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ON NIL-ALGEBRAS AND AN INVARIANT OF FIELDS

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ABSTRACT. About 30 years ago T. J. Laffey defined a field invariant and connected it with the existence of large commutative subalgebras of nil-algebras over fields.

In this note we are going to compare the invariant of Laffey with the well known u-invariant. We are also going to extend his result on nil-algebras over fields to the case of nil-algebras over arbitrary commutative, unital ring.

1. Square dependent rings

In this note F stands for a commutative ring with $1 \neq 0$. Associative F-algebras, not necessarily with 1, will be called *algebras* for short. If A is an algebra and $1 \notin A$ then by A^1 we denote the standard extension of A to a unital algebra with the help of F. If $1 \in A$ then let us take simply $A^1 = A$. In both cases any algebra ideal of A is an ideal of A^1. An algebra is called *reduced* if it has no nontrivial nilpotent elements. Subalgebras with zero multiplication will be called *zero subalgebras*. For further information about algebras we refer to [1, 4] and for basics about quadratic forms to [7].

As in [6] let us call a field F *square dependent* if for any infinite sequence $a_1, a_2, \dots, a_n, \dots$ of elements from F there exists $n \geqslant 1$ such that the equation $\sum\limits_{i=1}^{n} a_i x_i^2 = 0$ has a nontrivial solution in F. Fields being not square dependent will be called here *square independent*. In [6], among other results, T. J. Laffey proved the following one:

Theorem 1.1. *Let F be a field. Then F is square dependent if and only if every infinite dimensional nil-algebra over F contains an infinite dimensional zero subalgebra.*

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As a part of the proof L a f f e y provided the following simple but useful construction:

EXAMPLE 1.2. Let F be a square independent field. By definition we then have a sequence $a_1, a_2, \dots$ of elements from F such that for every $n \in \mathbb{N}$ the equation $a_1x_1^2 + \dots + a_nx_n^2 = 0$ has only the trivial solution in F. For our convenience let us add $a_0 = 0$.

Let V be an infinite dimensional vector space over F with basis $\{v_0, v_1, v_2, \dots\}$. Consider V as an algebra with multiplication given by the following rules:

$$v_iv_j = \begin{cases} 0 & \text{if } i \neq j, \\ a_iv_0 & \text{if } i = j, \end{cases} \quad \text{for} \quad i, j \in \{0, 1, 2, \dots\}.$$

Then it is easy to check, that $V^3 = 0$ and the only zero subalgebra of V is Fv_0, because for any $v = b_0v_0 + \dots + b_nv_n$ where $b_i \in F$ we have

$$v^2 = (a_1b_1^2 + \dots + a_nb_n^2)v_0 .$$

In addition, for every $n \geqslant 1$ the subspace spanned by $\{v_0, v_n\}$ is an ideal of the algebra V.

The notion of square dependence has an interesting connection with field invariants induced by quadratic forms. Following approach of D. W. H o f f m a n n (see [3, page 81]), let us agree that $\mathcal{C}_d(F)$ is the class of all diagonal quadratic forms over a field F and

$$d(F) = \sup\{\dim \phi \colon \phi \text{ anisotropic form over } F,\ \phi \in \mathcal{C}_d(F)\}.$$

It is well known that for F of characteristic not 2 any quadratic form is diagonalizable. Hence, for such fields $d(F) = u(F)$, but for fields of characteristic 2 we only have $d(F) \leqslant u(F)$. Here u-invariant is understood as in [7, § 11.4].

Immediately from the definition we have:

PROPOSITION 1.3. *Let F be a field.*

- *If $d(F) < \infty$ then F is square dependent;*
- *If F is formally real, then F is not square dependent.*

EXAMPLE 1.4. Let K be a non formally real field and let $F = K(t_1, t_2, \dots)$ be the field of rational functions of infinitely many indeterminates $t_1, t_2, \dots$ Then, using the infinite sequence $(t_1, t_2, \dots)$ and the diagonal forms $\phi_n = \langle t_1, t_2, \dots, t_n \rangle$ for $n \in \mathbb{N}$, one can see that F is square independent, hence $d(F) = \infty$. However, F is not formally real.

