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GENERALIZED BOOLEAN ALGEBRA EXTENSIONS OF LATTICE ORDERED GROUPS

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ABSTRACT. To each pair $(A,\ B)$, where A is a lattice ordered group and B is a generalized Boolean algebra, there corresponds a lattice ordered group G; the construction of G is due to Conrad and Darnel. In this paper we deal with the relations between higher degrees of distributivity of the partially ordered structures G and B. Further, we investigate direct product decomposition of G in the case when A is a linearly ordered group.

1. Introduction

According to the construction described by Conrad and Darnel [3], to each pair (A, B), where A is a lattice ordered group and B is a generalized Boolean algebra, there corresponds a lattice ordered group which is denoted by A_B; it is called the generalized Boolean algebra extension of A.

We remark that if A is equal to $\mathbb{Z}$ (the additive group of integers with the natural linear order) then A_B is a Specker lattice ordered group (cf. Conrad and Darnel [4] and the author [7]).

Further, if A is the additive group of all reals with the natural linear order, then A_B is a Carathéodory vector lattice (cf. Gofman [5] and the author [6], [8], [9]).

In the present paper we deal with the relations between higher degrees of distributivity of B and of A_B. In this direction, the investigation of Specker lattice ordered groups and of Carathéodory vector lattices was performed in [4] and [8].

We prove that if A is a linearly ordered group, then there is a one-to-one correspondence between internal direct factors of A_B and those of B. Further, in this case, each direct product decomposition of A_B has only a finite number

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of nonzero direct factors. For related results concerning Carathéodory vector lattice cf. [9].

2. Preliminaries and auxiliary results

For lattices and lattice ordered groups we apply the notation and the terminology as in Birkhoff [1] and Conrad [2].

A generalized Boolean algebra is defined to be a distributive lattice B with the least element 0 such that, for each element $x \in B$, the interval $[0,x]$ is a Boolean algebra.

We remark that we apply the symbol 0 for denoting the zero real number, the neutral element of a lattice ordered group and the least element of a generalized Boolean algebra; the meaning of this symbol will be always clear from the context.

Assume that B is a generalized Boolean algebra and that $A \neq \{0\}$ is a lattice ordered group. We recall the definition of the lattice ordered group A_B as given in [3].

Let Λ be the set of all proper ultrafilters on B. Each element $b \in B$ will be identified with the set $\{\lambda \in \Lambda \colon b \in \lambda\}$. For $a \in A$ and $b \in B$ let $a[b]$ be the element of $\prod_\Lambda A$ defined by

$$a[b](\lambda) = \begin{cases} a & \text{if } \lambda \in b, \\ 0 & \text{if } \lambda \notin b. \end{cases}$$

Further, let A_B be the subgroup of $\prod_\Lambda A$ generated by the set $\{a[b] \colon a \in A$ and $b \in B\}$. The set A_B is partially ordered by the relation of partial order induced from $\prod_\Lambda A$.

We mention the following results on A_B:

(α_1) A_B is an ℓ-subgroup of $\prod_\Lambda A$. ([3], Propos. 3.2.)

(α_2) Let b_1 and b_2 be nonzero elements of B. Then we have $0 \leqq a_1[b_1] \leqq a_2[b_2]$ if and only if $0 \leqq a_1 \leqq a_2$ and $b_1 \leqq b_2$. ([3], Propos. 3.3.)

(α_3) For any $0 \neq g \in A_B$ there are finite subsets $\{c_1,\dots,c_n\} \subseteq B$ and $a_1,\dots,a_n\} \subseteq A$ such that $c_i \neq 0$, $a_i \neq 0$ for $i = 1,2,\dots,n$, $g = a_1[c_1] + \dots + a_n[c_n]$ and $c_i \wedge c_j = 0$ whenever $i \neq j$. Moreover, there exists a finite subset $\{d_1,\dots,d_n\}$ of B and a finite subset $\{a'_1,\dots,a'_n\}$ of A such that $d_i \neq 0$, $a'_i \neq 0$ for $i = 1,2,\dots,n$, $g = a'_1[d_1] + \dots + a_n[d_n]$ and if $i \neq j$, then $d_i \wedge d_j = 0$, $a'_i \neq a'_j$. ([3], Propos. 3.1.)

Under the notation as above, the representation of g using a_i, b_i (or a'_i, d_i, respectively) is called the *Specker representation* (or the *standard Specker representation*, respectively). The standard Specker representation of g is unique.

From (α_3) and from the basic properties of Specker lattice ordered groups (cf. [4]) we immediately obtain that for each generalized Boolean algebra B, the lattice ordered group $\mathbb{Z}_B$ is the Specker lattice ordered group corresponding to B.

Similarly, (α_3) yields that R_B is the Carathéodory vector lattice corresponding to B.

Let $0 < u \in A$. We put $B_u = \{u[b] : b \in B\}$.

Lemma 2.1. *The mapping $b \to u[b]$ is an isomorphism of B onto B_u (where the partial order on B_u is induced by that of A_B).*

Proof. This is a consequence of (α_2). □

Proposition 2.2. *B_u is a closed sublattice of A_B.*

Proof. Let $X = \{u[b_i]\}_{i\in I}$ be a nonempty subset of B_u. Put $A_B = G$. Assume that an element $g \in G$ is the supremum of the set X in G. We have to verify that g belongs to B_u. The case $g = 0$ being obvious we can suppose that $g > 0$. In view of (α_3) there exists a Specker representation

$$g = a_1[c_1] + \dots + a_n[c_n]$$

of g. Thus $c_i > 0$ and $a_i \neq 0$ for $i \in 1, 2, \dots, n$. From the fact that the system $\{c_1, \dots, c_n\}$ is orthogonal and from the relation $g > 0$ we infer that $a_i > 0$ for each $i \in \{1, 2, \dots, n\}$.

If for each $i \in I$ and for each $\lambda \in b_i$ we have $\lambda \notin c_1$, then for $g' = a_2[c_2] + \dots + a_n[c_n]$ the relations $g' < g$ and

$$u[b_i] \leqq g' \qquad \text{for each } i \in I$$

would be valid, which is a contradiction. Thus there exist $i_1 \in I$ and $\lambda_1 \in b_{i_1}$ with $\lambda_1 \in c_1$. Then

$$a_1[c_1](\lambda_1) = a_1 > 0 .$$

Let $j \in \{2, \dots, n\}$. According to the orthogonality of $\{c_1, \dots, c_n\}$ we obtain $a_1[c_1] \wedge a_j[c_j] = 0$, whence $\big(a_j[c_j]\big)(\lambda_1) = 0$. This yields $g(\lambda_1) = \big(a_1[c_1]\big)(\lambda_1) = a_1$. Since $u[c_{i_1}] \leqq g$ we get

$$u = \big(u[c_1]\big)(\lambda_1) \leqq g(\lambda_1) = a_1 .$$

Analogously we have $u \leqq a_2, \dots, u \leqq a_n$.

