeneous.

To prove the additivity of $M$, first assume that $y \in M(A+B)$. Then $y \in M_n(A+B) = M_n(A) + M_n(B)$ for every $n$. In other words, $\forall n\, \exists\, y_{n,1} \in M_n(A)$ $\exists\, y_{n,2} \in M_n(B)$: $y = y_{n,1} + y_{n,2}$. By the compactness of the sets $M_n(A)$ and $M_n(B)$, and the monotonicity $M_1 \supset M_2 \supset \ldots$ we infer that there exist $y_1, y_2 \in Y$ such that $y_{n,1} \to y_1$, $y_{n,2} \to y_2$, $y_1 \in M_k(A)$ and $y_2 \in M_k(B)$ for each $k$. Moreover, $y = y_1 + y_2$. Consequently, $y \in M(A) + M(B)$. Thus, $M(A+B) \subset M(A) + M(B)$. Conversely, assume that $y = y_1 + y_2$, $y_1 \in M(A)$ and $y_2 \in M(B)$. Then for every $n$ we have $y_1 + y_2 \in M_n(A) + M_n(B) = M_n(A+B)$. Therefore $y \in \bigcap_n M_n(A+B) = M(A+B)$. Consequently, $M(A) + M(B) \subset M(A+B)$. This completes the proof of additivity.

To prove that $M$ is positively homogeneous, assume that $\lambda > 0$ and $z \in M(\lambda A) = \bigcap_n \lambda M_n(A)$. In other words: $\forall n\, \exists\, y_n \in M_n(A)$: $z = \lambda y_n$. By the compactness and the monotonicity there exists $y_0 \in M(A)$ such that $y_0 \in M_k(A)$ for all $k$, $y_n \to y_0$. Thus $z = \lambda y_0$, and $z \in \lambda M(A)$. Conversely, if $z \in \lambda M(A)$ then $z = \lambda y$ for some $y \in M(A)$. Thus $z \in M_n(\lambda A)$ for all $n$. Consequently, $z \in M(\lambda A)$. This completes the proof. □

The set $M_n(A)$ will be called the *$M_n$-face* of $A$, $n = 0, 1, 2, \ldots$.

Now, ask the question: Under what assumption on the functionals $\varphi_n$ the intersection $M(A) = \bigcap_n M_n(A)$ is a singleton? To give an answer, we will consider sequences of functionals separating the points of $Y$.

## 3. Sequences of functionals separating points

We say that a sequence of continuous linear functionals $\varphi_k \in Y^*$, $k = 1, 2, \ldots$, is total on $Y$, or separates the points of $Y$, provided that for every two distinct points $y, z \in Y$ there exists $k$ such that $\varphi_k(y) \neq \varphi_k(z)$. Equivalently, $(\varphi_k)$ separates the points of $Y$ if $\varphi_k(y) = 0$, for all $k$, implies that $y = 0$.

**Remarks.** 1. If $\dim Y = n < \infty$ then the coordinate functionals $\varphi_k\big((t_1, \ldots$ $\ldots, t_n)\big) = t_k$, $k = 1, \ldots, n$, form a finite total sequence on $Y$. Of course, the infinite sequence $\varphi_1, \ldots, \varphi_{n-1}, \varphi_n, \varphi_n, \cdots \in Y^*$ is also total on $Y$. Note that the coordinate functionals correspond with the lexicographic order used in [StZe].

2. Let $(y_k)$ be a Schauder basis of a normed space $Y$ and $(\varphi_k)$ be its biorthogonal functionals. Then every vector $y \in Y$ has the following unique representation $y = \sum_k \varphi_k(y)\, y_k$. It is clear that $(\varphi_k)$ separates the points of $Y$. Recall also that: (1) every Banach space with a Schauder basis is separable, and (2) each complete orthonormal system in a separable Hilbert space is a Schauder basis. However, not every separable Banach space has a Schauder basis ([Enfl]).

The following result is known.

