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CANCELLATION LAW IN CLASS OF MONOUNARY ALGEBRAS

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ABSTRACT. In this paper we deal with the validity of the implications $AB \cong AC \Rightarrow B \cong C$ and $A^k \cong B^k \Rightarrow A \cong B$ $(k \in \mathbb{N})$ for some classes of monounary algebras.

0. Introduction

This paper is a continuation of [2], where the validity of the implication

$$AB \cong AC \Rightarrow B \cong C \tag{1}$$

for some classes of monounary algebras was investigated. We extend the main results of [2] (Thm. 3.8 and Thm. 3.9) for larger classes of monounary algebras.

Further, we consider the validity of the implication

$$A^k \cong B^k \Rightarrow A \cong B \tag{2}$$

for $k \in \mathbb{N}$.

The cancellation law (1) for some classes of algebraic structures has been dealt with in [1], [9], [3]–[7]. In some of these papers, the relation (2) has been considered as well.

1. Auxiliary results

If $\mathcal{C}$ is a class of algebras such that for each $A, B, C \in \mathcal{C}$ the implication (1) holds, then we say that the cancellation law (1) is valid in $\mathcal{C}$. Similarly, we say that (2) holds in a class $\mathcal{C}$, if the implication (2) is valid for each $A, B \in \mathcal{C}$.

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Let D be a given monounary algebra. We denote by $\mathcal{K}(D)$ the class of all monounary algebras A such that

(i) A is a disjoint union of a finite number of algebras $E_1, E_2, \dots, E_k$;

(ii) if $j \in \{1, \dots, k\}$, then E_j can be expressed as a direct product of a finite number of subalgebras $A_{j1}, \dots, A_{jm_j}$ of D such that no A_{ji} ($i = 1, \dots, m_j$) is a cycle.

Let us remark that often we will not distinguish a monounary algebra (A, f) and the corresponding set A.

Let $\mathbb{Z}$ and $\mathbb{N}$ be the set of all integers and of all positive integers, respectively. We denote by $\mathbb{Z} = (\mathbb{Z}, f)$ the monounary algebra such that $f(x) = x + 1$ for each $x \in \mathbb{Z}$.

Let $n \in \mathbb{N}$. Then $D_n = (D_n, f)$ denotes a connected monounary algebra such that $D_n = \mathbb{Z}_n \cup \mathbb{N}$, where $\mathbb{Z}_n = \{0_n, 1_n, \dots, (n-1)_n\}$ is the set of all integers modulo n (here $i_n = \bigl\{j \in \mathbb{Z} \colon j \equiv i \pmod{n}\bigr\}$ for $i \in \mathbb{Z}$) and the operation f on D_n is defined as follows:

$$f(a) = \begin{cases} a + 1_n & \text{if } a \in \mathbb{Z}_n\,, \\ a - 1 & \text{if } a \in \mathbb{N},\ a \neq 1\,, \\ 0_n & \text{if } a = 1\,. \end{cases}$$

The notion of degree $s(x)$ of an element x of a monounary algebra (A, f) was introduced in [8] as follows. Let us denote by $A^{(\infty)}$ the set of all elements $x \in A$ such that there exists a sequence $\{x_n\}_{n \in \mathbb{N} \cup \{0\}}$ of elements belonging to A with the property $x_0 = x$ and $f(x_n) = x_{n-1}$ for each $n \in \mathbb{N}$. Further, we put $A^{(0)} = \{x \in A \colon f^{-1}(x) = \emptyset\}$. Now we define a set $A^{(\lambda)} \subseteq A$ for each ordinal λ by induction. Assume that we have defined $A^{(\alpha)}$ for each ordinal λ. Then we put

$$A^{(\lambda)} = \Bigl\{x \in A - \bigcup_{\alpha < \lambda} A^{(\alpha)} \colon f^{-1}(x) \subseteq \bigcup_{\alpha < \lambda} A^{(\alpha)}\Bigr\}.$$

The sets $A^{(\lambda)}$ are pairwise disjoint. For each $x \in A$, either $x \in A^{(\infty)}$ or there is an ordinal λ with $x \in A^{(\lambda)}$. In the former case we put $s_f(x) = \infty$, in the latter case we set $s_f(x) = \lambda$. We put $\lambda < \infty$ for each ordinal λ.

