Tatra Mt. Math. Publ. **32** (2005), 41–51

TATRA MOUNTAINS
Mathematical Publications

NOTES ON THE EXPRESSION OF NATURAL NUMBERS AS SUM OF POWERS

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ABSTRACT. This paper concerns problems of expression of natural numbers as a sum of powers of natural numbers.

1. Introduction

It is known (Waring Theorem) that for every given natural $k \geqslant 2$ there exists $m \geqslant k$ such that any natural number n can be expressed in the form $n = n_1^k + n_2^k + \cdots + n_m^k$, where $n_i \in \mathbb{N} \cup \{0\}$, $i = 1, 2, \ldots, m$. On the contrary, if $m < k$, then the set of natural numbers, which can be written in this form, has the asymptotic density equal to zero. Issues, connected with numbers in this form, are treated, for example, in [3], [4], [5], [6].

This paper deals with a more general case, namely with natural numbers in the form $n = n_1^{k_1} + n_2^{k_2} + \cdots + n_m^{k_m}$, where $k_1, k_2, \ldots, k_m \in \mathbb{N}$ are given and $n_1, n_2, \ldots, n_m$ are arbitrary nonnegative integers.

We will prove that in the case $\sum_{i=1}^{m} \frac{1}{k_i} < 1$ not only there are some natural numbers, which can not be expressed in the form $n = n_1^{k_1} + n_2^{k_2} + \cdots + n_m^{k_m}$, but the set of natural numbers which can be, has the asymptotic density equal to zero.

But first we will investigate some properties of the asymptotic density of sets in the special form $A = \bigcup_{i \in M} A_i$ where A_i and M are subsets of $\mathbb{N}$, because sets of natural numbers $n = n_1^{k_1} + n_2^{k_2} + \cdots + n_m^{k_m}$ use to be in this form.

2000 Mathematics Subject Classification: 11P99.
Keywords: asymptotic density. additive base, lattice points.

Supported by the grant 201/04/0381 of the Czech Grant Agency.

2. Densities of sets of natural numbers expressed as sum of powers

Denotation 2.1. Let $\mathbb{N}$ is the set of natural numbers and A is a subset of $\mathbb{N}$. We denote by $A(n)$ the number of elements of the set A which are less or equal to n and by $d(A)$ the asymptotic density of A. By $[x]$ we denote the integral part of number x.

Theorem 2.1. *Let $M \subseteq \mathbb{N}$ and $A \subseteq \mathbb{N}$ is a set which could be expressed as an union of sets $A_i \subseteq \mathbb{N}$, $i \in M$, i.e., $A = \bigcup_{i \in M} A_i$. If for every $n \in \mathbb{N}$, $i \in M$, holds $A_i(n) = 0$ for $i > n$ and exists a function $f(n)$ such that for every $k \in \mathbb{N}$ and for every $i \in M$ holds $A_i(k) \leqslant f(k)$, and if at least one of the following properties holds:*

a) $f(n) = O(n^\alpha)$ *and* $M(n) = O(n^\beta)$, $0 \leqslant \alpha + \beta < 1$;

b) $f(n) = o(n^\alpha)$ *and* $M(n) = O(n^\beta)$, $0 \leqslant \alpha + \beta \leqslant 1$;

c) $f(n) = O(n^\alpha)$ *and* $M(n) = o(n^\beta)$, $0 \leqslant \alpha + \beta \leqslant 1$,

then $d(A) = 0$.