Denote $g_1 = u[c_1] + \dots + u[c_n]$. Let $i \in I$ and $\lambda \in b_i$; thus $\big(u[b_i]\big)(\lambda) = u$. In view of the relation $u[b_i] \leqq g$ we infer that there exists $k \in \{1, 2, \dots, n\}$ with $\lambda \in c_k$. Then $u \leqq a_k$ and, moreover, $\lambda \notin c_{k_1}$ for $k_1 \in \{1, 2, \dots, n\}$, $k_1 \neq k$. Thus

$$\big(u[b_i]\big)(\lambda) = u = \big(u[c_k]\big)(\lambda) \leqq g_1(\lambda) .$$

Hence g_1 is an upper bound of the set $\{u[b_i]\}_{i\in I}$. Clearly $g_1 \leqq g$. Therefore $g_1 = g$. In view of the definition of g_1 we get $g_1 \in B_u$.

Similarly we verify the corresponding condition for infima of elements of B_u. □

A sublattice L_1 of a lattice L is called regular if, whenever $x \in L_1$, $X \subseteq L_1$ and x is the supremum (or infimum) of X in L_1, then x is also the supremum (or infimum, respectively) of X in L.

PROPOSITION 2.3. *B_u is a regular sublattice of A_B.*

Proof. Let X be as in the Proof of 2.2 and suppose that x is the supremum of the set X with respect to the lattice B_u. By way of contradiction, suppose that x fails to be the supremum of the set X with respect to the lattice A_B. Then there exists $g \in A_B$ such that g is an upper bound of the set X and $g < x$.

We apply the notation as in the Proof of 2.2. By using the same steps as in the mentioned proof we obtain that there exists $g_1 \in B_u$ such that g_1 is an upper bound of X and $g_1 \leqq g$. Then we would have $g_1 < x$, which is a contradiction. Similarly we prove the assertion dealing with infima of elements of B_u. □

For the related results concerning Carathéodory vector lattices cf. 3.9 and 4.4 in [8].

LEMMA 2.4.1. *Let $0 < b \in B$, $x \in A$, $y \in A$. Then $x[b] \vee y[b] = (x \vee y)[b]$, $x[b] \wedge y[b] = (x \wedge y)[b]$.*

Proof. We have clearly $x[b] \leqq (x \vee y)[b]$ and $y[b] \leqq (x \vee y)[b]$, hence $x[b] \vee y[b] \leqq (x \vee y)[b]$.

Let $g \in A_B$. Assume that $x[b] \leqq g$, $y[b] \leqq g$. Hence for each $\lambda \in \Lambda$, $(x[b])(\lambda) \leqq g(\lambda)$ and $y[b](\lambda) \leqq g(\lambda)$. Let $\lambda \notin b$. Then $(x[b])(\lambda) = 0 = (y[b])(\lambda)$, whence $g(\lambda) \geqq 0$. Further, we have $((x \vee y)[b])(\lambda) = 0$.

Next, let $\lambda \in b$. Thus $(x[b])(\lambda) = x$, $(y[b])(\lambda) = y$, $g(\lambda) \geqq x$, $g(\lambda) \geqq y$. Then $g(\lambda) \geqq x \vee y = ((x \vee y)[b])(\lambda)$.

Therefore $g(\lambda) \geqq ((x \vee y)[b])(\lambda)$ for each $\lambda \in \Lambda$. This yields that $g \geqq (x \vee y)[b]$, which completes the proof for the operation $\vee$. The proof for the operation $\wedge$ is analogous. □

Let $0 < b \in B$. We put $A_b = \{x[b] : x \in A\}$.

LEMMA 2.4.2. *Let $\{x_i[b]\}_{i\in I}$ be a nonempty subset of A_b. Let $g \in G$ and suppose that the relation $g = \bigvee_{i\in I} (x_i[b])$ is valid in G and that $g = a_1[c_1]$ for some $a_1 \in A$ $c_1 \in C$. Then $g \in A_b$.*

P r o o f . If $g = 0$, then $g \in A_b$. Suppose that $g \neq 0$. Then $a_1 \neq 0$. By way of contradiction, assume that g does not belong to A_b.

a) At first suppose that $x_i \leqq 0$ for each $i \in I$. Then $x_i[b] \leqq 0$ for each $i \in I$, whence $g \leqq 0$. In view of the assumption we have $g \neq 0$, thus $g < 0$. This yields that $x_i[b] < 0$ and so $x_i < 0$ for each $i \in I$.

Assume that $\lambda \in c_1$, $\lambda \notin b$. Then

$$0 = \big(x_i[b]\big)(\lambda) \leqq \big(a_1[c_1]\big)(\lambda) = a_1 .$$

Since $a_1 \neq 0$ we get $0 < a_1$. Then $g > 0$; we arrived at a contradiction. Therefore $c_1 \leqq b$. In view of $g \notin A_b$ we get $c_1 < b$.

Choose $\lambda \in c_1$ and let $i \in I$. Then

$$x_i = \big(x_i[b]\big)(\lambda) \leqq \big(a_1[c_1]\big)(\lambda) = a_1 .$$

Therefore we have

$$x_i[b] \leqq a_1[b] \qquad \text{for each } i \in I . \tag{1}$$

In view of $g \leqq 0$, $a_1 = g(\lambda) \leqq 0$. Since $a_1 \neq 0$, we obtain $a_1 < 0$. From this and from $c_1 < b$ we conclude that $a_1[b] \leqq a_1[c_1]$. There exists $\lambda_1 \in b$ with $\lambda_1 \notin c_1$. Then

$$\big(a_1[b]\big)(\lambda_1) = a_1 < 0 , \qquad \big(a_1[c_1]\big)(\lambda_1) = 0 ,$$

hence $a_1[b] < a_1[c_1]$. Since $a_1[c_1] = \bigvee\limits_{i \in I} \big(x_i[b]\big)$, in view of (1) we arrived at a contradiction. Thus $g \in A_b$.

b) Now suppose that there exists $i_0 \in I$ with $x_{i_0} \nleqq 0$. Hence $0 \vee x_{i_0} > 0$. For each $i \in I$ we denote $x_i' = 0 \vee x_i$. Further, we put $g' = 0 \vee g$. In view of 2.4.1, we have

$$x_i'[b] = (x_i \vee 0)[b] = x_i[b] \vee 0[b] ,$$

where $0[b]$ is the neutral element of G; thus $x_i'[b] \geqq 0$ and $x_{i_0}'[b] > 0$. Next,

$$\begin{aligned} g' = 0 \vee g = 0 \vee \Big(\bigvee_{i \in I} x_i[b] \Big) &= 0[b] \vee \Big(\bigvee_{i \in I} x_i[b] \Big) \\ = \bigvee_{i \in I} \big(0[b] \vee x_i[b]\big) = \bigvee_{i \in I} (0 \vee x_i)[b] &= \bigvee_{i \in I} \big(x_i'[b]\big) . \end{aligned} \tag{2}$$

Put $0 \vee a_1 = a_1'$. Then $a_1' \geqq 0$ and

$$g' = 0 \vee g = 0[c_1] \vee a_1[c_1] = (0 \vee a_1)[c_1] = a_1'[c_1] .$$

We have $0 < x_{i_0}'[b] \leqq a_1'[c_1]$, hence $a_1' > 0$. Then (α_2) yields $b \leqq c_1$. Since $g \notin A_b$, we must have $b < c_1$. Thus $a_1'[b] < a_1'[c]$.