In the remaining part of this section we assume that $n \in \mathbb{N}$, $n > 1$. Further, in 1.1–1.5 let $A_1, \dots, A_m$ be subalgebras of D_n such that no A_i $(i = 1, \dots, m)$ is a cycle. Put

$$E = A_1 A_2 \cdots A_m\,.$$

Let $\overline{0}_n$ be an element of E,

$$\overline{0}_n = (0_n, 0_n, \dots, 0_n)\,.$$

Let $x = (x_1, \dots, x_m) \in E$. It is obvious that x is cyclic if and only if $x_i \in \mathbb{Z}_n$ for $i = 1, 2, \dots, m$.

If $i \in \{1, \dots, m\}$, then the assumption yields that either $A_i = D_n$ or $A_i = \mathbb{Z}_n \cup \{1, \dots, l\}$ for some $l \in \mathbb{N}$. In the first case we will write $h(A_i) = \infty$ and in the second one, $h(A_i) = l$; $h(A_i)$ will be called the height of A_i. For each $i = \{1, \dots, m\}$ we put

$$\alpha_i = h(A_i).$$

Without loss of generality we can suppose that

$$\alpha_1 \le \alpha_2 \le \dots \le \alpha_m . \tag{$*$}$$

In more detail, we can assume that there are positive integers $j_1, j_2, \dots, j_k$ such that $j_1 + j_2 + \dots + j_k = m$,

$$\begin{gathered} \alpha_1 = \alpha_2 = \dots = \alpha_{j_1} < \alpha_{j_1+1} = \alpha_{j_1+2} = \dots = \alpha_{j_1+j_2} < \dots \\ \dots < \alpha_{j_1+\dots+j_{k-1}+1} = \alpha_{j_1+\dots+j_{k-1}+2} = \dots = \alpha_{j_1+\dots+j_{k-1}+j_k} . \end{gathered} \tag{$**$}$$

Denote $\beta_1 = \alpha_{j_1}$, $\beta_2 = \alpha_{j_1+j_2}, \dots,$ $\beta_k = \alpha_{j_1+j_2+\dots+j_k}$. Therefore $\beta_1 < \beta_2 < \dots < \beta_k$ and the system of the algebras $A_1, \dots, A_m$ consists of a system δ_1 of j_1 algebras of height β_1; δ_2 of j_2 algebras of height $\beta_2, \dots$; system δ_k of j_k algebras of height β_k.

Lemma 1.1. *Let F be a connected component of E such that $\overline{0}_n \in F$. Assume that c is a cyclic element of F. Then $c = \overline{0}_n$ if and only if $|f^{-1}(c)| > 1$.*

P r o o f. Let $a, b \in E$ be such that

$$\begin{aligned} a &= (1, -1_n, -1_n, \dots, -1_n), \\ b &= (-1_n, -1_n, \dots, -1_n). \end{aligned}$$

The element b is cyclic, a is not cyclic, thus $b \ne a$. We have $f(a) = \overline{0}_n = f(b)$, i.e., $|f^{-1}(\overline{0}_n)| > 1$.

Suppose that $\overline{0}_n \ne c = (c(1), \dots, c(m))$. Since $\overline{0}_n \in F$ and c is cyclic, there is $j \in \{0, \dots, n-1\}$ with $c = f^j(\overline{0}_n)$. Then $(c(1), \dots, c(m)) = c = (j_n, j_n, \dots, j_n)$. In each of the algebras $A_1, \dots, A_m$ we have $f^{-1}(j_n) = \{(j-1)_n\}$, thus $f^{-1}(c) = \{((j-1)_n, \dots, (j-1)_n)\}$, $|f^{-1}(c)| = 1$. $\square$

Lemma 1.2. *Let F be a connected component of E. Then $\overline{0}_n \notin F$ if and only if there exist cyclic elements $u, v \in F$ such that $u \ne v$ and $|f^{-1}(u)| > 1$, $|f^{-1}(v)| > 1$.*