P r o o f. The proof of Theorem 2.1 is based on the following inequality:

$$\frac{\left(\bigcup_{i \in M} A_i\right)(n)}{n} \leqslant \frac{\sum_{i \in M} A_i(n)}{n}.$$

And because of $A_i(n) = 0$ for $i > n$ and $A_i(n) \leqslant f(n)$ for every $i \in M$, we can write

$$\frac{\sum_{i \in M} A_i(n)}{n} = \frac{\sum_{i \in M, i \leqslant n} A_i(n)}{n} \leqslant \frac{\sum_{i \in M, i \leqslant n} f(n)}{n} = \frac{f(n)M(n)}{n}.$$

In the case a) it means, that there is an $\varepsilon > 0$ such that

$$\frac{\left(\bigcup_{i \in M} A_i\right)(n)}{n} \leqslant \frac{O(n^\alpha)O(n^\beta)}{n^{\alpha+\beta+\varepsilon}} = O(n^{-\varepsilon}) = o(1).$$

In the case b) it is easy to see, that

$$\frac{\left(\bigcup_{i \in M} A_i\right)(n)}{n} \leqslant \frac{o(n^\alpha)O(n^\beta)}{n} = o(1).$$

Finally, the case c) is analogical with case b). □

EXAMPLE 2.1. We can use Theorem 2.1 to prove, that set $A = \{n_1^3 + n_2^2 \colon n_1, n_2 \in \mathbb{N} \cup \{0\},\ (n_1, n_2) \neq (0,0)\}$ has asymptotic density $d(A) = 0$.

Proof. Denote $A_0 = \{0 + n_2^2 \colon n_2 \in \mathbb{N}\}$ and $A_i = \{i + n_2^2 \colon n_2 \in \mathbb{N} \cup \{0\}\}$ Then $A = A_0 \cup \left(\bigcup_{i \in M} A_i\right)$, where $M = \{n_1^3 \colon n_1 \in \mathbb{N}\}$. It is clear, that $M(n) = O\left(n^{\frac{1}{3}}\right)$ and for every $i \in M$ and $n \in \mathbb{N}$ holds $A_i(n) \leqslant f(n) = n^{\frac{1}{2}}$ and $A_i(n) = 0$ for $i > n$. Because $0 \leqslant \frac{1}{3} + \frac{1}{2} < 1$ and according to Theorem 2.1 we obtain $d\left(\bigcup_{i \in M} A_i\right) = 0$ and so $d(A) = 0$. □

LEMMA 2.1. *Let $k_1, k_2, \dots, k_m \in \mathbb{N}$ be given natural numbers and $A = \{n_1^{k_1} + n_2^{k_2} + \dots + n_m^{k_m} \colon n_i \in \mathbb{N} \cup \{0\},\ i = 1, 2 \dots, m, (n_1, n_2, \dots, n_m) \neq (0, 0, \dots, 0)\}$, then $A(n) = O\left(n^{\sum_{i=1}^{m} \frac{1}{k_i}}\right)$.*

Proof. By induction. In case of $m = 1$ the Lemma 2.1 is evidently true because $A = \{n_1^{k_1} \colon n_1 \in \mathbb{N}\}$. So $A(n) \leqslant n^{\frac{1}{k_1}} = O\left(n^{\frac{1}{k_1}}\right)$.

Now we can assume, that Lemma 2.1 holds for $m - 1$. Let us denote $M = \{n_1^{k_1} + n_2^{k_2} + \dots + n_{m-1}^{k_{m-1}} \colon n_i \in \mathbb{N} \cup \{0\},\ i = 1, 2 \dots, m-1, (n_1, n_2, \dots, n_{m-1}) \neq (0, 0, \dots, 0)\}$. By induction $M(n) = O\left(n^{\sum_{i=1}^{k_i} \frac{1}{k_i}}\right)$.

Let $A_0 = \{0 + n_m^{k_m} \colon n_m \in \mathbb{N}\}$ and $A_i = \{i + n_m^{k_m} \colon n_m \in \mathbb{N} \cup \{0\}\}$ for $i \in M$. Then $A = A_0 \cup \left(\bigcup_{i \in M} A_i\right)$. For $i > n$ there holds $A_i(n) = 0$ and for every $i \in M$ there holds $A_i(n) \leqslant n^{\frac{1}{k_m}} = 0\left(n^{\frac{1}{k_m}}\right)$. For $A(n)$ then it must hold