For each $i \in I$, $x_i'[b] \leqq a_i'[c_1]$, hence $x_i' \leqq a_i'$ and so $x_i'[b] \leqq a_i'[b]$. In view of (2), we arrived at a contradiction. Therefore $g \in A_b$. $\square$

LEMMA 2.4.3. *Let $X = \{x_i[b]\}_{i\in I}$ be a nonempty subset of A_b. Assume that in the lattice A_B we have $\bigvee_{i\in I}(x_i[b]) = g$. Then $g \in A_b$.*

P r o o f. By way of contradiction, assume that g does not belong to A_b. Hence $g \neq 0$. Then g has a standard Specker representation $g = a_1[c_1] + \cdots + a_n[c_n]$. The case $n = 1$ has been treated in 2.4.2; it remains to deal with the case $n > 1$.

a) First suppose that $x_i[b] \leqq 0$ for each $i \in I$. If $x_i[b] = 0$ for some $i \in I$, then $g = 0$, which is a contradiction. Thus $x_i[b] < 0$ for each $i \in I$. This is equivalent with the condition that $x_i < 0$ for each $i \in I$. Also, $a_1 < 0, \dots, a_n < 0$.

Let $\lambda_1 \in c_1$. Then $\lambda_1 \notin c_k$ for $k = 2, \dots, n$. Hence

$$(a_1[c_1])(\lambda_1) = a_1\,, \quad (a_k[c_k])(\lambda_1) = 0 \qquad \text{for } k = 2, \dots, n\,.$$

Thus $g(\lambda_1) = a_1$. If $\lambda_1 \notin b$, then $(x_i[b])(\lambda_1) = 0$ for each $i \in I$, hence $x_i[b] \nleqq g$, which is impossible. We obtain $\lambda_1 \in b$ for each $\lambda_1 \in c_1$, whence $c_1 \leqq b$. Analogously we have $c_2 \leqq b, \dots, c_n \leqq b$. Hence if $k \in \{1, 2, \dots, n\}$ and $\lambda_k \in c_k$, then for $i \in I$ we get

$$x_i = (x_i[b])(\lambda_k) \leqq g(\lambda_k) = (a_k[c_k])(\lambda_k) = a_k\,.$$

Denote $a_0 = a_1 \wedge \cdots \wedge a_n$ and $g' = a_0[c_1] + \cdots + a_0[c_n]$. Then $g' < g$; moreover, for each $i \in I$ and each $\lambda \in \Lambda$ we have $(x_i[b])(\lambda) \leqq g'(\lambda)$. Therefore $x_i[b] \leqq g'$ for each $i \in I$, which is a contradiction.

b) Next, assume that $x_i[b] \geqq 0$ for each $i \in I$. Then $g > 0$. Denote $I_1 = \{i \in I \colon x_i > 0\}$. Then we have $I_1 \neq \emptyset$. Also, $a_j > 0$ for each $j \in J = \{1, 2, \dots, n\}$. Let $\lambda \in b$ and $i \in I_1$. Thus

$$0 < x_i = (x_i[b])(\lambda) \leqq g(\lambda) = (a_1[c_1])(\lambda) + \cdots + (a_n[c_n])(\lambda)\,.$$

If $(a_j[c_j])(\lambda) = 0$ for each $j \in J$, then $g(\lambda) = 0$, which is impossible. Hence there exists $j(1) \in J$ with $(a_{j(1)}[c_{j(1)}])(\lambda) \neq 0$. In such case we have $(a_{j(1)}[c_{j(1)}])(\lambda) = a_{j(1)}$ and $(a_{j(2)}[c_{j(2)}])(\lambda) = 0$ whenever $j(2) \in J$, $j(2) \neq j(1)$; therefore $x_i \leqq a_{j(1)}$.

We put

$$J_0 = \{j \in J \colon \lambda \in c_j \ \text{ for some } j \in b\}\,,$$
$$g' = \sum_{j\in J_0} a_j[c_j]\,.$$

For each $\lambda \in \Lambda$ and each $i \in I_1$ we have $(a_i[b])(\lambda) \leqq g'(\lambda)$, whence $a_i[b] \leqq g'$. If $J_0 \neq J$, then $g' < g$ and so we arrived at a contradiction. Hence $J_0 = J$.

Therefore for each $i \in I_1$ and each $j \in J$ the relation $x_i \leqq a_j$ is valid. Put $a_0 = \bigwedge_{j \in J} a_j$. Then $x_i \leqq a_0$ for each $i \in I$. Denote

$$g'' = \sum_{j \in J} a_0[c_j] .$$

Since $n > 1$, we get $g'' < g$. Further, for each $\lambda \in \Lambda$ and $i \in I_1$ we have $\bigl(x_i[b]\bigr)(\lambda) \leqq g'(\lambda)$. Again, we arrived at a contradiction.

c) Let X be any nonempty subset of A_b; we apply the notation as above. Denote

$$X_1 = \{x \wedge 0 \colon x \in X\} , \quad X_2 = \{x \vee 0 \colon x \in X\} ,$$
$$g_1 = g \wedge 0 , \quad g_2 = g \vee 0 .$$

By applying the infinite distributivity of G we obtain

$$g_1 = \sup X_1 , \quad g_2 = \sup X_2 .$$

Further, we have $g_2 - g = 0 - g_1$, whence $g = g_1 + g_2$.