Recently, K. J. B e c h e r and D. W. H o f f m a n n solved in [2] an old problem attributed to H. G r o s s. From our point of view, for any field F_0

they constructed a field extension $F_0 \subseteq F$, that is square dependent, but $d(F) = u(F) = \infty$. In this situation my question, asked during the Number Theory Conference in Terchová in 2004, (see Problem 2 in [8]) is completely solved.

If F is an integral domain with the field of fractions K, then it is natural to agree that $d(F) = d(K)$. We are not going to define d-invariant for more general rings. However, for our generalization of Theorem 1.1 (Theorems 2.9 and 3.2) we define an arbitrary commutative ring F with unity to be *square dependent* if the factor ring F/M is a square dependent field for every maximal ideal M of F.

EXAMPLE 1.5. Let K be an algebraically closed field and let $F = K[t_1, t_2, \dots]$ be the polynomial ring of indeterminates $t_1, t_2, \dots$ over K. Then, by definition, $d(K) = 1$, but, by Example 1.2 and Proposition 1.3, $d(F) = \infty$.

If K is nondenumerable, then F is square dependent, by a version of the Hilbert Nullstellensatz ([1]).

If K is denumerable, then there exists an epimorphism of F onto the field $K(t_1, t_2, \dots)$. Hence, by Example 1.2, F is square independent.

2. Main results

In this section we are going to prove a version of Theorem 1.1 for algebras over an arbitrary ring F. For this we provide first some necessary definitions, notations and results.

In the sequel F-modules will be called modules, for short. By $\mathcal{F}$ we denote the class of all modules of finite length. If a module M belongs to $\mathcal{F}$, then the length of M will be denoted by $l(M)$.

Algebras will often be identified with their underlying modules. For example, if A is an algebra of finite length as a module, then we will often write $A \in \mathcal{F}$. If X is a subset of an algebra A, then let us put, as in [6]

$$\mathrm{Ann}_A(X) = \{a \in A \colon aX = Xa = 0\}.$$

If X is an ideal of A then certainly $\mathrm{Ann}_A(X)$ is an ideal of A. Our further notation, terminology and results are rather standard, as for example in [1, 4].

Let $M, N \in \mathcal{F}$ be modules. Then it is well known (see [1]) that the module $\mathrm{Hom}_F(M, N)$ also belongs to $\mathcal{F}$. As a consequence we have

LEMMA 2.1. *Let $B \in \mathcal{F}$ be an ideal of an algebra A and C be the left, (right) annihilator of B in A. Then C is an ideal of A and the algebra A/C belongs to $\mathcal{F}$. Thus the algebra $A/(\mathrm{Ann}_A(B))$ also belongs to $\mathcal{F}$.*

It can be seen that for every module $M \in \mathcal{F}$ there exists a finite product G of maximal ideals in F with $GM = 0$. This observation often allows to reduce considerations on algebras from $\mathcal{F}$ to finite dimensional algebras over fields. For example we have

PROPOSITION 2.2. *Let $A \in \mathcal{F}$ be a nil-algebra. If $l(A) = n$ then $A^n = 0$.*

In the sequel, if A is an algebra, then by $N(A)$ we denote the largest nil-ideal of A and by $P(A)$ the smallest semiprime ideal (the prime radical) of A. Certainly $P(A) \subseteq N(A)$ and both these ideals are algebra ideals of A. As a generalization of a result from [5] we have

LEMMA 2.3. *Let $N = P(N)$ be an algebra and I be a maximal zero ideal of N. Put $J = \mathrm{Ann}_N(I)$. Then:*

1. *$J^3 = 0$;*
2. *If $I \in \mathcal{F}$ then N is nilpotent and $N/J \in \mathcal{F}$.*