$$
\begin{aligned}
A(n) \leqslant A_0(n) + \sum_{i \in M} A_i(n) &= A_0(n) + \sum_{i \in M, i \leqslant n} A_i(n) \\
&\leqslant O\left(n^{\frac{1}{k_m}}\right) + \sum_{i \in M, i \leqslant n} n^{\frac{1}{k_m}} \\
&= O\left(n^{\frac{1}{k_m}}\right) + M(n) n^{\frac{1}{k_m}} \\
&= O\left(n^{\frac{1}{k_m}}\right) + O\left(n^{\frac{1}{k_m}}\right) O\left(n^{\sum_{i=1}^{k_i} \frac{1}{k_i}}\right) \\
&= O\left(n^{\sum_{i=1}^{k_i} \frac{1}{k_i}}\right).
\end{aligned}
$$

□

THEOREM 2.2. *Let $k_1, k_2, \dots, k_m \in \mathbb{N}$ be given natural numbers and $A = \{n_1^{k_1} + n_2^{k_2} + \dots + n_m^{k_m} \colon n_i \in \mathbb{N} \cup \{0\},\ i = 1, 2 \dots, m, (n_1, n_2, \dots, n_m) \neq (0, 0, \dots, 0)\}$. If $\sum_{i=1}^{m} \frac{1}{k_i} < 1$, then $d(A) = 0$.*

Proof. We obtain $A(n) = O\left(n^{\sum\limits_{i=1}^{m} \frac{1}{k_i}}\right)$ from Lemma 2.1. Since $\sum\limits_{i=1}^{m} \frac{1}{k_i} < 1$, we can see that there is an $\varepsilon > 0$ such that

$$\frac{A(n)}{n} = \frac{O\left(n^{\sum\limits_{i=1}^{m} \frac{1}{k_i}}\right)}{n^{\sum\limits_{i=1}^{m} \frac{1}{k_i}} n^{\varepsilon}} = O(n^{-\varepsilon}) = o(1)\,.$$

□

COROLLARY 2.1. *If we denote* $A_m = \left\{n_1^m + n_2^m + \cdots + n_{m-1}^m : n_i \in \mathbb{N} \cup \{0\},\ i = 1, 2 \ldots, m-1,\ (n_1, n_2, \ldots, n_{m-1}) \neq (0, 0, \ldots, 0)\right\}$, *then for any given* $m \geqslant 2$ *there holds that* $d(A_m) = 0$.

COROLLARY 2.2. *From Theorem 2.2 it follows, that if we denote* $A_k = \left\{n_1^k + n_2^k + \cdots + n_m^k : n_i \in \mathbb{N} \cup \{0\},\ i = 1, 2 \ldots, m,\ (n_1, n_2, \ldots, n_m) \neq (0, 0, \ldots, 0)\right\}$ *for some given* m, *then for any* $k = 2, 3, \ldots$ *there holds, that if* $k > m$, *then* $d(A_k) = 0$. *Hence we obtain a known result, that set* $B_k = \{n^k : n \in \mathbb{N}\}$ *can not be additive base of* $\mathbb{N}$ *of degree less than* m.

Remark 2.1. Let $A = \left\{n_1^{k_1} + n_2^{k_2} + \cdots + n_m^{k_m} : n_i \in \mathbb{N} \cup \{0\},\ i = 1, 2 \ldots, m,\ (n_1, n_2, \ldots, n_m) \neq (0, 0, \ldots, 0)\right\}$. The condition $\sum\limits_{i=1}^{m} \frac{1}{k_i} < 1$ is sufficient to $d(A) = 0$, but not necessary.

We will prove, that the set $D = \left\{n_1^2 + n_2^2 : n_1, n_2 \in \mathbb{N} \cup \{0\}, (n_1, n_2) \neq (0, 0)\right\}$ has the asymptotic density equal to zero. We will show two different proofs of the fact that $d(D) = 0$.

THEOREM 2.3. *Let* $D = \left\{n_1^2 + n_2^2 : n_1, n_2 \in \mathbb{N} \cup \{0\},\ (n_1, n_2) \neq (0, 0)\right\}$ *then* $d(D) = 0$.