**Lemma 3.1.** *On every separable normed space $Y$ there exists a total sequence of continuous linear functionals.*

**Remarks.** 1. Jayne and Rogers ([JaRo]) remarked that if the dual of a normed space $Y$ contains a total sequence $(\varphi_k)$, then the map $\lambda\colon Y \to \mathbb{R}^{\mathbb{N}}$ defined by $\lambda(y) = \big(\varphi_k(y)\big)$ is linear injective (and weakly continuous). From this we infer that in this case we necessarily have $\dim Y \le \mathfrak{c}$, and we conclude that for every normed space $Y$ with the dimension greater than $\mathfrak{c}$ the dual space contains no total sequence.

2. If the dual of a normed space $Y$ contains a total sequence then there exists an equivalent norm on $Y$, $\|\cdot\|_1$, such that the space $\big(Y, \|\cdot\|_1\big)$ is strictly convex.

3. The space $l^\infty$ with the usual supremum norm is not separable but the coordinate functionals $\varphi_k\big((t_n)\big) = t_k$, $(t_n) \in l^\infty$, $k = 1, 2, \ldots$, are linear, continuous and the sequence $(\varphi_k)$ is total on $l^\infty$. Note that $\dim l^\infty = \mathfrak{c}$ and $l^\infty$ is not strictly convex.

4. If for a normed space $Y$ its dual contains a sequence $(\varphi_k)$ separating the points of $Y$, then there exists a separable metrizable linear topology $\alpha$ on $Y$ weaker than the original one. $\alpha$ can be taken simply as the coarsest topology on $Y$ such that the functionals $\varphi_k$ are continuous.

**Lemma 3.2.** *Let $Y$ be an arbitrary normed space, $\varphi_n \in Y^*$ a total sequence, and $(M_n)$ be a sequence of functions defined as in Lemma 2.2. Then the function $m\colon \mathrm{Kv}(Y) \to Y$ defined by the formula*

$$\big\{m(A)\big\} = M(A) = \bigcap_{n=1}^{\infty} M_n(A), \qquad A \in \mathrm{Kv}(Y),$$

*is additive positively homogeneous and $m(A) \in A$.*

P r o o f. In virtue of Lemma 2.2, we need only to prove the single-valuedness of $M(A)$ (this is due to the fact in [Cole]). Suppose that for some $A$ there exist two different points $y_1, y_2 \in M(A)$. Then $y_1, y_2 \in M_n(A) = \big\{x \in M_{n-1}\colon \varphi_n(x) = \sup\{\varphi_n(y)\colon y \in M_{n-1}\}\big\}$ for all $n$. By the separation property there exists $k$ such that $\varphi_k(y_1) \ne \varphi_k(y_2)$. This implies that $y_1 \notin M_k(A)$ or $y_2 \notin M_k(A)$, a contradiction. □

**Remarks.** 1. If $(\varphi_n)$ is a total sequence on $Y$ then $m(A)$ defined in Lemma 3.2 is an extreme point of $A$. Indeed, denote $y_0 = m(A)$ and suppose that $y_0$ is not extreme. There exist two distinct points $a_1, a_2 \in A\backslash\{y_0\}$ and $\lambda \in [0,1]$ such that $y_0 = \lambda a_1 + (1-\lambda)a_2$. Then there exists $n_0$ such that $\varphi_{n_0}(a_1) \ne \varphi_{n_0}(a_2)$. Assume that $\varphi_{n_0}(a_1) < \varphi_{n_0}(a_2)$. Now we obtain $\varphi_{n_0}(y_0) = \lambda\varphi_{n_0}(a_1) + (1-\lambda)\varphi_{n_0}(a_2)$

$> \varphi_{n_0}(a_1)$. On the other hand, we have $\varphi_{n_0}(y_0) \leq \varphi_{n_0}(a_1)$. Thus, $\varphi_{n_0}(a_1) < \varphi_{n_0}(a_1)$, and we arrived at a contradiction.