P r o o f. If $\overline{0}_n \in F$, then by 1.1 the relation $|f^{-1}(c)| > 1$ for some cyclic element c yields that $c = \overline{0}_n$. Conversely, suppose that $\overline{0}_n \notin F$. There is at least one cyclic element in F, say $a = (a(1), a(2), \dots, a(n))$. There are two distinct coordinates in a (without loss of generality, $a(1) \neq a(2)$), since if $a(1) = a(2) = \dots = a(m)$, then a belongs to the same connected component as $\overline{0}_n$. Put

$$\begin{aligned} u &= \bigl(0_n,\ a(2) - a(1), \dots,\ a(n) - a(1)\bigr)\,, \\ v &= \bigl(a(1) - a(2),\ 0_n,\ a(3) - a(2), \dots,\ a(n) - a(2)\bigr)\,. \end{aligned}$$

If α, β belong to the classes $a(1), a(2)$ modulo n, respectively, we get

$$f^{\alpha}(u) = a = f^{\beta}(v)\,.$$

Hence u, v are cyclic elements belonging to F. Next, the relation $u = v$ implies that the coordinates are equal, thus $a(1) = a(2)$, a contradiction; therefore u, v are distinct.

The elements

$$\begin{aligned} &\bigl(1,\ a(2) - a(1) - 1_n, \dots,\ a(n) - a(1) - 1_n\bigr)\,, \\ &\bigl(-1_n,\ a(2) - a(1) - 1_n, \dots,\ a(n) - a(1) - 1_n\bigr) \end{aligned}$$

belong to $f^{-1}(u)$ and are distinct; similarly,

$$\begin{aligned} &\bigl(a(1) - a(2) - 1_n,\ -1_n,\ a(3) - a(2) - 1_n, \dots,\ a(n) - a(2) - 1_n\bigr)\,, \\ &\bigl(a(1) - a(2) - 1_n,\ 1\,,\ a(3) - a(2) - 1_n, \dots,\ a(n) - a(2) - 1_n\bigr) \end{aligned}$$

are distinct elements of $f^{-1}(v)$. ☐

LEMMA 1.3. *Let F be a connected component of E with $\overline{0}_n \in F$ and let $c = \bigl(c(1), \dots, c(m)\bigr)$ be a non-cyclic element of F such that $f(c)$ is cyclic. Then $\{-1_n\} \neq \bigl\{c(1), \dots, c(m)\bigr\} \subseteq \{-1_n, 1\}$. Further,*

$$s(c) = \min\bigl\{\alpha_i - 1 \colon c(i) = 1\,,\ i \in \{1, \dots, m\}\bigr\}\,.$$

Proof. The first assertion follows immediately from the definition of D_n.

Denote $\gamma = \min\bigl\{\alpha_i - 1 \colon c(i) = 1\,,\ i \in \{1, \dots, m\}\bigr\}$. If $\gamma = \infty$, then $A_i = D_n$ for each $i \in \{1, \dots, m\}$ and $s(1) = \infty$ in each of the algebras A_i. Since $s(-1_n) = \infty$, too, we get $s\bigl(c(i)\bigr) = \infty$ for each $i \in \{1, \dots, m\}$, thus $s(c) = \infty = \gamma$. Now let $\gamma \neq \infty$, $\gamma = \alpha_j - 1 \in \mathbb{N} \cup \{0\}$ for some $j \in \{1, \dots, m\}$. To prove that $s(c) = \gamma$ we need to show the following two relations:

(1) $f^{-\gamma}(c) = \emptyset$.

(2) $f^{-(\gamma+1)}(c) = \emptyset$.

Since $\alpha_1 \leq \dots \leq \alpha_j \leq \dots \leq \alpha_m$, we have $c(l) = -1_n$ for each $l < j$ and for $l \geq j$ the element $\gamma + 1 = \alpha_j < \alpha_l$ belongs to A_l. Hence the element d with

$$d(l) = \begin{cases} (1-\gamma)_n & \text{if } \ l < j\,, \\ \gamma + 1 & \text{if } \ l \geq j\,, \end{cases}$$

belongs to $f^{-\gamma}(c)$ and (1) is valid. Suppose that (2) does not hold and that $\big(e(1),\dots,e(m)\big) = e \in f^{-(\gamma+1)}(c)$. Then the relation

$$1 = c(j) = f^{\gamma+1}\big(e(j)\big) = f^{\alpha_j}\big(e(j)\big)\,, \quad f^{-\alpha_j}(1) \neq \emptyset\,,$$

holds in the algebra A_j, which is a contradiction. □

Lemma 1.3. and $(**)$ yield that the set of all $c \in F$ with $\overline{0}_n \in F$ such that c is non-cyclic and $f(c)$ is cyclic, is a disjoint union of sets $C_1, C_2, \dots, C_k$, where, for $l = 1,\dots,k$,

$$C_l = \big\{c \in F\colon c\text{-non-cyclic}\,,\ f(c)\text{-cyclic},\ s(c) = \beta_l - 1\big\}\,.$$