Proofs.

Proof I. We will use Landau's result (see [1] and [7]): $D(n) = \frac{c.n}{\sqrt{\log n}}\big(1 + o(1)\big)$ and so

$$d(D) = \lim_{n \to \infty} \frac{D(n)}{n} = 0\,.$$

□

Proof II. Now we will prove Theorem 2.3 without using the above mentioned Landau's result. This proof is based on the following lemma, which could be found in [2].

LEMMA 2.2. *A natural number* n *could be expressed in form* $n = n_1^2 + n_2^2$, *where* n_1, n_2 *are integers, if and only if there is no prime number* $p = 4k + 3$, $k \in \mathbb{N} \cup \{0\}$ *with odd exponent in the canonical decomposition of* n.

For example: $486 = 2 \cdot 3^5$ therefore there are no integers n_1, n_2, such that $486 = n_1^2 + n_2^2$.

To prove Theorem 2.3, we will also need following lemmas.

Lemma 2.3. *Let us denote: $p_1 < p_2 < \dots < p_i < \dots$ the sequence of all prime numbers in the form $p_i = 4k+3$, $k \in \mathbb{N} \cup \{0\}$, and $D_i = \{m \cdot p_i^{2j} : j \in \mathbb{N} \cup \{0\},\ m \in \mathbb{N},\ \gcd(m, p_i) = 1\}$, i.e., D_i is a set of natural numbers, which have not the prime number p_i with odd exponent in their canonical decompositions. Then the asymptotic density $d(D_i) = \frac{p_i}{p_i+1}$.*

Proof. At first we define the set B_k as a set of natural numbers, which have the prime number p_i with exponent greater or equal to k. Therefore we can write

$$B_0 = \mathbb{N} = \{m : m \in \mathbb{N}\} \quad \text{and} \quad B_0(n) = [n],$$

$$B_1 = \{p_i \cdot m : m \in \mathbb{N}\} \quad \text{and} \quad B_1(n) = \left[\frac{n}{p_i}\right],$$

$$B_2 = \{p_i^2 \cdot m : m \in \mathbb{N}\} \quad \text{and} \quad B_2(n) = \left[\frac{n}{p_i^2}\right],$$

$$\vdots$$

$$B_k = \{p_i^k \cdot m : m \in \mathbb{N}\} \quad \text{and} \quad B_k(n) = \left[\frac{n}{p_i^k}\right].$$

Now we are going to define sets C_k and Q_k, $k = 1, 2, \dots$ recurrently: $C_1 = B_0$, $Q_k = C_k - B_{2k-1}$, $C_k = Q_{k-1} \cup B_{2(k-1)}$.

It is easy to see, that

a) In case of $k = 1$ and $n \in \mathbb{N}$ there holds

$$C_1 = B_0 = \mathbb{N} \supseteq D_i \supseteq Q_1 = B_0 - B_1,$$

so we can write

$$C_1(n) = [n] \geqslant D_i(n) \geqslant Q_1(n) = [n] - \left[\frac{n}{p_i}\right].$$

b) In case of $k = 2$ and $n \in \mathbb{N}$ there holds

$$C_2 = Q_1 \cup B_2 \supseteq D_i \supseteq Q_2 = C_2 - B_3,$$

so we can write

$$C_2(n) = [n] - \left[\frac{n}{p_i}\right] + \left[\frac{n}{p_i^2}\right] \geqslant D_i(n) \geqslant Q_2(n) = [n] - \left[\frac{n}{p_i}\right] + \left[\frac{n}{p_i^2}\right] - \left[\frac{n}{p_i^3}\right].$$

c) Finally we can generally say, that for every $k \in \mathbb{N}$ and $n \in \mathbb{N}$ there holds

$$C_k \supseteq D_i \supseteq Q_k,$$

so we can write

$$C_k(n) = \sum_{j=0}^{2(k-1)} (-1)^j \left[\frac{n}{p_i^j}\right] \geqslant D_i(n) \geqslant Q_k(n) = \sum_{j=0}^{2k-1} (-1)^j \left[\frac{n}{p_i^j}\right].$$