2. Conversely, each extreme point of $A$ can be obtained as $m(A)$ for some total sequence $(\varphi_n)$ on $Y$ provided that $Y^*$ contains a total sequence. Indeed, let $y_0$ be an extreme point of $A$. There exists a supporting hyperplane $H_1$ of $A$ at $y_0$ and the corresponding functional $\varphi_1 \in Y^*$ such that $y_0 \in M_{\varphi_1}(A)$. Moreover, $y_0$ is an extreme point of $M_{\varphi_1}(A)$. Thus we can repeat the above arguments to get a functional $\varphi_2 \in Y^*$ such that $y_0 \in M_{\varphi_2}(A)$ and $y_0$ is an extreme point of $M_{\varphi_2}(A)$. In this way, we get a sequence $\varphi_k \in Y^*$ such that $y_0 \in \bigcap\limits_k M_{\varphi_k}(A)$.

We can assume that the sequence $(\varphi_k)$ is total, otherwise we are able to extend this sequence to a total one. Therefore we obtain $\{y_0\} = \bigcap\limits_k M_{\varphi_k}(A)$. See [StZe].

## 4. Translation invariant metrics on $\mathrm{Kv}(Y)$

In this section we construct a translation invariant and positively homogeneous metric $d_H^*$ and a semimetric $d^*$ on $\mathrm{Kv}(Y)$, and then we embed the metric convex cone $\big(\mathrm{Kv}(Y), d^*\big)$ and the semimetric convex cone $\big(\mathrm{Kv}(Y), d_H^*\big)$ into some normed spaces. Recall that such embeddings are possible because the considered cones of subsets possess the law of cancellation.

First we prove some technical formulas.

**Lemma 4.1.** *Let $(A_n)$ and $(B_n)$ be two nonincreasing sequences of compact subsets of $Y$. Then:*

$$\bigcap_{n=1}^{\infty}(A_n + C) = \left(\bigcap_{n=1}^{\infty} A_n\right) + C\,, \qquad \text{for every closed } C \subset Y\,, \tag{1}$$

$$\sup_{n\geq 0} d_H(A_n, B_n) \geq d_H\left(\bigcap_{n=0}^{\infty} A_n\,,\ \bigcap_{n=0}^{\infty} B_n\right), \tag{2}$$

*where $d_H$ is the Hausdorff metric.*

P r o o f. 1. It is obvious that $\bigcap_n (A_n + C) \supset \left(\bigcap_n A_n\right) + C$. Conversely, let $y \in \bigcap_n (A_n + C)$. Then for every $n$ there exist $y_n \in A_n$ and $u_n \in C$ such that $y = y_n + u_n$. The sequence $(y_n)$ has a convergent subsequence. In virtue of the monotonicity of $(A_n)$ we may assume that $y_n \to y_0$ for some $y_0$ and $y_0 \in A_k$ for all $k$. Thus $(u_n)$ converges to some $u_0 \in C$. From this we get $y = y_0 + u_0 \in \left(\bigcap_n A_n\right) + U$, completing the proof of (1).

2. Denote: $\alpha_n = d_H(A_n, B_n)$ and $\alpha = \sup d_H(A_n, B_n)$. We show that

$$d_H\left(\bigcap_n A_n\,,\,\bigcap_n B_n\right) \le \alpha\,.$$

We have $A_n \subset B_n + U(\alpha)$ and $B_n \subset A_n + U(\alpha)$ for all $n$, where $U(\alpha)$ is the open ball with center at the origin and radius $\alpha$. Then $\bigcap_n A_n \subset \bigcap_n \big(B_n + U(\alpha)\big)$ and $\bigcap_n B_n \subset \bigcap_n \big(A_n + U(\alpha)\big)$. In virtue of (1) we get $\bigcap_n A_n \subset \Big(\bigcap_n B_n\Big) + U(\alpha)$ and $\bigcap_n B_n \subset \Big(\bigcap_n A_n\Big) + U(\alpha)$. This means that $d_H\Big(\bigcap_n A_n,\ \bigcap_n B_n\Big) \le \alpha$. □

**Remarks.** The formula (1) in Lemma 4.1 is a particular case of a more general theorem (see Theorem 2.6 in [LeZi]).

**Lemma 4.2.** *Let $\varphi_n \in Y^*$ be a total sequence, $(M_n)$ the sequence of functions defined in Lemma 2.2. Let $M_0(A) = A$. Let $m$ be defined as in Lemma 3.2. Then the function*

$$d_H^*(A, B) = \sup_{n \ge 0} d_H\big(M_n(A),\ M_n(B)\big)$$

*is a metric on $\mathrm{Kv}(Y)$, and the function*

$$d^*(A, B) = \sup_{n \ge 1} d_H\big(M_n(A),\ M_n(B)\big)$$

*is a semimetric on $\mathrm{Kv}(Y)$. Moreover,*

$$d_H^*(A, B) \ge d_H\big(m(A), m(B)\big)\,, \quad d^*(A, B) \ge d_H\big(m(A), m(B)\big)\,,$$

