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NOTE ON AN EXISTENCE OF SMALL DEGREE VERTICES WITH AT MOST ONE BIG DEGREE NEIGHBOUR IN PLANAR GRAPHS

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ABSTRACT. Let G be a planar graph of minimum degree at least three that does not contain an edge with degree sum of its endvertices at most 8. We prove that every such graph G contains a vertex of degree d, $d \in \{3,4,5\}$, which has at least $d-1$ neighbours each of which has degree at most 20. The bound 20 is tight.

1. Introduction

In this note we consider planar graphs. Loops or multiple edges are not allowed in graphs but they can appear in multigraphs. We use standard graph theory terminology according to the book by Bondy and Murty [1].

Motivations for the present note come from recent papers. Harant and Jendrol' [2] and independently van den Heuvel and McGuinness [3] proved that every planar graph of minimum degree 3 contains a vertex of degree d (a *d-vertex* in the sequel), $d \in \{3,4,5\}$, which has at least $d-2$ neighbours each of which has degree bounded by a constant which does not depend on the graph. The graph of bipyramid (and infinitely many other graphs of minimum degree 4) has the property that every its 4-vertex has at least two neighbours of "unbounded" degree. On the other side, in [4] we proved that any planar graph of minimum degree 5 contains a 5-vertex having four neighbours each of which has degree at most 10.

In this note we prove the existence of d-vertex, $d \in \{3,4,5\}$, with at least $d-1$ neighbours of bounded degree in any planar graph of minimum degree at least 3 which does not contain an edge $e = uv$ with weight $w(e) = \deg(u) + \deg(v) \le 8$. The graphs $K_{1,r}$ and $K_{2,r}$, $r \ge 3$, that are planar show that the requirement

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on minimum degree at least 3 is necessary in the main result of this note. Our next theorem supplements results from the recent paper [5] by Mohar *et al.*

THEOREM 1. *Each planar graph of minimum degree at least* 3 *and minimum edge weight at least* 9 *contains a vertex of degree* d, *for* $d \in \{3,4,5\}$, *which is adjacent with at least* $d-1$ *vertices of degree at most* 20. *The bound* 20 *is the best possible.*

2. Proof of theorem

We prove the theorem for a larger family $\mathcal{G}$ of multigraphs which satisfy the following restrictions:

1. The minimum vertex degree is at least 3 and the minimum edge weight is at least 9.
2. Each face is of size at least 3.
3. No multiple edge is incident with a vertex of degree 3, 4 or 5.

Suppose that $G \in \mathcal{G}$ is a counterexample with p vertices and maximum number of edges among all counterexamples with p vertices. For the purposes of this proof let us call every vertex of degree at least 21 the *big* vertex. A k-vertex y which is adjacent to a vertex x is called a k-*neighbour* of x.

CLAIM. G *is a triangulation.*

Proof. Suppose that G has a face α of size $k \geq 4$. If α contains a 3-vertex u, then all three its neighbours v, w and z of u (note that these neighbours are distinct) are of degree at least 6 and at least two of them are big. If there is a pair of nonadjacent neighbours of u on α—say v and w— then a diagonal vw can be inserted into α without creating a face of size 2. Suppose now that there exist all three edges vw, wz, zv. Then at least one edge incident with u is not incident with α, say uz. In this case, a diagonal vw can be inserted into α, again without creating a face of size 2. Thus, we obtain a multigraph $G' \in \mathcal{G}$, which is again a counterexample, but it has one edge more than G, a contradiction.

Suppose now that α contains no 3-vertices. If it contains a 4-vertex u, then all its neighbours v, w, z, y are distinct and of degree at least 5. Then we can repeat the considerations on possible adjacency of these four neighbours; we get again that there are two of them on α and a diagonal joining them can be inserted into α providing the counterexample from $\mathcal{G}$ contradicting the maximality of G. So, let all vertices of α be of degree at least 5. Then there exists at least one pair of non-adjacent vertices on α which allows to join them by a diagonal, again yielding a multigraph which contradicts the maximality of G. $\square$

According to the consequence of Euler formula

$$\sum_{v\in V(G)} \big(\deg_G(v) - 6\big) = -12$$

assign to each vertex $v \in V(G)$ the initial charge $g(v) = \deg_G(v) - 6$. Thus $\sum\limits_{v\in V(G)} g(v) = -12$. We will perform the local redistribution of charges such that their sum remains the same. This is performed by the following rule:

Rule. Let $e = xy$ be an edge of G with $\deg_G(x) \le 5$ and $\deg_G(y) \ge 21$. Then the vertex y sends to x the charge

(a) $\frac{3}{2}$ if x is a 3-vertex except of the case when x is adjacent with another two big vertices. In this exceptional case, y sends 1 to x.