P r o o f . By our definition, $I \subseteq J$ and J is an ideal of N. It is also well known (see [4]) that N is a union of an increasing chain of ideals N_α, where N_1 is the sum of all nilpotent ideals of N, $N_{\alpha+1}$ is the sum of all ideals of N being nilpotent modulo N_α for all $\alpha \geqslant 1$ and $N_\beta = \sum_{\alpha<\beta} N_\alpha$ if β is a limit ordinal. It is enough to show that for any ordinal α we have

$$J_\alpha^3 = 0\,, \quad \text{where } J_\alpha = J \cap N_\alpha\,.$$

Let $a, b, c \in J_1$ and let K be the ideal of N generated by a, b and c. Then $K \subseteq N_1$ is a finitely generated ideal. Hence $K^m = 0$ for some m, and we can assume, that m is minimal with this property.

Suppose $m > 3$. If $L = K^{m-2}$ then certainly

$$L^2 = K^{2m-4} = 0 = IL = LI, \quad \text{thus } (I + L)^2 = 0\,.$$

By the maximality of I we then have $L \subseteq I$. Hence

$$K^{m-1} = LK \subseteq IK \subseteq IJ = 0\,.$$

This is a contradiction with the choice of m. Hence it has to be $m \leqslant 3$. In particular, $abc = 0$. This means, that $J_1^3 = 0$, because our elements $a, b, c \in J_1$ were arbitrarily chosen.

Assume, that $J_\alpha^3 = 0$ for all $\alpha < \beta$. If β is a limit ordinal, then certainly $J_\beta^3 = 0$. Let $\beta = \alpha + 1$ and let K be a finitely generated ideal contained in J_β. Then, by definition, $K^m \subseteq N_\alpha$ for some $m \geqslant 1$ and $K^m \subseteq J$. Thus $K^m \subseteq J_\alpha$ and, by our assumption, $K^{3m} = 0$. Hence $K \subseteq N_1$ and $K^3 = 0$ by the first step of our proof. In this way we proved that $J_\beta^3 = 0$, as required.

Now let us assume that $I \in \mathcal{F}$. Then, by Lemma 2.1, one can see that $N/J \in \mathcal{F}$. □

Let A be an algebra. Then, as in [5], by $\Delta(A)$ we denote the sum of all such ideals I of A that $I \in \mathcal{F}$. As in [5] one can check

Lemma 2.4. *Let A be an algebra and $a \in A$. Then $a \in \Delta(A)$ if and only if the ideal $A^1 a A^1 \in \mathcal{F}$.*

Lemma 2.5. *Let A be an algebra. Then*

1. $\Delta(\Delta(A)) = \Delta(A)$;
2. *If $B \subseteq A$ is a subalgebra then $B \cap (\Delta(A)) \subset \Delta(B)$;*
3. *If I is an ideal of A then $(\Delta(A) + I)/I \subseteq \Delta(A/I)$.*

As in [5] let us agree that an algebra A is an *FZS-algebra* if any its zero subalgebra belongs to $\mathcal{F}$.

Reduced algebras and algebras from the class $\mathcal{F}$ are certainly FZS-algebras. One can also easily deduce the following observation

Proposition 2.6. *Let A be an algebra.*

- *If A is an FZS-algebra and $B \subseteq A$ is a subalgebra then B is an FZS-algebra;*
- *If $I \subseteq A$ is an algebra ideal such that I and A/I are FZS-algebras then A is an FZS-algebra.*

The following generalization of Lemma 3.5 from [6], proved in [5], is very useful for our further considerations.

Lemma 2.7. *Let A be an FZS-algebra. Then the algebra $A/(\Delta(A))$ is reduced. In particular, $\Delta(A)$ contains all nilpotent elements of A.*

The following result was motivated by some lemmas from [6, 5]. We give here an independent proof.