Let $x = x_i[b] \in X$. According to 2.4.1,

$$x \vee 0 = x_i[b] \vee 0[b] = (x_i \vee 0)[b] \in A_b , \quad x \wedge 0 = (x_i \wedge 0)[b] \in A_b .$$

Therefore in view of a), b) and according to 2.4.2 we obtain $g_1 \in A_b$, $g_2 \in A_b$. Thus g belongs to A_b. □

Lemma 2.4.4. *Let $X = \bigl\{x_i[b]\bigr\}_{i \in I}$ be a nonempty subset of A_b. Let $g \in A_B$ such that $g = \inf X$. Then $g \in A_b$.*

P r o o f. From $g = \bigwedge_{i \in I} \bigl(x_i[b]\bigr)$ we obtain

$$-g = \bigvee_{i \in I} \bigl((-x_i)[b]\bigr) .$$

Then according to 2.4.3, $-g \in A_b$ yielding that $g \in A_b$. □

Proposition 2.4.5. *Let $A \neq \{0\}$ be a lattice ordered group and B be a generalized Boolean algebra. Let $b \in B$. Then A_b is a closed sublattice of A_B.*

P r o o f. This is a consequence of 2.4.3 and 2.4.4. □

From 2.4.3 we also obtain:

Lemma 2.5. *Let $X = \bigl\{x_i[b]\bigr\}_{i \in I} \subseteq A_b$, $g \in A_B$, $g \notin A_b$, such that g is an upper bound of X. Then there exists $g_1 \in A_b$ such that g_1 is an upper bound of X and $g_1 < g$.*

Similarly, in view of 2.4.4 we have

LEMMA 2.6. *Let $X = \{x_i[b]\}_{i \in I} \subseteq A_b$, $g \in A_B$, $g \notin A_b$ such that g_1 is a lower bound of X. Then there exists $g_1 \in A_b$ such that g_1 is a lower bound of X and $g < g_1$.*

PROPOSITION 2.7. *A_b is a regular sublattice of A_B.*

Proof. This is a consequence of 2.5 and 2.6. □

3. The (α, β)-distributivity

Assume that α and β are nonzero cardinals. We recall the definition of (α, β)-distributivity of a lattice.

Let T and S be nonempty sets of indices with $\operatorname{card} T \leqq \alpha$, $\operatorname{card} S \leqq \beta$. A lattice L is called (α, β)-distributive if the identities

$$\bigwedge_{t \in T} \bigvee_{s \in S} x_{t,s} = \bigvee_{\varphi \in S^T} \bigwedge_{t \in T} x_{t,\varphi(t)}, \tag{1}$$

$$\bigvee_{t \in T} \bigwedge_{s \in S} x_{t,s} = \bigwedge_{\varphi \in S^T} \bigvee_{t \in T} x_{t,\varphi(t)} \tag{2}$$

hold in L under the assumption that all joins and meets standing in (1) or in (2) exist in L. If L is (α, α)-distributive, then it is said to be (α)-distributive; L is completely distributive if it is (α)-distributive for each cardinal α.

If G is a lattice ordered group, then the underlying lattice of G is self-dual; hence for defining the (α, β)-distributivity of G it suffices to consider the relation (1).

Each interval of a generalized Boolean algebra B is self-dual; hence also in this case it suffices to deal with the relation (1).

PROPOSITION 3.1. *Let $A \neq \{0\}$ be a lattice ordered group and B be a generalized Boolean algebra. Assume that $G = A_B$ is (α, β)-distributive. Then both A and B are (α, β)-distributive.*

Proof. The case $B = \{0\}$ being trivial we can suppose that there exists $b \in B$ with $b \neq 0$. Also, there exists $0 < u \in A$.

By way of contradiction, assume that B is not (α, β)-distributive. Let B_u be as in Section 2. Since $B \simeq B_u$, we obtain that B_u is not (α, β)-distributive. Thus there exist $(x_{t,s})_{t \in T, s \in S}$ in B_u such that $\operatorname{card} T \leqq \alpha$, $\operatorname{card} S \leqq \beta$, all joins and meets standing in (1) exist in B_u and (1) is not valid. According to 2.3, B_u is a regular sublattice of G and therefore (1) fails to be valid in G. We arrived at a contradiction. Thus B is (α, β)-distributive.

We apply the same method for A (using 2.4.5 and the fact that the underlying lattice of A is isomorphic to A_b). $\square$

As a corollary we obtain a result from [8] concerning Carathéodory vector lattices.

A lattice L with the least element 0 will be called *atomic* if for each $0 < x \in L$ there exists an atom a of L with $a \leqq x$.

Lemma 3.2. *Let A be a lattice ordered group and let u be an atom of the lattice A^+. Further, let B be a generalized Boolean algebra and $G = A_B$. Then B_u is an ideal of the lattice G^+.*

Proof. We have $B_u = \{u[b] : b \in B\}$. Let $b \in B$, $0 < g \in G$ such that $g \leqq u[b]$. In view of (α_3) in Section 2, g has a Specker representation of the form $g = a_1[c_1] + \cdots + a_n[c_n]$. Since $g > 0$, we get $a_1 > 0, \ldots, a_n > 0$. Hence $0 < a_i[c_i] \leqq u[b]$ for $i = 1, 2, \ldots, n$. Then according to (α_2) we obtain $a_i \leqq u$ and $c_i \leqq b$. Since u is an atom in A^+ we have $a_i = u$, hence $g = u[c_1] + \cdots + u[c_n]$. In view of the orthogonality of the system $\{c_1, \ldots, c_n\}$, the relation

$$g = u[c_1] \vee \cdots \vee u[c_n] = u[c_1 \vee \cdots \vee c_n]$$

is valid. Since $c_1 \vee \cdots \vee c_n \in B$, we get $g \in B_u$. Further, because B_u is a sublattice of G^+, we conclude that it is an ideal of G^+. $\square$

Proposition 3.3. *Assume that $A \neq \{0\}$ is a lattice ordered group such that the lattice A^+ is atomic. Let $B \neq \{0\}$ be a generalized Boolean algebra and $G = A_B$. Then the following conditions are equivalent:*

(i) *G is (α, β)-distributive.*

(ii) *B is (α, β)-distributive.*

Proof. In view of 3.1 we have (i)$\Rightarrow$(ii). Assume that (ii) is invalid. By way of contradiction, suppose that (i) fails to hold. Hence there exists a nonempty indexed set $(x_{t,s})_{t \in T, s \in S}$ of elements of G such that $\operatorname{card} T \leqq \alpha$, $\operatorname{card} S \leqq \beta$, all joins and meets standing in (1) exist in G and

$$u_0 = \bigvee_{\varphi \in S^T} \bigwedge_{t \in T} x_{t,\varphi(t)} < v = \bigwedge_{t \in T} \bigvee_{s \in S} x_{t,s} .$$

