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ON THE CONVERGENCE OF SOME PARTICULAR SERIES

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ABSTRACT. We investigate the interrelationships of eight closely related properties of sequences in $]0,1]$ formulated in terms of the convergence of some particular series.

The obtained results can be used to easily prove some slight extensions of the essential parts of our former convergence theorems on the series $\sum a_n^{1-x_n/n}$ and $\sum a_n^{1-x_n/\log(1+n)}$.

1. Four equivalent statements for sequences

Throughout this paper, we denote by $\mathcal{S}$ the family of all sequences in the unit interval $]0,1]$.

Having in mind the generalized harmonic series $\sum 1/n^\alpha$, we start with the next simple lemma.

Lemma 1.1. *If $(x_n) \in \mathcal{S}$ such that $\sum\limits_{n=1}^{\infty} x_n^\alpha < +\infty$ for some $\alpha > 0$, then there exists $(a_n) \in \mathcal{S}$ such that $\sum\limits_{n=1}^{\infty} a_n < +\infty$ and $0 < \inf\limits_{n\in\mathbb{N}} a_n^{x_n}$.*

Proof. Define $a_n = x_n^\alpha$ for all $n \in \mathbb{N}$. Then, by the assumptions, it is clear that $(a_n) \in \mathcal{S}$ and $\sum\limits_{n=1}^{\infty} a_n < +\infty$. Moreover, since $\lim\limits_{n\to\infty} a_n = 0$, and hence $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} a_n^{1/\alpha} = 0$, it is clear that $\lim\limits_{n\to\infty} a_n^{x_n} = \lim\limits_{n\to\infty} \left(x_n^{x_n}\right)^\alpha = 1$. (Namely, by L'Hospital's rule, we have $\lim\limits_{x\to 0} x^x = 1$.) Therefore, $0 < \inf\limits_{n\in\mathbb{N}} a_n^{x_n}$ is also true. $\square$

The following example shows that the converse of the above lemma need not be true.

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EXAMPLE 1.2. Let $x_1 = x_2 = 1$ and $x_n = 1/\log(n)$ for all $n \geq 3$. Moreover, let $a_n = 1/n^2$ for all $n \in \mathbb{N}$. Then, $(x_n) \in \mathcal{S}$ and $(a_n) \in \mathcal{S}$ such that $\sum_{n=1}^{\infty} a_n < +\infty$ and $0 < \inf_{n\in\mathbb{N}} a_n^{x_n}$, but $\sum_{n=1}^{\infty} x_n^{\alpha} = +\infty$ for all $\alpha > 0$.

Namely, we have $a_n^{x_n} = \left(1/n^2\right)^{1/\log(n)} = \left(e^{-2\log(n)}\right)^{1/\log(n)} = e^{-2}$ for all $n \geq 3$. Moreover, if $\alpha > 0$, then by using L'Hospital's rule, we can see that $\lim_{x\to\infty} x^{\alpha}/e^{x} = 0$. Therefore, there exists $n_0 \in \mathbb{N}$ such that $\log(n)^{\alpha}/n \leq 1$, and hence $1/n \leq x_n^{\alpha}$ for all $n \geq n_0$.

Simple reformulations of the conclusion of Lemma 1.1 give the following

THEOREM 1.3. *If $(x_n) \in \mathcal{S}$, then the following assertions are equivalent:*

(1) *there exists $(a_n) \in \mathcal{S}$ such that $\sum_{n=1}^{\infty} a_n < +\infty$ and $0 < \inf_{n\in\mathbb{N}} a_n^{x_n}$;*

(2) *there exists $a > 0$ such that $\sum_{n=1}^{\infty} a^{1/x_n} < +\infty$;*

(3) *there exist $(a_n) \in \mathcal{S}$ and $a > 0$ such that $\sum_{n=1}^{\infty} a_n < +\infty$ and $ab_n^{1-x_n} \leq \max\{a_n, b_n\}$ for all $(b_n) \in \mathcal{S}$ and $n \in \mathbb{N}$;*

(4) *there exist $(a_n) \in \mathcal{S}$ and $a > 0$ such that $\sum_{n=1}^{\infty} a_n < +\infty$ and $aa_n^{1-x_n} \leq a_n$ for all $n \in \mathbb{N}$.*

Proof. If the assertion (1) holds, then by defining $a = \inf_{n\in\mathbb{N}} a_n^{x_n}$ we can see at once that $0 < a$ and $\sum_{n=1}^{\infty} a^{1/x_n} \leq \sum_{n=1}^{\infty} a_n < +\infty$. Therefore, the assertion (2) also holds.