Lemma 2.8. *Let $J \notin \mathcal{F}$ be an algebra with $J^3 = 0$, I be a maximal zero subalgebra of J and $W = \mathrm{Ann}_J(I)$.*

- *If $x \in W$ is such that $x^2 = 0$, then $x \in I$;*
- *If G is the annihilator of J^2 in F, then $GJ \subseteq I$;*
- *If $I \in \mathcal{F}$, then $F/G \in \mathcal{F}$ and there exists an infinite sequence $v_1, v_2, \ldots$ of elements from W such that for every $i, j \geqslant 1$ we have $v_i^2 \neq 0$, but $v_i v_j = 0$ for $i \neq j$.*

P r o o f . If $x \in W$ is such, that $x^2 = 0$, then $(I + Fx)^2 = 0$ and, by the choice of I we have $x \in I$.

If G is the annihilator of J^2 in F, then $(I + GJ)^2 \subseteq I^2 + GJ^2 = 0$, hence $GJ \subseteq I$, again by maximality of I.

Let $I \in \mathcal{F}$. From the assumptions we then have that $J^2 \subseteq I \in \mathcal{F}$, hence $F/G \in \mathcal{F}$. Moreover, $Fx \in \mathcal{F}$ for any $x \in J$.

Let $V_0 = I$ and $W_0 = W$. Then $V_0 \in \mathcal{F}$ and, by Lemma 2.1, $J/W_0 \in \mathcal{F}$. Thus $W_0 \notin \mathcal{F}$ and $W_0 \not\subseteq V_0$. Let us take any $v_1 \in W_0 \setminus V_0$. Then, by the first conclusion, $v_1^2 \neq 0$. Also $V_1 = V_0 + Fv_1 \in \mathcal{F}$ and, by Lemma 2.1, $W_1 = \operatorname{Ann}_J(V_1)$ is such that $J/W_1 \in \mathcal{F}$. Thus $W_1 \notin \mathcal{F}$. Hence we can take as v_2 any element from $W_1 \setminus V_1$. In this way we can continue this procedure to obtain a required sequence of elements v_i with the help of sequences V_i, W_i of ideals of J. $\square$

Theorem 2.9. *For our basic ring F the following conditions are equivalent:*

1. *F is square dependent;*
2. *Any nilpotent FZS-algebra N with $l(N^2) \leqslant 1$ belongs to $\mathcal{F}$;*
3. *Any nilpotent FZS-algebra N with $N^3 = 0$ belongs to $\mathcal{F}$;*
4. *Any nil-algebra with property FZS belongs to $\mathcal{F}$.*

P r o o f . Clearly $4 \Rightarrow 3 \Rightarrow 2$. The implication $2 \Rightarrow 1$ follows by Example 1.2, because for any maximal ideal $G \subset F$ we can consider every algebra over F/G as an F-algebra.

$1 \Rightarrow 2$. Let J be a nilpotent FZS-algebra with $l(J^2) \leqslant 1$. If $l(J^2) = 0$ then $J^2 = 0$ and certainly $J \in \mathcal{F}$, by the assumed property FZS.

Now let $l(J^2) = 1$. Then there exists an element $v_0 \in J^2$ and a maximal ideal G of F such that $J^2 = Fv_0 \simeq F/G$. Certainly $J^3 = 0$. If I is a maximal zero subalgebra of J then, by assumption, $I \in \mathcal{F}$.

Suppose $J \notin \mathcal{F}$. Hence, by Lemma, 2.8, we have an infinite sequence of elements $v_i \in J \setminus I$ such that for any $i, j \geqslant 1$ we have $0 \neq v_i^2 = a_i v_0$ for some $a_i \in F \setminus G$ and $v_i v_j = 0$ for $i \neq j$. By assumption on F the field F/G is square dependent. Thus, for some $n > 1$ there are elements $b_1, \ldots, b_n \in F$ with $b_n \notin G$, for which the expression $\sum_{i=1}^{n} a_i b_i^2$ belongs to G. If we put $v = \sum_{i=1}^{n} b_i v_i$, then

$$v^2 = \left(\sum_{i=1}^{n} a_i b_i^2 \right) v_0 \in Gv_0 = 0 .$$

Thus, from Lemma 2.8, $v \in I$. In particular, in the notation from the proof of this lemma, $b_n v_n \in V_{n-1}$. On the other hand, $b_n \notin G$. Hence, there are elements $f \in F$ and $g \in G$ with $1 = fb_n + g$. Thus

$$v_n = (fb_n + g)v_n = fb_n v_n + gv_n \in V_{n-1} ,$$

a contradiction with the choice of v_n. Hence, it has to be $J \in \mathcal{F}$, as required.