For $t \in T$ and $s \in S$ we put

$$x'_{t,s} = \big((x_{t,s} - u_0) \vee u_0\big) \wedge v .$$

Then we have

$$0 = \bigvee_{\varphi \in S^T} \bigwedge_{t \in T} x'_{t,\varphi(t)} , \quad v' = \bigwedge_{t \in T} \bigvee_{s \in S} x'_{t,s} ,$$

where $v' = v - u_0$. Since $v' > 0$, there exists an atom u of G^+ such that $u \leqq v'$. Denote $x''_{t,s} = x'_{t,s} \wedge u$ for $t \in T$, $s \in S$. Hence

$$0 = \bigvee_{\varphi \in S^T} \bigwedge_{t \in T} x''_{t,\varphi(t)}, \quad u = \bigwedge_{t \in T} \bigvee_{s \in S} x''_{t,s}.$$

Thus the interval $[0,u]$ of G fails to be (α,β)-distributive. But in view of 3.2 we have $[0,u] \subseteq B_u \simeq B$, whence according to (ii) we conclude that $[0,u]$ is (α,β)-distributive; we arrived at a contradiction. □

COROLLARY 3.4. *Let A, B and G be as in 3.3. Then the following conditions are equivalent:*

(i) *G is completely distributive.*

(ii) *B is completely distributive.*

We remark that the assumption of atomicity of A cannot be omitted in 3.3. In fact, let α be an infinite cardinal. There exists a lattice ordered group A which is not (α)-distributive. Further, there exists a Boolean algebra $B \neq \{0\}$ which is completely distributive. Put $G = A_B$. Then B is (α)-distributive and in view of 3.1, G is not (α)-distributive.

4. Direct product decompositions

In this section we assume that A is a nonzero linearly ordered group and that B is a generalized Boolean algebra. We investigate some relations between direct product decompositions of $G = A_B$ and those of B.

We start by recalling the notation and some definitions. For the direct product of lattice ordered groups we apply the notation $\prod_{i \in I} G_i$ (or $G_1 \times \dots \times G_n$ if the number of direct factors is finite).

Let φ be an isomorphism of a lattice ordered group G onto $\prod_{i \in I} G_i$; then we say that φ is a direct product decomposition of G. In such situations we will assume that $G_i \neq \{0\}$ for each $i \in I$.

Let φ be as above and let $i \in I$. We denote by G_i^0 the set of all $g \in G$ such that $\bigl(\varphi(g)\bigr)_j = 0$ whenever $j \in I$, $j \neq i$. Then G_i^0 is a convex ℓ-subgroup of G. For $x^i \in G_i$ let x^{i0} be the element of G_i^0 such that $\bigl(\varphi(x^{i0})\bigr)_i = x^i$. Then the mapping $\varphi_i \colon G_i \to G_i^0$ defined by $\varphi_i(x^i) = x^{i0}$ is an isomorphism of G_i onto G_i^0. For each $x \in G$ with $\varphi(x) = (x_i)_{i \in I}$ we put

$$\varphi_0(x) = \bigl(\varphi_i(x_i)\bigr)_{i \in I}.$$

Then the mapping

$$\varphi_0 \colon G \to \prod_{i \in I} G_i^0$$

is a direct product decomposition of G. We say that φ_0 is an *internal direct product decomposition* of G. The lattice ordered groups G_i^0 are called *internal direct factors* of G. The collection of all internal direct product decompositions of G is a set, while the collection of all direct product decompositions of G is a proper class. To each direct product decomposition φ of G there corresponds an internal direct product decomposition φ_0 of G defined as above; up to isomorphism, φ and φ_0 do not differ. The same method can be applied for defining the notions of internal direct product decomposition and of internal direct factor of a generalized Boolean algebra.

We derive some auxiliary results concerning convex ℓ-subgroups of a lattice ordered group G and ideals of a generalized Boolean algebra B; we suppose that $G = A_B$, where A is a linearly ordered group. The notation and the facts given in Section 2 will be applied.

We denote by $c(G)$ the set of all convex ℓ-subgroups of G; further, let $J(B)$ be the set of all ideals of B. Both $c(G)$ and $J(B)$ are partially ordered by the set-theoretical inclusion.

For $X \in c(G)$ let $\varphi(X)$ be the set of all $b \in B$ such that there exists $0 \neq a \in A$ with $a[b] \in X$.

LEMMA 4.1. *Let $X \in c(G)$. Then $\varphi(X) \in J(B)$.*

Proof.

a) Let $b \in \varphi(X)$ and $b_1 \in B$, $b_1 \leqq b$. There is $0 \neq a \in A$ with $a[b] \in X$. If $a < 0$, then $-(a[b]) = (-a)[b]$, hence $(-a)[b] \in X$; thus without loss of generality we can suppose that $a > 0$. Hence $0 \leqq a[b_1] \leqq a[b]$; therefore $a[b_1] \in X$ and so $b_1 \in \varphi(X)$.

b) Let $b_1, b_2 \in \varphi(X)$. Hence there are nonzero elements $a_1, a_2 \in A$ with $a_1[b_1], a_2[b_2] \in X$. Similarly as in a), we can suppose that $a_1 > 0$, $a_2 > 0$. Also, without loss of generality we can assume that $a_1 \leqq a_2$. Then $0 < a_1[b_2] \leqq a_2[b_2]$. Hence, in view of a), $a_1[b_2] \in X$.

Let b_1' be the complement of $b_1 \wedge b_2$ in the interval $[0, b_1]$ and let the meaning of b_2' be analogous. We have

$$\begin{aligned} a_1[b_1 \vee b_2] &= a_1\bigl[b_1' \vee (b_1 \wedge b_2) \vee b_2'\bigr] = a_1[b_1'] + a_1[b_1 \wedge b_2] + a_1[b_2'], \\ a_1[b_1] + a_1[b_2] &= a_1\bigl[b_1' \vee (b_1 \wedge b_2)\bigr] + a_1\bigl[b_2' \vee (b_1 \wedge b_2)\bigr] \\ &= a_1[b_1'] + 2\bigl(a_1[b_1 \wedge b_2]\bigr) + a_1[b_2']. \end{aligned}$$

Thus we obtain

$$0 < a_1[b_1 \vee b_2] \leqq a_1[b_1] + a_1[b_2].$$

Since $a_1[b_1] + a_1[b_2]$ belongs to X, in view of a) we get $a_1[b_1 \vee b_2] \in X$. Thus $b_1 \vee b_2 \in \varphi(X)$. ☐

For $Y \in J(B)$ we denote by $\psi(Y)$ the set of all $g \in G$ such that either $g = 0$ or g has a Specker representation $g = a_1[b_1] + \dots + a_n[b_n]$, where $b_1, \dots, b_n \in Y$.