While if the assertion (2) holds, then we necessarily have $a < 1$. Therefore, by defining $a_n = a^{1/x_n}$ for all $n \in \mathbb{N}$, we evidently have not only $\sum_{n=1}^{\infty} a_n < +\infty$, but also $(a_n) \in \mathcal{S}$. Moreover, if $(b_n) \in \mathcal{S}$, then

$$a_n \leq b_n \quad \text{implies} \quad ab_n^{1-x_n} = ab_n^{-x_n} b_n \leq aa_n^{-x_n} b_n = b_n \leq \max\{a_n, b_n\},$$

and

$$b_n < a_n \quad \text{implies} \quad ab_n^{1-x_n} \leq aa_n^{1-x_n} = aa_n^{-x_n} a_n = a_n \leq \max\{a_n, b_n\}.$$

Therefore, the assertion (3) also holds. Moreover, if the assertion (3) holds, then by writing (a_n) in position of (b_n), we can see at once that the assertion (4) also holds.

Finally, if the assertion (4) holds, then we evidently have $a \leq a_n a_n^{x_n - 1} = a_n^{x_n}$ for all $n \in \mathbb{N}$, and hence $a \leq \inf_{n\in\mathbb{N}} a_n^{x_n}$. Therefore, the assertion (1) also holds. $\square$

2. Consequences of the assertion 1.3 (2)

Lemma 2.1. *If* $(x_n) \in \mathcal{S}$ *such that* $\sum_{n=1}^{\infty} a^{1/x_n} < +\infty$ *for some* $a > 0$, *then* $\sum_{n=1}^{\infty} a_n^{1/x_n} < +\infty$ *for all* $(a_n) \in \mathcal{S}$ *with* $\lim_{n\to\infty} a_n = 0$.

P r o o f. If (a_n) is as above, then there exists $n_0 \in \mathbb{N}$ such that $a_n \leq a$, and hence $a_n^{1/x_n} \leq a^{1/x_n}$ for all $n \geq n_0$. Therefore, $\sum_{n=1}^{\infty} a_n^{1/x_n} < +\infty$ is also true. □

Theorem 2.2. *If* $(x_n) \in \mathcal{S}$ *such that* $\sum_{n=1}^{\infty} a_n^{1/x_n} < +\infty$ *for all* $(a_n) \in \mathcal{S}$ *with* $\lim_{n\to\infty} a_n = 0$, *then* $\lim_{n\to\infty} x_n = 0$.

P r o o f. Assume, on the contrary, that (x_n) is not a null sequence. Then, there exists $\varepsilon > 0$ such that for each $n \in \mathbb{N}$ there exists $k \geq n$ such that $x_k \geq \varepsilon$. Therefore, there exists a strictly increasing function k of $\mathbb{N}$ into itself such that $x_{k(n)} \geq \varepsilon$ for all $n \in \mathbb{N}$. Define

$$a_n = \begin{cases} \dfrac{1}{k^{-1}(n)^{\varepsilon}} & \text{for } n \in k(\mathbb{N}), \\ \dfrac{1}{n^{\varepsilon}} & \text{for } n \in \mathbb{N} \setminus k(\mathbb{N}). \end{cases}$$

Then, it is clear that $0 < a_n \leq 1$ for all $n \in \mathbb{N}$, and thus $(a_n) \in \mathcal{S}$. Moreover, by considering the two cases, it can be easily seen that

$$a_n \leq \frac{1}{n_0^{\varepsilon}} \qquad \text{for all } n_0, n \in \mathbb{N} \quad \text{with } n \geq k(n_0).$$

Therefore, (a_n) is a null sequence, and thus $\sum_{n=1}^{\infty} a_n^{1/x_n} < +\infty$. On the other hand, we can easily see that

$$\frac{1}{n} = a_{k(n)}^{1/\varepsilon} \leq a_{k(n)}^{1/x_{k(n)}}$$

for all $n \in \mathbb{N}$. Hence, it follows that

$$+\infty = \sum_{n=1}^{\infty} \frac{1}{n} \leq \sum_{n=1}^{\infty} a_{k(n)}^{1/x_{k(n)}} \leq \sum_{n=1}^{\infty} a_n^{1/x_n} < +\infty.$$

This contradiction proves the required assertion. □

The following example shows that the converse of the above theorem need not be true.