Hence for every $k, n \in \mathbb{N}$ there holds

$$\frac{\sum\limits_{j=0}^{2(k-1)} (-1)^j \left[\frac{n}{p_i^j}\right]}{n} \geqslant \frac{D_i(n)}{n} \geqslant \frac{\sum\limits_{j=0}^{2k-1} (-1)^j \left[\frac{n}{p_i^j}\right]}{n}.$$

With $n \to \infty$ we obtain for arbitrary fixed $k \in \mathbb{N}$

$$\lim_{n\to\infty} \frac{\sum\limits_{j=0}^{2(k-1)} (-1)^j \left[\frac{n}{p_i^j}\right]}{n} \geqslant \limsup_{n\to\infty} \frac{D_i(n)}{n} \geqslant \liminf_{n\to\infty} \frac{D_i(n)}{n} \geqslant \lim_{n\to\infty} \frac{\sum\limits_{j=0}^{2k-1} (-1)^j \left[\frac{n}{p_i^j}\right]}{n}$$

and because k is fixed, we receive

$$\sum_{j=0}^{2(k-1)} (-1)^j \frac{1}{p_i^j} \geqslant \limsup_{n\to\infty} \frac{D_i(n)}{n} \geqslant \liminf_{n\to\infty} \frac{D_i(n)}{n} \geqslant \sum_{j=0}^{2k-1} (-1)^j \frac{1}{p_i^j}.$$

With $k \to \infty$ we obtain

$$\lim_{k\to\infty} \sum_{j=0}^{2(k-1)} (-1)^j \frac{1}{p_i^j} \geqslant \limsup_{n\to\infty} \frac{D_i(n)}{n} \geqslant \liminf_{n\to\infty} \frac{D_i(n)}{n} \geqslant \lim_{k\to\infty} \sum_{j=0}^{2k-1} (-1)^j \frac{1}{p_i^j}$$

and finally

$$\frac{p_i}{p_i+1} \geqslant \limsup_{n\to\infty} \frac{D_i(n)}{n} \geqslant \liminf_{n\to\infty} \frac{D_i(n)}{n} \geqslant \frac{p_i}{p_i+1}.$$

□

Lemma 2.4. *Let D_i be sets defined as in previous lemma. Then for every $n \in \mathbb{N}$ there holds, that the asymptotic density* $d\Bigl(\bigcap\limits_{i=1}^{n} D_i\Bigr) = \prod\limits_{i=1}^{n} d(D_i) = \prod\limits_{i=1}^{n} \frac{p_i}{p_i+1}$.

P r o o f . We will prove this lemma by induction. The proof for $n = 1$ is trivial and for $n = 2$ is analogous to the previous one: If we realise, that $D_1 \cap D_2 = \{p_1^{2j_1} p_2^{2j_2} \cdot m \colon j_1, j_2 \in \mathbb{N} \cup \{0\},\ m \in \mathbb{N},\ \gcd(m, p_1) = \gcd(m, p_2) = 1\}$, then to prove, that $d(D_1 \cap D_2) = \frac{p_1}{p_1+1}\frac{p_2}{p_2+1}$, we have to define sets B_k in the following way

$$B_0 = D_2 = \{m \colon m \in D_2\} \quad \text{and so} \quad B_0(n) = D_2(n),$$

$$B_1 = \{p_1 \cdot m \colon m \in D_2\} \quad \text{and so} \quad B_1(n) = D_2\left(\left[\frac{n}{p_1}\right]\right),$$

$$B_2 = \{p_1^2 \cdot m \colon m \in D_2\} \quad \text{and so} \quad B_2(n) = D_2\left(\left[\frac{n}{p_1^2}\right]\right),$$

$$\vdots$$

$$B_k = \{p_1^k \cdot m \colon m \in D_2\} \quad \text{and so} \quad B_k(n) = D_2\left(\left[\frac{n}{p_1^k}\right]\right).$$

Now we are going to define sets C_k and Q_k, $k = 1, 2, \dots$, recurrently: $C_1 = B_0$, $Q_k = C_k - B_{2k-1}$, $C_k = Q_{k-1} \cup B_{2(k-1)}$.