LEMMA 4.2. *Let $Y \in J(B)$. Then $\psi(Y) \in c(G)$.*

Proof.

a) It is obvious that $\psi(Y)$ is a subgroup of the group G. It remains to verify that $\psi(Y)$ is a convex sublattice of G.

b) Let $0 \neq g \in \psi(Y)$. Hence g has a Specker representation $g = a_1[b_1] + \dots + a_n[b_n]$ with $b_1, \dots, b_n \in Y$. Put

$$N_1 = \{i \in \{1, 2, \dots, n\} \colon a_i > 0\}.$$

If $N_1 = \emptyset$, then $g \vee 0 = 0$; otherwise we have

$$g \vee 0 = \sum_{i \in N_1} a_i[b_i].$$

Thus $g \vee 0 \in \psi(Y)$. From this and from a) we conclude that $\psi(Y)$ is an ℓ-subgroup of G.

c) Let $0 < g \in \psi(Y)$, $0 < g_1 \in G$, $g_1 \leqq g$. Hence g has a Specker representation as in b); moreover, we have $a_1 > 0, \dots, a_n > 0$. Thus

$$g = a_1[b_1] \vee \dots \vee a_n[b_n],$$
$$g_1 = g_1 \wedge g = \bigvee_{i=1,2,\dots,n} (g_1 \wedge a_i[b_i]).$$

Let $g_1 = \sum\limits_{j \in J} a'_j[b'_j]$ $(J = \{1, 2, \dots, m\})$ be a Specker representation for g_1. We get

$$g_1 = \bigvee_{i \in I} \bigvee_{j \in J} (a'_j[b'_j] \wedge a_i[b_i]),$$

where $I = \{1, 2, \dots, .n\}$.

Let $i \in I$ and $j \in J$. If $b'_j \wedge b_i = 0$, then

$$a'_j[b'_j] \wedge a_i[b_i] = 0;$$

otherwise we have

$$a'_j[b'_j] \wedge a_i[b_i] = (a_j \wedge a_i)[b'_j \wedge b_i].$$

Since $b'_j \wedge b_i \in Y$ and $a'_j \wedge a_i > 0$, we get $a'_j[b'_j] \wedge a_i[b_i] \in \psi(Y)$. Therefore in view of b) we obtain $g_1 \in \psi(Y)$, which completes the proof. $\square$

From the definitions of φ and ψ we immediately obtain

LEMMA 4.3. *If $X_1, X_2 \in c(G)$, $X_1 \subseteq X_2$, then $\varphi(X_1) \subseteq \varphi(X_2)$. If $Y_1, Y_2 \in J(B)$, $Y_1 \subseteq Y_2$, then $\psi(Y_1) \subseteq \psi(Y_2)$.*

For each $X \in c(G)$ we can consider the relation between X and $\psi(\varphi(X))$; similarly, for $Y \in J(B)$ we can consider the relation between Y and $\varphi(\psi(Y))$.

Lemma 4.4.1. *Let $X \in c(G)$. Then $X \subseteq \psi(\varphi(X))$.*

Proof. Let $0 \neq x \in X$ and let $x = a_1[b_1] + \dots + a_n[b_n]$ be the Specker representation of x. Then $|x| \in X$ and $|x| = a'_1[b_1] + \dots + a'_n[b_n]$, where $a'_i = |a_i|$ for $i = 1, 2, \dots, n$. Thus $|x| \geqq a'_i[b_i] > 0$, whence $a'_i[b_i] \in X$. Then $b_i \in \varphi(X)$ and therefore $x \in \psi(\varphi(X))$. □

By examples we can show that, in general, X need not be equal to $\psi(\varphi(X))$. In fact, if $0 \neq a \in A$, $0 \neq b \in B$ and $a[b] \in X$, then there can exist $a_1 \in A$ with $a_1[b] \notin X$; in such case we have $a_1[b] \in \psi(\varphi(X))$.

The proofs of the following two assertions are simple; they will be omitted.

Lemma 4.4.2. *Let X be an element of $c(G)$ such that, whenever a, a_1 are nonzero elements of A, b is a nonzero element of B with $a[b] \in X$, then $a_1[b] \in X$. In this case we have $\psi(\varphi(X)) = X$.*

Lemma 4.4.3. *Let $Y \in J(B)$. Then $\varphi(\psi(Y)) = Y$.*

Now let G be any lattice ordered group. It is well-known that the system $c(G)$ is a complete distributive lattice; if $X, Y \in c(G)$, then $X \wedge Y = X \cap Y$ and $X \vee Y$ is the set of all $g \in G$ which can be expressed in the form $g = z_1 + \dots + z_n$, where $z_i \in X \cup Y$ for each $i \in \{1, 2, \dots, n\}$.

Proposition 4.5. *Let G be a lattice ordered group and $X, Y \in c(G)$. Then the following conditions are equivalent:*

(i) *G is an internal direct product of X and Y.*

(ii) *$X \wedge Y = \{0\}$ and $X \vee Y = G$.*

Proof.

The implication (i)$\Rightarrow$(ii) is obvious. Assume that (ii) is valid.

a) Let $x \in X$ and $y \in Y$. From the relation $X \wedge Y = \{0\}$ we obtain $|x| \wedge |y| = 0$. This yields $x + y = y + x$.

b) Let $g \in G$. In view of $X \vee Y = G$ there exist $z_1, \dots, z_n \in X \cup Y$ such that $g = z_1 + \dots + z_n$. From this and from a) we infer that there are $x \in X$ and $y \in Y$ with $g = x + y$.

c) The representation $g = x + y$ from b) is uniquely determined. In fact, if $x_1 \in X$, $y_1 \in Y$ and $g = x_1 + y_1$, then we have $x + y = x_1 + y_1$, hence $x - x_1 = y - y_1$. Since $x - x_1 \in X$ and $y - y_1 \in Y$, we get $x - x_1 = 0 = y - y_1$.

d) Let $g = x + y$ be as above. Further, let $g' \in G$, $x' \in X$, $y' \in Y$, $g' = x' + y'$. Then $x + x' \in X$, $y + y' \in Y$ and $g + g' = (x + x') + (y + y')$. If $x \leqq x'$ and $y \leqq y'$, then $g \leqq g'$. Conversely, assume that $g \leqq g'$. Hence $g'' = g' - g \geqq 0$. We have $g'' = x'' + y''$, where $x'' = x' - x$, $y'' = y' - y$. Thus

$$0 \leqq g'' = (x'')^+ - (x'')^- + (y'')^+ - (y'')^-.$$

By applying a) we obtain

$$(x'')^+ + (y'')^+ \geqq (x'')^- + (y'')^- \geqq 0\,.$$

Also, $(x'')^- \wedge (x'')^+ = 0$, $(x'')^- \wedge (y'')^+ = 0$. Moreover, $(x'')^+ \wedge (y'')^+ = 0$, whence

$$(x'')^+ + (y'')^+ = (x'')^+ \vee (y'')^+.$$

Thus we obtain

$$(x'')^- = (x'')^- \wedge \big((x'')^+ \vee (y'')^+\big) = \big((x'')^- \wedge (x'')^+\big) \vee \big((x'')^- \wedge (y'')^+\big) = 0\,.$$