EXAMPLE 2.3. Let $x_n = a_n = 1$ for all $n \in \{1, 2, \dots, 3^3 - 1\}$ and

$$x_n = \frac{1}{\log(\log(n))} \qquad \text{and} \qquad a_n = \frac{1}{n^{x_n}}$$

for all $n \geq 3^3$. Then, $(x_n) \in \mathcal{S}$ and $(a_n) \in \mathcal{S}$ such that $\lim_{n\to\infty} x_n = 0$ and $\lim_{n\to\infty} a_n = 0$, but $\sum_{n=1}^{\infty} a_n^{1/x_n} = +\infty$.

Namely, we have $a_n = n^{-x_n} = e^{-x_n \log(n)} = e^{-\log(n)/\log(\log(n))}$ for all $n \geq 3^3$, and moreover $\sum_{n=1}^{\infty} a_n^{1/x_n} \geq \sum_{n=1}^{\infty} 1/n = +\infty$.

Finally, as an immediate consequence of Lemma 2.1 and Theorem 2.2, we can also state

COROLLARY 2.4. *If $(x_n) \in \mathcal{S}$ such that $\sum_{n=1}^{\infty} a^{1/x_n} < +\infty$ for some $a > 0$, then $\lim_{n\to\infty} x_n = 0$.*

3. Consequences of Assertion 1.3 (3)

LEMMA 3.1. *If $(x_n) \in \mathcal{S}$ such that there exist $(a_n) \in \mathcal{S}$ and $a > 0$ such that $\sum_{n=1}^{\infty} a_n < +\infty$ and $a b_n^{1-x_n} \leq \max\{a_n, b_n\}$ for all $(b_n) \in \mathcal{S}$ and $n \in \mathbb{N}$, then $\sum_{n=1}^{\infty} b_n^{1-x_n} < +\infty$ for all $(b_n) \in \mathcal{S}$ with $\sum_{n=1}^{\infty} b_n < +\infty$.*

P r o o f. If (b_n) is as above, then $b_n^{1-x_n} \leq a^{-1} \max\{a_n, b_n\} \leq a^{-1} a_n + a^{-1} b_n$ for all $n \in \mathbb{N}$. Therefore, $\sum_{n=1}^{\infty} b_n^{1-x_n} \leq a^{-1} \sum_{n=1}^{\infty} a_n + a^{-1} \sum_{n=1}^{\infty} b_n < +\infty$. $\square$

THEOREM 3.2. *If $(x_n) \in \mathcal{S}$ such that $\sum_{n=1}^{\infty} a_n^{1-x_n} < +\infty$ for all $(a_n) \in \mathcal{S}$ with $\sum_{n=1}^{\infty} a_n < +\infty$, then $\lim_{n\to\infty} x_n = 0$.*

P r o o f. Assume, on the contrary, that (x_n) is not a null sequence. Then, there exist an $\varepsilon \in \,]0, 1[$ and a strictly increasing function k of $\mathbb{N}$ into itself such that $x_{k(n)} \geq \varepsilon$ for all $n \in \mathbb{N}$. Define $\alpha = 1/\sqrt{1-\varepsilon}$, and

$$a_n = \begin{cases} \dfrac{1}{k^{-1}(n)^{\alpha}} & \text{for } n \in k(\mathbb{N}), \\ \dfrac{1}{n^{\alpha}} & \text{for } n \in \mathbb{N} \setminus k(\mathbb{N}). \end{cases}$$

Then, it is clear that $(a_n) \in \mathcal{S}$. Moreover, since $1 < \alpha$, we can easily see that

$$\sum_{n=1}^{\infty} a_n = \sum_{n \in k(\mathbb{N})} a_n + \sum_{n \in \mathbb{N} \setminus k(\mathbb{N})} a_n \leq \sum_{n=1}^{\infty} \frac{1}{n^{\alpha}} + \sum_{n=1}^{\infty} \frac{1}{n^{\alpha}} < +\infty .$$