It is easy to see, that

a) In case of $k = 1$ and $n \in \mathbb{N}$ there holds

$$C_1 = B_0 = Q_2 \supseteq (D_1 \cap D_2) \supseteq Q_1 = B_0 - B_1$$

so we can write

$$C_1(n) = D_2([n]) \geqslant (D_1 \cap D_2)(n) \geqslant Q_1(n) = D_2([n]) - D_2\left(\left[\frac{n}{p_1}\right]\right)$$

b) In case of $k = 2$ and $n \in \mathbb{N}$ there holds

$$C_2 = Q_1 \cup B_2 \supseteq D_1 \cap D_2 \supseteq Q_2 = C_2 - B_3$$

so we can write

$$C_2(n) = \sum_{j=0}^{2} (-1)^j D_2\left(\left[\frac{n}{p_1^j}\right]\right) \geqslant (D_1 \cap D_2)(n) \geqslant Q_2(n) = \sum_{j=0}^{3} (-1)^j D_2\left(\left[\frac{n}{p_1^j}\right]\right).$$

c) Finally we can generally say, that for every $k \in \mathbb{N}$ and $n \in \mathbb{N}$ there holds

$$C_k \supseteq D_1 \cap D_2 \supseteq Q_k$$

so we can write

$$C_k(n) = \sum_{j=0}^{2(k-1)} (-1)^j D_2\left(\left[\frac{n}{p_1^j}\right]\right) \geqslant (D_1 \cap D_2)(n) \geqslant Q_k(n) = \sum_{j=0}^{2k-1} (-1)^j D_2\left(\left[\frac{n}{p_1^j}\right]\right).$$

Hence for every $k, n \in \mathbb{N}$ there holds

$$\frac{\sum_{j=0}^{2(k-1)} (-1)^j D_2\left(\left[\frac{n}{p_1^j}\right]\right)}{n} \geqslant \frac{(D_1 \cap D_2)(n)}{n} \geqslant \frac{\sum_{j=0}^{2k-1} (-1)^j D_2\left(\left[\frac{n}{p_1^j}\right]\right)}{n}.$$

If we denote $D^* = D_1 \cap D_2$, $\bar{d}(D^*)$ the upper asymptotic density of D^* and $\underline{d}(D^*)$ the lower asymptotic density of D^*, respectively, then with $n \to \infty$ we obtain for every fixed $k \in \mathbb{N}$ the following inequalities.

$$\lim_{n\to\infty} \frac{\sum\limits_{j=0}^{2(k-1)} (-1)^j D_2\left(\left[\frac{n}{p_1^j}\right]\right)}{n} \geqslant \bar{d}(D^*) \geqslant \underline{d}(D^*) \geqslant \lim_{n\to\infty} \frac{\sum\limits_{j=0}^{2k-1} (-1)^j D_2\left(\left[\frac{n}{p_1^j}\right]\right)}{n},$$

$$\lim_{n\to\infty} \sum_{j=0}^{2(k-1)} \frac{(-1)^j D_2\left(\left[\frac{n}{p_1^j}\right]\right)}{n} \frac{\left[\frac{n}{p_1^j}\right]}{\left[\frac{n}{p_1^j}\right]} \geqslant \bar{d}(D^*) \geqslant \underline{d}(D^*)$$

$$\geqslant \lim_{n\to\infty} \sum_{j=0}^{2k-1} \frac{(-1)^j D_2\left(\left[\frac{n}{p_1^j}\right]\right)}{n} \frac{\left[\frac{n}{p_1^j}\right]}{\left[\frac{n}{p_1^j}\right]},$$

$$d(D_2) \sum_{j=0}^{2(k-1)} (-1)^j \frac{1}{p_1^j} \geqslant \bar{d}(D^*) \geqslant \underline{d}(D^*) \geqslant d(D_2) \sum_{j=0}^{2k-1} (-1)^j \frac{1}{p_1^j}.$$