Analogously we get $(y'')^- = 0$. Hence $x'' \geqq 0$ and $y'' \geqq 0$. Therefore $x \leqq x'$ and $y \leqq y'$.

e) Let $g \in G$. If $x \in X$, $y \in Y$ and $g = x + y$, then we put $\varphi(g) = (x, y)$. In view of a) – d), φ determines an internal direct product decomposition of G with direct factors X and Y. □

Proposition 4.6. *Let $\{0\} \neq X_1, \dots, \{0\} \neq X_n \in c(G)$, $n \geqq 2$. Assume that $X_1 \vee \dots \vee X_n = G$ and $X_{i(1)} \wedge X_{i(2)} = 0$ whenever $i(1), i(2) \in \{1, 2, \dots, n\}$, $i(1) \neq i(2)$. Then G is an internal direct product of $X_1, \dots, X_n$.*

P r o o f. We proceed by induction with respect to n. If $n = 2$, then the assertion is valid in view of 4.5. Let $n > 2$. Put $X_1' = X_2 \vee \dots \vee X_n$. We have $X_1 \vee X_1' = G$ and

$$X_1 \wedge X_1' = X_1 \wedge (X_2 \vee \dots \vee X_n) = (X_1 \wedge X_2) \vee \dots \vee (X_1 \wedge X_n) = \{0\}\,.$$

Now it suffices to apply the induction assumption for $X_2 \vee \dots \vee X_n$. □

Let us remark that if $X \in c(G)$ and Y is as in 4.5, then Y is uniquely determined; this is a consequence of the distributivity of $c(G)$. Thus we have

Proposition 4.7. *An element $X \in c(G)$ is an internal direct factor of G if and only if X has a complement in $c(G)$; in this case, the corresponding complement is uniquely determined.*

Further, let B be a generalized Boolean algebra. It is easy to verify that $J(B)$ is a complete distributive lattice. For $X, Y \in J(B)$ we have $X \wedge Y = X \cap Y$ and

$$X \vee Y = \{x \vee y \colon x \in X \text{ and } y \in Y\}\,.$$

Proposition 4.8. *Let X and Y be elements of $J(B)$. Then the following conditions are equivalent:*

(i) *B is an internal direct product of X and Y.*

(ii) *$X \wedge Y = \{0\}$ and $X \vee Y = B$.*

P r o o f . The validity of the implication (i)$\Rightarrow$(ii) is obvious. Assume that (ii) holds.

a) Let $b \in B$. In view of the relation $B = X \vee Y$ there exist $x \in X$ and $y \in Y$ such that $b = x \vee y$. If $x_1 \in X$, $y_1 \in Y$, $b = x_1 \vee y_1$, then $x = x \wedge (x_1 \vee y_1) = (x \wedge x_1) \vee (x \wedge y_1)$. But since $X \wedge Y = \{0\}$ we get $x \wedge y_1 = 0$, thus $x = x \wedge x_1$, whence $x \leqq x_1$. In the same way we obtain $x_1 \leqq x$, thus $x_1 = x$. Analogously, $y_1 = y$. Therefore the elements x and y are uniquely determined.

b) Let b, x and y be as above. Further, let $b' \in B$, $x' \in X$, $y' \in Y$, $b' = x' \vee y'$. If $x \leqq x'$ and $y \leqq y'$, then $b \leqq b'$. Assume that $b \leqq b'$. Since $x \wedge y' = 0$ we get

$$x = x \wedge b' = (x \wedge x') \vee (x \wedge y') = x \wedge x'.$$

Thus $x \leqq x'$. Similarly, $y \leqq y'$.

c) In view of a) and b), the mapping $b \to (x, y)$ determines an internal direct product decomposition of B with direct factors X and Y. $\square$

COROLLARY 4.9. *An element X of $J(B)$ is an internal direct factor of B if and only if it has a complement in $J(B)$.*

By analogous steps as in the proof of 4.6 we can verify

PROPOSITION 4.10. *Let $X_1, \dots, X_n \in J(B)$, $n \geqq 2$, such that $X_1 \vee \dots \vee X_n = B$ and $X_{i(1)} \wedge X_{i(2)} = \{0\}$ whenever $i(1)$ and $i(2)$ are distinct elements of $\{1, 2, \dots, n\}$. Then B is an internal direct product of $X_1, \dots, X_n$.*

Now let us return to the case $G = A_B$, where $A \neq \{0\}$ is a linearly ordered group and B is a generalized Boolean algebra.

LEMMA 4.11. *Let X be an internal direct factor of G. Assume that a, a_1 are nonzero elements of A and b is a nonzero element of B such that $a[b] \in X$. Then $a_1[b] \in X$.*

P r o o f . By way of contradiction, suppose that $a_1[b]$ does not belong to X. Without loss of generality we can assume that $0 < a$ and $0 < a_1$. There exists an internal direct factor Y of G such that G is an internal direct product of X and Y.

Since $a_1[b]$ does not belong to X there exists $0 < a_2[b_1] \in Y$ such that $a_2[b_1] \leqq a_1[b]$. Then we have $b_1 \leqq b$, whence $0 < a[b_1] \leqq a[b]$ and thus $a[b_1] \in X$. Put $a_0 = \min\{a, a_2\}$. Then $0 < a_0$ and $0 < a_0[b_1] \in X$, $a_0[b_1] \in Y$, which is a contradiction. $\square$

Lemma 4.12. *Assume that X is an internal direct factor of G. Then $\psi(\varphi(X)) = X$.*

P r o o f . This is a consequence of 4.11 and 4.4.2. ☐

Lemma 4.13. *Assume that G is an internal direct product of X and X_1. Then B is an internal direct product of $\varphi(X)$ and $\varphi(X_1)$.*

P r o o f . $\varphi(X)$ and $\varphi(X_1)$ are elements of $J(B)$. From the relation $X \wedge X_1 = \{0\}$ and from 4.12 we obtain

$$\varphi(X) \wedge \varphi(X_1) = \{0\}. \tag{1}$$

Let b be any nonzero element of B. In view of (1) and 4.8 it suffices to verify that b is a join of some element of $\varphi(X)$ and of some element of $\varphi(X_1)$.