Therefore, by the assumption of the theorem, we also have $\sum_{n=1}^{\infty} a_n^{1-x_n} < +\infty$. On the other hand, we can also see that

$$1 - x_{k(n)} \leq 1 - \varepsilon = 1/\alpha^2 < 1/\alpha ,$$

and thus

$$1/n = a_{k(n)}^{1/\alpha} \leq a_{k(n)}^{1-x_{k(n)}}$$

for all $n \in \mathbb{N}$. Therefore,

$$+\infty = \sum_{n=1}^{\infty} \frac{1}{n} \leq \sum_{n=1}^{\infty} a_n^{1-x_n} < +\infty .$$

This contradiction proves the required assertion. □

The following example shows that the converse of the above theorem need not be true.

EXAMPLE 3.3. Let $x_n = a_n = 1$ for all $n \in \{1, 2, \ldots, 3^3 - 1\}$ and

$$x_n = \frac{1}{\log(\log(n))} \qquad \text{and} \qquad a_n = \frac{1}{n \log(n)^2}$$

for all $n \geq 3^3$. Then, $\lim_{n \to \infty} x_n = 0$. Moreover, the series $\sum_{n=1}^{\infty} a_n < +\infty$, but $\sum_{n=1}^{\infty} a_n^{1-x_n} = +\infty$.

Namely, by using Cauchy's condensation test, it can be easily seen that $\sum_{n=1}^{\infty} a_n < +\infty$, but $\sum_{n=1}^{\infty} a_n \log(n) = +\infty$ ([3, p. 399]). Therefore, to prove that $\sum_{n=1}^{\infty} a_n^{1-x_n} = +\infty$, it is sufficient only to show that

$$\frac{1}{n \log(n)} \leq a_n^{1-x_n}$$

for all $n \geq 3^3$. For this, assume that $n \geq 3^3$. Then, by using that the function log is increasing and $\log(x)^2 \leq x$ for all $x > 0$, we can easily see that

$$1 \leq \log(n) , \quad n \leq n \log(n) , \quad \log(\log(n)) \leq \log(\log(n \log(n))) ,$$

and hence

$$\log\bigl(\log(n)\bigr)^2 \le \log\bigl(\log(n\log(n))\bigr)^2 \le \log\bigl(n\log(n)\bigr)\,.$$

Therefore,

$$\begin{aligned}\log\bigl(\log(n)^{1/x_n-1}\bigr) &= (1/x_n-1)\log\bigl(\log(n)\bigr)\\ &= \bigl(\log(\log(n))-1\bigr)\log\bigl(\log(n)\bigr) < \log\bigl(\log(n)\bigr)^2 \le \log\bigl(n\log(n)\bigr)\,,\end{aligned}$$

hence

$$\bigl(\log(n)\bigr)^{1/x_n-1} < n\log(n)\,,\quad \bigl(\log(n)\bigr)^{1/x_n} < a_n^{-1},\quad \log(n) < a_n^{-x_n}\,.$$

Therefore,

$$\frac{1}{n\log(n)} = a_n\log(n) < a_n^{1-x_n}\,.$$

Finally, as an immediate consequence of Lemma 3.1 and Theorem 3.2, we can also state

Corollary 3.4. *If $(x_n)\in\mathcal{S}$ such that there exist $(a_n)\in\mathcal{S}$ and $a>0$ such that $\sum\limits_{n=1}^{\infty} a_n<+\infty$ and $ab_n^{1-x_n}\le\max\{a_n,b_n\}$ for all $(b_n)\in\mathcal{S}$ and $n\in\mathbb{N}$, then $\lim\limits_{n\to\infty} x_n=0$.*

4. An application of Theorem 1.3 and Lemma 3.1

Lemma 4.1. *If $(\alpha_n)\in\mathcal{S}$ such that $\sum\limits_{n=1}^{\infty}\alpha_n<+\infty$, then we have $\sum\limits_{n=1}^{\infty}\alpha_n^{1-x_n/n}<+\infty$ for all bounded sequence (x_n) in $\mathbb{R}$ with $(x_n/n)\in\mathcal{S}$.*