(b) 1 if x is a 4-vertex.

(c) $\frac{1}{2}$ if x is a 5-vertex except of the case when this 5-vertex is adjacent to another two big vertices. In this exceptional case y sends $\frac{1}{3}$ to x.

Next we will show that after redistribution the new charge $h(y)$ is non-negative for each vertex y of G. Since every d-vertex x, $d \in \{3, 4, 5\}$, is adjacent to at least two big vertices or, in the exceptional case, to at least three big vertices, the new charge $h(x)$ of x is non-negative. Note that the new charge $h(x)$ is the sum of the original charge $g(x) = d - 6$ of x and charges received from the big neighbours of x according to the Rule.

Since the charge of every vertex z of degree between 6 and 20 is not changed, it is also $h(z) = g(z) = \deg_G(z) - 6 \ge 0$. So, it remains to determine the new charge $h(x)$ for the vertices of degree at least 21.

Let y be a d-vertex, $d \ge 21$, and let $e_1, e_2, \dots, e_d$ be edges incident with y, $e_i = yx_i$; note that it can happen that $x_i = x_j$ for $i \ne j$, and then e_i, e_j are two distinct edges. Let r_3, r_4, and r_5 be the number of transfers of a charge from y to all its 3-, 4-, and 5-neighbours. Since the edge weight of G is at least 9, we obtain

$$2r_3 + r_4 + r_5 \le \begin{cases} d-1 & \text{for } r_3 \ge 1 \text{ and } r_3 \ne \frac{d}{2}, \\ d & \text{otherwise}. \end{cases} \tag{1}$$

Note, however, that

$$r_3 \le \left\lfloor \frac{d}{2} \right\rfloor, \quad r_4 \le \left\lfloor \frac{d}{2} \right\rfloor \qquad \text{and} \qquad r_3 + r_4 \le \left\lfloor \frac{d}{2} \right\rfloor. \tag{2}$$

Next we distinguish several cases.

1. Let $\deg_G(y) = d = 21$. Then $g(y) = 15$.

1.1. Let $r_3 \geq 1$. If $\frac{3}{2}r_3 + r_4 + \frac{r_5}{2} \leq 15$, then clearly $h(y) \geq 0$. So suppose that $\frac{3}{2}r_3+r_4+\frac{r_5}{2} > 15$. This, using (1), provides $15 < 2r_3+r_4+r_5-\frac{r_3+r_5}{2} \leq 20-\frac{r_3+r_5}{2}$ and hence $r_3+r_5 < 10$. On the other side also $15 < r_3+r_4+\frac{r_3+r_5}{2} < \frac{10}{2}+r_3+r_4$ which gives $r_3+r_4 > 10$, a contradiction with (2). Hence in all cases we obtained $h(y) \geq 0$.

1.2. Suppose now that $r_3 = 0$. Then $r_4+r_5 \leq 21$. By (2) we know that $r_4 \leq 10$. If $r_4 + r_5 \leq 20$ or $r_4 \leq 9$ we immediately have $h(y) = 15 - r_4 - \frac{1}{2}r_5 \geq 0$. Let $r_4 = 10$ and $r_5 = 11$. Since each 4-vertex of G is adjacent to at least two big vertices and no two 4-vertices are adjacent, we see that at least nine 5-neighbours of y are adjacent to precisely three big vertices. Thus, the exception in Rule (c) is applied at least 9 times and we actually get $h(x) \geq 15-1\times10-2\times\frac{1}{2}-9\times\frac{1}{3} = 1 > 0$.

2. Let $\deg(y) = d = 22$. Then $g(y) = 16$. Our aim is to show that the new charge $h(y) \geq 0$.

2.1. Consider first the case when $r_3 = 11$. Then every second neighbour of y is a 3-vertex. Since every 3-vertex is adjacent with at least two vertices of degree at least 21, it follows that there is a 3-neighbour x of y having both its remaining neighbours of degree at least 21. Thus, the exception in Rule (a) is applied to the edge xy and we get $h(y) \geq 16 - \frac{3}{2} \times 10 - 1 = 0$.