With $k \to \infty$ we obtain:

$$\lim_{k\to\infty} d(D_2) \sum_{j=0}^{2(k-1)} (-1)^j \frac{1}{p_1^j} \geqslant \bar{d}(D^*) \geqslant \underline{d}(D^*) \geqslant \lim_{k\to\infty} d(D_2) \sum_{j=0}^{2k-1} (-1)^j \frac{1}{p_1^j},$$

$$d(D_2)\frac{p_1}{p_1+1} \geqslant \bar{d}(D^*) \geqslant \underline{d}(D^*) \geqslant d(D_2)\frac{p_1}{p_1+1},$$

$$d(D_2)d(D_1) \geqslant \bar{d}(D^*) \geqslant \underline{d}(D^*) \geqslant d(D_2)d(D_1).$$

Now we have to do the induction step. Let us denote $D^* = \bigcap\limits_{i=1}^{r-1} D_i$. By the induction assumption we obtain $d(D^*) = \prod\limits_{i=1}^{r-1} d(D_i) = \prod\limits_{i=1}^{r-1} \frac{p_i}{p_i+1}$ and we are going to prove that $d(D^* \cap D_r) = \prod\limits_{i=1}^{r} d(D_i) = \prod\limits_{i=1}^{r} \frac{p_i}{p_i+1}$. The proof of this fact is analogous to the proof in the case of $n = 2$. We just have to define sets B_k for $k = 1, 2, \dots$ in the following way

$$B_0 = \{m \colon m \in D^*\} \quad \text{and so} \quad B_0(n) = D^*(n),$$

$$B_1 = \{p_r \cdot m \colon m \in D^*\} \quad \text{and so} \quad B_1(n) = D^*\left(\left[\frac{n}{p_r}\right]\right),$$

$$B_2 = \{p_r^2 \cdot m \colon m \in D^*\} \quad \text{and so} \quad B_2(n) = D^*\left(\left[\frac{n}{p_r^2}\right]\right),$$

$$\vdots$$

$$B_k = \{p_r^k \cdot m\colon m \in D^*\} \quad \text{and so} \quad B_k(n) = D^*\left(\left[\frac{n}{p_r^k}\right]\right),$$

and to define sets C_k and Q_k, $k = 1, 2, \dots$, recurrently: $C_1 = B_0$, $Q_k = C_k - B_{2k-1}$, $C_k = Q_{k-1} \cup B_{2(k-1)}$.

Then, as in the case of $n = 2$, we have for every $k, n \in \mathbb{N}$

$$\frac{\sum_{j=0}^{2(k-1)} (-1)^j D^*\left(\left[\frac{n}{p_r^j}\right]\right)}{n} \geqslant \frac{(D^* \cap D_r)(n)}{n} \geqslant \frac{\sum_{j=0}^{2k-1} (-1)^j D^*\left(\left[\frac{n}{p_r^j}\right]\right)}{n}.$$

If we denote $D^{**} = D^* \cap D_r$, then with $n \to \infty$ we obtain for arbitrary fixed $k \in \mathbb{N}$

$$\lim_{n\to\infty} \frac{\sum_{j=0}^{2(k-1)} (-1)^j D^*\left(\left[\frac{n}{p_r^j}\right]\right)}{n} \geqslant \bar{d}(D^{**}) \geqslant \underline{d}(D^{**}) \geqslant \lim_{n\to\infty} \frac{\sum_{j=0}^{2k-1} (-1)^j D^*\left(\left[\frac{n}{p_r^j}\right]\right)}{n},$$

$$\lim_{n\to\infty} \sum_{j=0}^{2(k-1)} \frac{(-1)^j D^*\left(\left[\frac{n}{p_r^j}\right]\right)}{n} \frac{\left[\frac{n}{p_r^j}\right]}{\left[\frac{n}{p_r^j}\right]} \geqslant \bar{d}(D^{**}) \geqslant \underline{d}(D^{**})$$

$$\geqslant \lim_{n\to\infty} \sum_{j=0}^{2k-1} \frac{(-1)^j D^*\left(\left[\frac{n}{p_r^j}\right]\right)}{n} \frac{\left[\frac{n}{p_r^j}\right]}{\left[\frac{n}{p_r^j}\right]}.$$