There exists $0 < a \in A$. The element $a[b]$ can be expressed in the form $a[b] = x + x_1 = x \vee x_1$ with $x \in X$, $x_1 \in X_1$, $x \wedge x_1 = 0$. Consider the standard Specker representations of x and x_1, respectively,

$$x = a_1[b_1] + \dots + a_n[b_n], \quad x_1 = a'_1[b'_1] + \dots + a'_m[b'_m].$$

Then the relation

$$a[b] = a_1[b_1] + \dots + a_n[b_n] + a'_1[b'_1] + \dots + a'_m[b'_m]$$

is a Specker representation of $a[b]$. Therefore we have

$$a = a_1 = \dots = a_n = a'_1 = \dots = a'_m.$$

Thus we obtain $x = a[b_1]$, $x_1 = a[b'_1]$, whence

$$a[b] = a[b_1] \vee a[b'_1] = a[b_1 \vee b'_1].$$

Therefore $b = b_1 \vee b'_1$. Since $b_1 \in \varphi(X)$ and $b'_1 \in \varphi(X_1)$, the proof is complete. ☐

Lemma 4.14. *Assume that B is an internal direct product of Y and Y_1. Then G is an internal direct product of $\psi(Y)$ and $\psi(Y_1)$.*

P r o o f . We have $\psi(Y), \psi(Y_1) \in c(G)$. Since $Y \wedge Y_1 = \{0\}$, in view of 4.4.3 we obtain

$$\psi(Y) \wedge \psi(Y_1) = \{0\}. \tag{1'}$$

According to (1') and in view of 4.5 it suffices to show that any element g of G is a sum of elements belonging to $\psi(Y) \cup \psi(Y_1)$. Further, we can restrict ourselves to the case $g > 0$.

Consider the standard Specker representation

$$g = a_1[b_1] + \dots + a_n[b_n]$$

of the element g. We have

$$g = a_1[b_1] \vee \cdots \vee a_n[b_n].$$

In view of the assumption, for each $i \in \{1, 2, \ldots, n\}$ we can write $b_i = b_{i0} \vee b_{i1}$ with $b_{i0} \in Y$, $b_{i1} \in Y_1$. Thus

$$a_i[b_i] = a_i[b_{i0} \vee b_{i1}] = a_i[b_{i0}] \vee a_i[b_{i1}],$$

whence

$$g = a_1[b_{10}] \vee \cdots \vee a_n[b_{n0}] \vee a_1[b_{11}] \vee \cdots \vee a_n[b_{n1}].$$

Denote

$$x = a_1[b_{10}] \vee \cdots \vee a_n[b_{n0}], x_1 = a_1[b_{11}] \vee \cdots \vee a_n[b_{n1}].$$

Then we have $g = x \vee y$, $x \wedge y = 0$, hence $g = x + y$. Further, $x \in \psi(Y)$ and $y \in \psi(Y_1)$. This completes the proof. □

THEOREM 4.15. *Assume that $G = A_B$, where $A \neq \{0\}$ is a linearly ordered group and B is a generalized Boolean algebra. Then the mapping φ determines a one-to-one correspondence between internal direct factors of G and internal direct factors of B.*

P r o o f . This is a consequence of 4.13 and 4.1.4. □

Further, in view of 4.15, 4.3, 4.9, 4.10 and 4.11 we obtain

COROLLARY 4.15.1. *Let G, A and B be as in* 4.15. *Then there is a one-to-one correspondence between finite internal direct product decompositions of G and finite internal direct product decompositions of B.*

If A, B and G are as above then, it can occur that B can be represented as a direct product of an infinite system of nonzero direct factors. We will show that for G, the situation is different.

If X is an internal direct factor of G and $g \in G$, then the component of g in X will be denoted by $g(X)$.

THEOREM 4.16. *Assume that G, A and B are as in* 4.15. *Further, suppose that G is isomorphic to a direct product of a system S of nonzero direct factors. Then the system S is finite.*

P r o o f . Without loss of generality we can suppose that S is a system of internal direct factors of G. By way of contradiction, suppose that the system S is infinite. Hence there exist distinct elements X_i $\big(i \in I = \{1, 2, \ldots, \}\big)$ of S. Thus if $i(1), i(2) \in I$, $i(1) \neq i(2)$, then $X_{i(1)} \cap X_{i(2)} = \{0\}$.

For each $i \in I$ there exists $0 < h^i \in X_i$; let h^i have a Specker representation

$$h^i = a_1^i[b_1^i] + \cdots + a_{n(i)}^i[b_{n(i)}^i].$$

Then $0 < a_1^i\bigl[b_1^i\bigr] \in X_i$.

We define by induction positive integers k_i $(i = 1, 2, \dots)$ as follows. We put $k_1 = 1$. Assume that $i > 1$ and that we have defined $k_1, \dots, k_{i-1}$ such that the elements

$$k_1 a_1^1,\, k_2 a_1^2, \dots, k_{i-1} a_1^{i-1}$$

are mutually distinct. We define k_i to be the least positive integer such that

$$k_i a_1^i \neq k_1 a_1^1, \dots, k_i a_1^i \neq k_{i-1} a_1^{i-1};$$

such positive integer k_i exists.

For each $i \in I$ denote

$$h_0^i = k_i\bigl(a_1^i\bigl[b_1^i\bigr]\bigr) = \bigl(k_i a_1^i\bigr)\bigl[b_1^i\bigr].$$

We have $h_0^i \in X_i$ for each $i \in I$. Thus there exists $h \in G$ such that

$$\begin{aligned} h(X_i) &= h_0^i \quad \text{for each } i \in I, \\ h(X) &= 0 \quad \text{whenever } X \in S \text{ and } X \neq X_i \text{ for each } i \in I. \end{aligned}$$

Let $i \in I$ be fixed. Then G can be represented as an internal direct product of X_i and X_i', where X_i' is an internal direct product of lattice ordered groups belonging to the system $S_i = S \setminus \{X_i\}$.

Choose $\lambda_i \in b_1^i$ (cf. the notation in Section 2). Each element $0 < y \in X_i'$ is orthogonal to $a_1^i\bigl[b_1^i\bigr]$, hence $y(\lambda_i) = 0$. Then $\bigl(h(X_i')\bigr)(\lambda_i) = 0$. Since $h = h(X_i) + h(X_i')$, we obtain

$$\begin{aligned} h(\lambda_i) &= \bigl(h(X_i)\bigr)(\lambda_i) + \bigl(h(X_i')\bigr)(\lambda_i) = \bigl(h(X_i)\bigr)(\lambda_i) \\ &= h_0^i(\lambda_i) = \bigl(\bigl(k_i a_1^i\bigr)\bigl[b_1^i\bigr]\bigr)(\lambda_i) = k_1 a_1^i. \end{aligned}$$

Therefore the set $\bigl\{h(\lambda) \colon \lambda \in \Lambda\bigr\}$ is infinite. But by applying the standard Specker representation we conclude that for each $g \in G$, the set $\bigl\{g(\lambda) \colon \lambda \in \Lambda\bigr\}$ is finite. Hence we arrived at a contradiction. $\square$

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