2.2. Let $1 \leq r_3 \leq 10$. By (1) there is $2r_3 + r_4 + r_5 \leq 21$. If $\frac{3}{2}r_3 + r_4 + \frac{r_5}{2} \leq 16$ then $h(y) \geq 0$. So suppose $\frac{3}{2}r_3+r_4+\frac{r_5}{2} > 16$. Then $16 < 2r_3+r_4+r_5-\frac{r_3+r_5}{2} \leq 21 - \frac{r_3+r_5}{2}$. Hence $r_3 + r_5 < 10$. On the other side also $16 < \frac{3}{2}r_3 + r_4 + \frac{r_5}{2} = r_3 + r_4 + \frac{r_3+r_5}{2} < r_3 + r_4 + \frac{10}{2}$ which gives $r_3 + r_4 > 11$, a contradiction.

2.3. Finally, let $r_3 = 0$; then $r_4 + r_5 \leq 22$. If $r_4 + \frac{r_5}{2} \leq 16$ then $h(y) \geq 0$. Suppose that $r_4 + \frac{r_5}{2} > 16$. Then $16 < \frac{r_4+r_5}{2} + \frac{r_4}{2} \leq \frac{22}{2} + \frac{r_4}{2}$, which yields $r_4 > 10$. Since $r_4 \leq \lfloor\frac{22}{2}\rfloor$, we have $r_4 = 11$ and $r_5 = 11$. As each vertex of degree 4 is adjacent to at least two big vertices and because G is a triangulation we can see that each 5-neighbour of y is adjacent to another two big vertices. Thus, the exception in Rule (c) is applied 11 times which leads actually to $h(y) \geq 16 - 11 \times 1 - 11 \times \frac{1}{3} = \frac{4}{3} > 0$.

3. Let $\deg(y) = d = 23$. Then $g(y) = 17$.

3.1. Let $r_3 \geq 1$; then, by (1), $2r_3 + r_4 + r_5 \leq 22$. If $\frac{3}{2}r_3 + r_4 + \frac{r_5}{2} \leq 17$ then $h(y) \geq 0$; so suppose $\frac{3}{2}r_3 + r_4 + \frac{r_5}{2} > 17$. Analogously as above we get $17 < 2r_3 + r_4 + r_5 - \frac{r_3+r_5}{2} \leq 22 - \frac{r_3+r_5}{2}$ which provides $r_3 + r_5 < 10$. Similarly from $17 < r_3+r_4+\frac{r_3+r_5}{2} < r_3+r_4+\frac{10}{2}$ we obtain $r_3+r_4 > 12$, a contradictions with (2).

3.2. Now suppose $r_3 = 0$. Then $r_4 + r_5 \leq 23$. If $r_4 + \frac{r_5}{2} \leq 17$, we are done, $h(y) \geq 0$. Suppose that $r_4 + \frac{r_5}{2} > 17$; then $17 < \frac{r_4+r_5}{2} + \frac{r_4}{2} \leq \frac{r_4}{2} + \frac{23}{2}$ which gives $r_4 \geq 12$, again a contradiction with (2).

4. Let $\deg(y) = d \geq 24$. Consider the following averaging of charges sent from y to its neighbours of degree 3, 4 and 5:

- if a 3-neighbour x_i of y receives $\frac{3}{2}$ from y then x_i donates $\frac{3}{8}$ to each of e_{i-1}, e_{i+1} and $\frac{3}{4}$ to e_i; if it receives 1 from y then it donates $\frac{1}{8}$ to each of e_{i-1}, e_{i+1} and $\frac{3}{4}$ to e_i
- if a 4-neighbour x_i of y receives 1 from y then it donates $\frac{1}{8}$ to each of e_{i-1}, e_{i+1} and $\frac{3}{4}$ to e_i.

Let e be an edge incident with y. If the second envertex of e is a 3-vertex or 4-vertex then e preserves $\frac{3}{4}$. If the second endvertex of e is a 5-vertex then it preserves at most $\frac{1}{2} + 2 \times \frac{1}{8} = \frac{3}{4}$, and at most $2 \times \frac{3}{8} = \frac{3}{4}$ if it is a vertex of degree at least 6. Thus, in average, y sends at most $\frac{3}{4}$ along every incident edge, which gives $h(y) \geq d - 6 - \frac{3}{4} \times d \geq 0$, since $d \geq 24$.

In the kleetope of the kleetope of an icosahedron, each 3-vertex is adjacent with a 20-vertex. Thus, the bound 20 is the best possible. Recall that the kleetope of a 3-connected plane graph G is a triangulation G' obtained from G in the following way: into each face of G insert a new vertex and join it with the new edges with all vertices on the face boundary.

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