*$d_H \le d_H^*$, $d_H^*$ and $d^*$ are translation invariant and positively homogeneous, i.e., $d_H^*(A + C, B + C) = d_H^*(A, B)$ and $d_H^*(\lambda A, \lambda B) = \lambda d_H^*(A, B)$ for all $A, B, C \in \mathrm{Kv}(Y)$ and all $\lambda \ge 0$. The same properties hold for $d^*$.*

P r o o f. It is easy to check that $d_H^*$ and $d^*$ are metric and semimetric, respectively. The two inequalities above are clear because of Lemma 4.1. (2). The inequality $d_H \le d_H^*$ is obvious. Now recall that the Hausdorff metric is translation invariant and positively homogeneous. Thus $d_H^*$ and $d^*$ have the same two properties because the functions $M_n$ are additive and positively homogeneous. □

**Remarks.** 1. If $Y = \mathbb{R}$ then $\mathrm{Kv}(\mathbb{R})$ contains only intervals and singletons. Thus $d^* \le d_H$, and consequently, $d_H^* = d_H$.

2. If $\dim Y > 1$ then $d_H$ and $d^*$ are not comparable. Indeed, let $Y = \mathbb{R}^2$, $\varphi_1\big((x, y)\big) = y$ and $\varphi_2\big((x, y)\big) = x$. Then the sequence $(\varphi_1, \varphi_2)$ separates the points of $\mathbb{R}^2$. If $A = \big\{(x, 0)\colon x \in [0, 1]\big\}$ and $B = \big\{(x, 1 - x)\colon x \in [0, 1]\big\}$, then $d_H(A, B) = 1 < d^*(A, B) = \sqrt{2}$. If $A = \big\{(0, y)\colon y \in [0, 1]\big\}$ and $B = \big\{(x, 1 - x)\colon x \in [0, 1]\big\}$, then $d_H(A, B) = 1 > d^*(A, B) = 0$.

## 5. Rådström type embeddings

In the next two lemmas, we proceed the same way as Rådström in [Rads]. Let $\widehat{Y} = \big(\mathrm{Kv}(Y) \times \mathrm{Kv}(Y)\big)/\sim$, where $\sim$ is an equivalent relation defined by: $(A,B) \sim (C,D) \iff A + D = B + C$.

**Lemma 5.1.** *The space $\widehat{Y}$ defined above is a vector space under the addition $\langle A,B\rangle + \langle C,D\rangle = \langle A+C, B+D\rangle$, the scalar multiplication*

$$\alpha\langle A,B\rangle = \begin{cases} \langle \alpha A, \alpha B\rangle, & \alpha \ge 0, \\ \big\langle |\alpha| B, |\alpha| A\big\rangle, & \alpha < 0, \end{cases}$$

*and the origin $\widehat{0} = \langle 0,0\rangle = \big\langle \{0\},\{0\}\big\rangle$. The functional $\big\|\langle A,B\rangle\big\|_H^* = d_H^*(A,B)$, is a norm on $\widehat{Y}$. Moreover, the map $\pi\colon \big(K(Y), d_H^*\big) \to \big(\widehat{Y}, \|\cdot\|_H^*\big)$ defined by $\pi(A) = \big\langle A, \{0\}\big\rangle$ is an algebraic and isometric embedding.*

Proof. The proof is the same as in [Rads]. For example, we show only that $\pi$ is an isometry. Indeed, we have $\pi(A) - \pi(B) = \langle A,0\rangle - \langle B,0\rangle = \langle A,0\rangle + \langle 0,B\rangle = \langle A,B\rangle$. Thus, $\big\|\pi(A) - \pi(B)\big\|_H^* = d_H^*(A,B)$. □

We will identify $\pi(A) = \langle A,0\rangle \in \widehat{Y}$ with $A \in \mathrm{Kv}(Y)$. In particular, we will write $m(A) - m(B)$ instead of $\big\langle m(B), m(A)\big\rangle = \big\langle m(A),0\big\rangle - \big\langle m(B),0\big\rangle = \big\langle m(A),0\big\rangle + (-1)\big\langle m(B),0\big\rangle$. We will also identify the point $m(A)$ with the singleton $\big\{m(A)\big\}$.