P r o o f. Suppose that (x_n) is as above. Then, we necessarily have $x_n>0$ for all $n\in\mathbb{N}$. Moreover, there exists $K>0$ such that $x_n\le K$ for all $n\in\mathbb{N}$. Hence, it is clear that

$$\sum_{n=1}^{\infty}(x_n/n)^2 \le \sum_{n=1}^{\infty} K^2/n^2 < +\infty\,.$$

Therefore, by Lemma 1.1, there exists $(a_n)\in\mathcal{S}$ such that

$$\sum_{n=1}^{\infty} a_n < +\infty \qquad \text{and} \qquad 0<\inf_{n\in\mathbb{N}} a_n^{x_n/n}.$$

Hence, by Theorem 1.3, it follows that there exist $(a_n) \in \mathcal{S}$ and $a > 0$ such that

$$\sum_{n=1}^{\infty} a_n < +\infty \qquad \text{and} \qquad a b_n^{1-x_n/n} \leq \max\{a_n, b_n\}$$

for all $(b_n) \in \mathcal{S}$ and $n \in \mathbb{N}$. Therefore, by Lemma 3.1, we also have

$$\sum_{n=1}^{\infty} b_n^{1-x_n/n} < +\infty$$

for all $(b_n) \in \mathcal{S}$ with $\sum\limits_{n=1}^{\infty} b_n < +\infty$. Thus, in particular, the required inequality $\sum\limits_{n=1}^{\infty} \alpha_n^{1-x_n/n} < +\infty$ is also true. □

Now, by using the above lemma, we can easily prove a slight extension of the (1)⇒(2) part of a convergence theorem of [1].

Theorem 4.2. *If (β_n) is a sequence in $]0, +\infty[$ such that $\sum\limits_{n=1}^{\infty} \beta_n < +\infty$, then $\sum\limits_{n=1}^{\infty} \beta_n^{1-y_n/n} < +\infty$ for any sequence (y_n) in $\mathbb{R}$ which is bounded above.*

Proof. Define

$$\alpha_n = \min\{\beta_n, 1\}$$

for all $n \in \mathbb{N}$. Then, it is clear that (α_n) is as in Lemma 4.1.

Suppose now that (y_n) is as in the theorem. Then, there exists $K > 0$ such that $y_n \leq K$ for all $n \in \mathbb{N}$. Define

$$x_n = \begin{cases} y_n & \text{if } 0 < y_n \leq n\,, \\ \min\{n, K\} & \text{otherwise}\,. \end{cases}$$

Then, it is clear that (x_n) is as in Lemma 4.1. Therefore,

$$\sum_{n=1}^{\infty} \alpha_n^{1-x_n/n} < +\infty\,.$$

On the other hand, since $\sum\limits_{n=1}^{\infty} \beta_n < +\infty$, and hence $\lim\limits_{n\to\infty} \beta_n = 0$, there exists $n_0 \geq K$ such that $\beta_n \leq 1$ for all $n \geq n_0$. Thus, we have

$$\alpha_n = \beta_n \qquad \text{and} \qquad y_n \leq x_n$$

for all $n \geq n_0$. (Namely, if $n \geq n_0$, then $y_n \leq K \leq n_0 \leq n$, and thus $x_n = y_n$ or $y_n \leq 0 < x_n$.) Hence, since $(\alpha_n) \in \mathcal{S}$, it is clear that

$$\beta_n^{1-y_n/n} = \alpha_n^{1-y_n/n} \leq \alpha_n^{1-x_n/n}$$

for all $n \geq n_0$. Therefore, $\sum\limits_{n=1}^{\infty} \beta_n^{1-y_n/n} < +\infty$ is also true. $\square$

Remark 4.3. In [1], we have also proved that if (β_n) is a sequence in $]0,+\infty[$ such that $\sum\limits_{n=1}^{\infty} \beta_n^{1-y_n/n} < +\infty$ for some bounded sequence (y_n) in $\mathbb{R}$, then $\sum\limits_{n=1}^{\infty} \beta_n < +\infty$.