$2 \Rightarrow 3$. Let N be an FZS-algebra with $N^3 = 0$. If I is a maximal zero subalgebra of N, then certainly $N^2 \subseteq I$. Thus, by assumption, $N^2 \subseteq I \in \mathcal{F}$ and $l(N^2) < \infty$.

We will proceed by induction on $l(N^2)$. If $l(N^2) = 0$ or $l(N^2) = 1$, then $N \in \mathcal{F}$, by the assumed condition 2. Now let $l(N^2) = k + 1 > 1$, and $V \subset N^2$ be a submodule with $l(V) = k$. From the assumed $N^3 = 0$ it is clear that V is an ideal of N. If $\overline{N} = N/V$, then it is visible that $l(\overline{N}^2) = 1$.

Assume first, that $\overline{N}$ is not an FZS-algebra. Then there exists a subalgebra $B \subset N$ containing V, such that $\overline{B} = B/V$ is a zero subalgebra of $\overline{N}$ not belonging to $\mathcal{F}$. Hence, $B \notin \mathcal{F}$, but $B^2 \subseteq V$. Clearly B is an FZS-algebra with $B^3 = 0$, a contradiction with our induction assumption. Hence, $\overline{N}$ has to be an FZS-algebra, and $\overline{N} \in \mathcal{F}$, by the assumed Condition 2. This means that $N \in \mathcal{F}$, by the choice of V.

$3 \Rightarrow 4$. Let N be a nil-algebra with property FZS. Then, by Lemma 2.7, $\Delta(N) = N$. By definition and by Proposition 2.2 it means, that $N = P(N)$. Lemma 2.3 implies, that N contains an ideal J with $J^3 = 0$ and $N/J \in \mathcal{F}$. Certainly J is an FZS-algebra. Hence, by assumption, $J \in \mathcal{F}$ and, as a consequence, $N \in \mathcal{F}$. □

3. Final remarks

In [5] (see also [6]) one can find some further information on FZS-algebras and the ideal Δ. In particular we have the following result:

Theorem 3.1. *Let A be a semiprime FZS-algebra, B be the ideal of A generated by all nilpotent elements and C be the annihilator of B in A. Then $A = B \oplus C$, $B \in \mathcal{F}$ and C is a reduced algebra.*

If A is an arbitrary FZS-algebra, then from Lemma 2.7 we know, that $A/\Delta(A)$ is also an FZS-algebra. However, the class of FZS-algebras is not homomorphically closed. Indeed, if F is a square independent field and A is the algebra from Example 1.2, then $\Delta(A) = A$, and A is an FZS-algebra with $A^3 = 0$, but A/A^2 is not an FZS-algebra. On the other hand, as a consequence of Theorems 2.9 and 3.1 we have:

Theorem 3.2. *Let F be square dependent ring and A be an FZS-algebra. Then $N(A) \in \mathcal{F}$ and A/I is an FZS-algebra for every ideal $I \subseteq N(A)$. Hence, there exists $n \geqslant 1$ such that $B^n = 0$ for every nil-subalgebra $B \subseteq A$.*

In connection with Lemma 2.7 and the above theorem one can ask:

QUESTION 1. *Let A be an FZS-algebra. Is $A/\big(N(A)\big)$ an FZS-algebra too? Does there exist $n \in \mathbb{N}$ such that $B^n = 0$ for any nil-subalgebra B of A?*

The answer to both questions is affirmative if either A is semiprime, or F is square dependent.

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