To proceed analogously with the semimetric $d^*$, first we consider an equivalent relation $\sim_1$ on $\mathrm{Kv}(Y)$. We define $A \sim_1 B \iff M_1(A) = M_1(B)$ and $\mathrm{Kv}_1(Y) = \mathrm{Kv}(Y)/{\sim_1} = \big\{[A]\colon A \in \mathrm{Kv}(Y)\big\}$, where $[A]$ denote the abstract class of $A$ and consists of sets which have the same $M_1$-face as the set $A$. Observe that $\mathrm{Kv}_1(Y)$ is a cone under the addition $[A]+[B] = [A+B]$ and the scalar multiplication $\alpha[A] = [\alpha A]$ for $\alpha \ge 0$. The law of cancellation on $\mathrm{Kv}_1(Y)$ holds, too. Indeed, let $[A]+[B] = [C]+[B]$. Then $M_1(A+B) = M_1(C+B)$ and consequently $M_1(A) = M_1(C)$ because of additivity of $M_1$. Thus $[A] = [C]$. Now, the function $d^*\big([A],[B]\big) = d^*(A,B)$ is well defined and is a metric on $\mathrm{Kv}_1(Y)$. For simplicity, we will identify $[A]$ with $A$ and treat $d^*$ as a metric on $\mathrm{Kv}(Y) \equiv \mathrm{Kv}_1(Y)$.

We have a similar lemma for $d^*$.

**Lemma 5.2.** *The functional $\big\|\langle A,B\rangle\big\|^* = d^*(A,B)$, is a norm on $\widehat{Y}$. Moreover, the map $\pi_1^*\colon \big(K(Y), d^*\big) \to \big(\widehat{Y}, \|\cdot\|^*\big)$ defined by $\pi_1^*(A) = \big\langle A,\{0\}\big\rangle$ is an algebraic and isometric embedding.*

Proof. The proof is the same as in Lemma 5.1. □

## 6. Linear continuous maps

Here we show that the additive and positively homogeneous map $m\colon \mathrm{Kv}(Y) \to Y$ is continuous and has a natural extension to a continuous and linear map $\widehat{m}\colon \widehat{Y} \to Y$.

Let $\varphi_n \in Y^*$ be a total sequence of functionals on $Y$, $(M_n)$ a sequence of functions defined as in Lemma 2.2, and $m$ be as in Lemma 3.2. It is clear, in virtue of Lemmas 3.2 and 4.2 that the functions $m\colon \big(\mathrm{Kv}(Y), d_H^*\big) \to \big(Y, \|\cdot\|\big)$ and $m\colon \big(\mathrm{Kv}(Y), d^*\big) \to \big(Y, \|\cdot\|\big)$ are additive, positively homogeneous and continuous.

Now, we consider an $\widehat{m}$ of the map $m\colon \mathrm{Kv}(Y) \to Y$ over the space $\widehat{Y}$. Namely, we define:

$$\widehat{m}\big(\langle A, B\rangle\big) = m(A) - m(B)\,, \qquad \langle A, B\rangle \in \widehat{Y}\,.$$

**Lemma 6.1.** *The maps*

$$\widehat{m}\colon \big(\widehat{Y}, \|\cdot\|_H^*\big) \to \big(Y, \|\cdot\|\big) \quad \text{and} \quad \widehat{m}\colon \big(\widehat{Y}, \|\cdot\|^*\big) \to \big(Y, \|\cdot\|\big)$$

*are linear and continuous. Consequently, these maps are Fréchet differentiable.*

Proof. The proof of algebraic properties is an easy calculation. The continuity follows by the equalities $\big\|\widehat{m}\big(\langle A, B\rangle\big)\big\| = \big\|m(A) - m(B)\big\| = \big\|\big\langle m(A), m(B)\big\rangle\big\|_H^* = \big\|\big\langle m(A), m(B)\big\rangle\big\|^*$. The differentiability is obvious. □

## 7. Differentiable selections

We complete the construction by formulating results on existence of differentiable selections whenever $Y$ is an arbitrary normed space such that the dual space $Y^*$ contains a total sequence.