5. A further application of Theorem 1.3 and Lemma 3.1

LEMMA 5.1. *If $(\alpha_n) \in \mathcal{S}$ such that $\sum\limits_{n=1}^{\infty} \alpha_n < +\infty$, then we have $\sum\limits_{n=1}^{\infty} \alpha_n^{1-x_n/\log(1+n)} < +\infty$ for all bounded sequence (x_n) in $\mathbb{R}$ with $\big(x_n/\log(1+n)\big) \in \mathcal{S}$.*

Proof. Suppose that (x_n) is as above. Then, we necessarily have $x_n > 0$ for all $n \in \mathbb{N}$. Moreover, there exists $K > 0$ such that $x_n \leq K$ for all $n \in \mathbb{N}$. Hence, it is clear that

$$\sum_{n=1}^{\infty} \big(e^{-2K}\big)^{\log(1+n)/x_n} \leq \sum_{n=1}^{\infty} \big(e^{-2K}\big)^{\log(1+n)/K} \leq \sum_{n=1}^{\infty} (1+n)^{-2} < +\infty \,.$$

Hence, by Theorem 1.3, it follows that there exist $(a_n) \in \mathcal{S}$ and $a > 0$ such that

$$\sum_{n=1}^{\infty} a_n < +\infty \qquad \text{and} \qquad a b_n^{1-x_n/\log(1+n)} \leq \max\{a_n, b_n\}$$

for all $(b_n) \in \mathcal{S}$ and $n \in \mathbb{N}$. Therefore, by Lemma 3.1, we also have

$$\sum_{n=1}^{\infty} b_n^{1-x_n/\log(1+n)} < +\infty$$

for all $(b_n) \in \mathcal{S}$ with $\sum\limits_{n=1}^{\infty} b_n < +\infty$. Thus, in particular, the required inequality $\sum\limits_{n=1}^{\infty} \alpha_n^{1-x_n/\log(1+n)} < +\infty$ is also true. $\square$

Now, by using the above lemma, we can easily prove a slight extension of the $(1) \Rightarrow (2)$ part of a convergence theorem of [2].

THEOREM 5.2. *If (β_n) is a sequence in $]0,+\infty[$ such that $\sum\limits_{n=1}^{\infty} \beta_n < +\infty$, then $\sum\limits_{n=1}^{\infty} \beta_n^{1-y_n/\log(1+n)} < +\infty$ for any sequence (y_n) in $\mathbb{R}$ which is bounded above.*

Proof. Define

$$\alpha_n = \min\{\beta_n, 1\}$$

for all $n \in \mathbb{N}$. Then, it is clear that (α_n) is as in Lemma 5.1.

Suppose now that (y_n) is as in the theorem. Then, there exists $K > 0$ such that $y_n \leq K$ for all $n \in \mathbb{N}$. Define

$$x_n = \begin{cases} y_n & \text{if } 0 < y_n \leq \log(1+n)\,, \\ \min\{\log(1+n), K\} & \text{otherwise}\,. \end{cases}$$

Then, it is clear that (x_n) is as in Lemma 5.1. Therefore,

$$\sum_{n=1}^{\infty} \alpha_n^{1-x_n/\log(1+n)} < +\infty\,.$$

On the other hand, since $\sum\limits_{n=1}^{\infty} \beta_n < +\infty$, and hence $\lim\limits_{n\to\infty} \beta_n = 0$, there exists $n_o \geq e^K - 1$ such that $\beta_n \leq 1$ for all $n \geq n_0$. Thus, we have

$$\alpha_n = \beta_n \qquad \text{and} \qquad y_n \leq x_n$$

for all $n \geq n_0$. (Namely, if $n \geq n_0$, then $y_n \leq K \leq \log(1+n_0) \leq \log(1+n)$, and thus $x_n = y_n$ or $y_n \leq 0 < x_n$.) Hence, since $(\alpha_n) \in \mathcal{S}$, it is clear that

$$\beta_n^{1-y_n/\log(1+n)} = \alpha_n^{1-y_n/\log(1+n)} \leq \alpha_n^{1-x_n/\log(1+n)}$$

for all $n \geq n_0$. Therefore, $\sum\limits_{n=1}^{\infty} \beta_n^{1-y_n/\log(1+n)} < +\infty$ is also true. $\square$

Remark 5.3. In [2], we have also proved that if (β_n) is a sequence in $]0, +\infty[$ such that $\sum\limits_{n=1}^{\infty} \beta_n^{1-y_n/\log(1+n)} < +\infty$ for some bounded sequence (y_n) in $\mathbb{R}$, then $\sum\limits_{n=1}^{\infty} \beta_n < +\infty$.

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