Because k is fixed, we can write

$$d(D^*). \sum_{j=0}^{2(k-1)} (-1)^j \frac{1}{p_r^j} \geqslant \bar{d}(D^{**}) \geqslant \underline{d}(D^{**}) \geqslant d(D^*). \sum_{j=0}^{2k-1} (-1)^j \frac{1}{p_r^j},$$

$$d(D^*)d(D_r) \geqslant \bar{d}(D^{**}) \geqslant \underline{d}(D^{**}) \geqslant d(D^*)d(D_r).$$

From the previous induction assumption we have

$$\left(\prod_{i=1}^{r-1} d(D_i)\right) d(D_r) \geqslant \bar{d}(D^{**}) \geqslant \underline{d}(D^{**}) \geqslant \left(\prod_{i=1}^{r-1} d(D_i)\right) d(D_r),$$

$$\prod_{i=1}^{r} d(D_i) \geqslant \limsup_{n\to\infty} \frac{\left(\bigcap_{i=1}^r D_i\right)(n)}{n} \geqslant \liminf_{n\to\infty} \frac{\left(\bigcap_{i=1}^r D_i\right)(n)}{n} \geqslant \prod_{i=1}^{r} d(D_i). \qquad \square$$

Using previous lemmas we can introduce the proof of Theorem 2.3. From Lemma 2.2 we conclude, that $D = \bigcap_{i=1}^{\infty} D_i$, hence for every $k \in \mathbb{N}$ there holds

$$D \subseteq \left(\bigcap_{i=1}^{k} D_i \right).$$

So for every $k \in \mathbb{N}$ and for every $n \in \mathbb{N}$ there holds

$$D(n) = \left(\bigcap_{i=1}^{\infty} D_i \right)(n) \leqslant \left(\bigcap_{i=1}^{k} D_i \right)(n),$$

$$\frac{D(n)}{n} \leqslant \frac{\left(\bigcap_{i=1}^{k} D_i \right)(n)}{n}.$$

Using Lemma 2.4 we can say, that for every fixed $k \in \mathbb{N}$ there holds

$$0 \leqslant \liminf_{n\to\infty} \frac{D(n)}{n} \leqslant \limsup_{n\to\infty} \frac{D(n)}{n} \leqslant \prod_{i=1}^{k} \frac{p_i}{p_i + 1}.$$

Now we have to prove that $\lim_{k\to\infty} \prod_{i=1}^{k} \frac{p_i}{p_i+1} = 0$. We can see that

$$0 \leqslant \prod_{i=1}^{k} \frac{p_i}{p_i + 1} = \prod_{i=1}^{k} \frac{1}{1 + \frac{1}{p_i}} = \frac{1}{\prod_{i=1}^{k} \left(1 + \frac{1}{p_i}\right)} \leqslant \frac{1}{\sum_{i=1}^{k} \frac{1}{p_i}}.$$

If $p_1 < p_2 < \dots < p_n < \dots$ is the sequence of all prime numbers in the form $p_n = 4k + 3$, $k \in \mathbb{N} \cup \{0\}$, then $\sum_{i=1}^{\infty} \frac{1}{p_i} = \infty$.

So we obtain

$$0 \leqslant \liminf_{n\to\infty} \frac{D(n)}{n} \leqslant \limsup_{n\to\infty} \frac{D(n)}{n} \leqslant 0.$$

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Received December 8, 2004

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