Remark that there is no $d_H$-uniformly continuous selection on $\mathrm{Kv}(Y)$ whenever $Y$ is infinite dimensional [PrYo]. Thus we formulate our results for stronger topologies or topologies not comparable with the $d_H$-topology.

**Theorem 7.1.** *Let $X$ and $Y$ be two normed spaces and the dual space $Y^*$ contains a total sequence, $U \subset X$ an open subset, $\mathrm{Kv}(Y)$ the family of all nonempty compact convex subsets of $Y$, and $F\colon U \to \mathrm{Kv}(Y)$ be $d_H^*$-differentiable or $d^*$-differentiable. Then $F$ has a differentiable selection.*

Proof. Let $\varphi_n \in Y^*$ be total. Let $\widehat{m}\colon \widehat{Y} \to Y$ be defined as in the previous section. Let $\widehat{F}(x) = \big\langle F(x), 0\big\rangle$ for all $x \in U$. Then the composition $f = \widehat{m} \circ \widehat{F}$ is the required differentiable selection. □

As a corollary we have the following main result.

**Theorem 7.2.** *If $Y$ is a separable normed space then every $d^*$-differentiable (or $d_H^*$-differentiable) function $F\colon U \to \mathrm{Kv}(Y)$ has a differentiable selection.*

Proof. In virtue of Lemma 3.1, there exists a total sequence $\varphi_n \in Y^*$. Now, we apply Theorem 7.1. □

**Remarks.** 1. According to the remarks stated after Lemma 3.2, the above obtained selections are extreme points selections.

2. If $Y$ is finite dimensional then for a given point $(x, y)$ of the graph of $F$ we can take the coordinate functionals and proceed similarly like Stefani and Zecca [StZe] to obtain a differentiable selection through the point $(x, y)$.

3. Continuous not $d_H$-differentiable set-valued functions may have differentiable selections. For example, consider functional $\varphi_1(y) = y$, $y \in \mathbb{R}$, which separates the points in $\mathbb{R}$. Then the multifunction $F(x) = \big[0, |x|\big]$, $x \in \mathbb{R}$, is $d^*$-differentiable but not $d_H^*$- and $d_H$-differentiable at $x = 0$. Our construction leads to the selection $f(x) = 0$ for all $x \in \mathbb{R}$. Moreover, $F$ has a differentiable selection through every point of the graph of $F$.

4. As it was shown by Gelman, c.f., [BGMO], each $\varepsilon$-hull of upper semicontinuous continuous function $F\colon U \to \mathrm{Kv}(\mathbb{R}^n)$ includes a $d_H$-differentiable set-valued function. Thus every such $F$ has a differentiable $\varepsilon$-selection.

5. Fréchet differentiability can be replaced by each concept of differentiability which guarantees differentiability of continuous linear maps and differentiability of the compositions of differentiable maps.

## 8. Lipschitz selections

In a similar way as above we may obtain Lipschitz selections from $d^*$- or $d_H^*$-Lipschitz multifunctions.

**Theorem 8.1.** *Let $X$ and $Y$ be two normed spaces and the dual space $Y^*$ contains a total sequence, $U \subset X$ an open subset, $\mathrm{Kv}(Y)$ the family of all nonempty compact convex subsets of $Y$, and $F\colon U \to \mathrm{Kv}(Y)$ be $d_H^*$-Lipschitz or $d^*$-Lipschitz, then $F$ has a Lipschitz selection.*

And as a corollary we get

**Theorem 8.2.** *If $Y$ is a separable normed space then each $d^*$-Lipschitz or $d_H^*$-Lipschitz function $F\colon U \to \mathrm{Kv}(Y)$ has a Lipschitz selection.*

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