Lemma 1.4. *Let F be a connected component of E with $\overline{0}_n \in F$ and let C_l for $l = 1,\dots,k$ be as above. Then $|C_l| = j_l \cdot 2^{m-1-(j_1+j_2+\dots+j_{l-1})}$ for each $l \in \{1,\dots,k\}$.*

Proof. Let $l \in \{1,\dots,k\}$. By 1.3, an element $c = \big(c(1),\dots,c(m)\big)$ belongs to C_l if and only if

$$\begin{aligned} -1_n &= c(1) = c(2) = \dots = c(j_1) \\ &= c(j_1+1) = \dots = c(j_1+j_2) = \dots \\ &= c(j_1+j_2+\dots+j_{l-1})\,, \end{aligned} \tag{1}$$

$$\{-1_n\} \neq \big\{c(j_1+\dots+j_{l-1}+1)\,, c(j_1+\dots+j_{l-1}+2), \dots c(j_1+\dots+j_{l-1}+j_l)\big\} \subseteq \{-1_n, 1\} \tag{2}$$

$$c(\iota) \in \{-1_n, 1\} \qquad \text{for each } \iota = j_1+\dots+j_l+1,\dots,m\,. \tag{3}$$

Thus

$$|C_l| = j_l \cdot 2^{j_l-1} \cdot 2^{m-(j_1+\dots+j_l)} = j_l \cdot 2^{m-1-(j_1+j_2+\dots+j_{l-1})}\,.$$

□

Corollary 1.5. *For each $l \in \{1,\dots,k\}$, the number j_l is uniquely determined.*

Proof. We have $|C_1| = j_1 \cdot 2^{m-1}$, thus j_1 is determined. Assume that we know $j_1,\dots,j_{l-1}$. then 1.4 determines the value for j_l uniquely. □

Proposition 1.6. *Let $A_1,\dots,A_m, B_1,\dots,B_m$ be subalgebras of D_n which are not cycles. If $A_1A_2\dots A_m \cong B_1B_2\dots B_m$, then there is a permutation α such that $B_i \cong A_{\alpha(i)}$ for each $i = 1,2,\dots,m$.*

Proof. Without loss of generality, suppose that $A_1,\dots,A_m$ are ordered with respect to their height, i.e.,

$$h(A_1) \le h(A_2) \le \dots \le h(A_m)\,.$$

There exists a permutation α such that $B_i = A'_{\alpha(i)}$ for each $= 1,\dots,m$ and that
$$h(A'_1) \le h(A'_2) \le \dots \le h(A'_m).$$
Denote $E = A_1 A_2 \dots A_m$, $E' = A'_1 A'_2 \dots A'_m$. Then $E \cong E'$. We will use the symbols $\alpha_1,\dots,\alpha_m$, $\beta_1,\dots,\beta_k$ as above for the algebra E; analogous symbols corresponding to E' will be denoted similarly, only adding $'$, e.g., α'_1, j'_1, k' etc.

Denote by F and F' the connected component of E and E', respectively, containing the element $\overline{0}_n$. By 1.2, F is uniquely determined by means of E, thus the sets F and F' coincide. In F, the element $\overline{0}_n$ is determined in view of 1.1, and so is $\overline{0}_n$ in F'. Next, the sets $C_1,\dots,C_k$ having $j_1,\dots,j_k$ elements are determined in F. If we consider $C'_1,\dots,C'_{k'}$ in F', then the relation $F = F'$ and 1.5 imply that $k = k'$ and $j_1 = j'_1,\dots,j_k = j'_k$. Therefore $A_1 = A'_1,\dots,A_m = A'_m$ and the assertion is valid. $\square$

2. Cancellation laws

The aim of this section is to investigate the validity of the cancellation law (1) and of the implication (2) in the classes $\mathcal{K}(\mathbb{Z})$ and $\mathcal{K}(D_n)$ for $n \in \mathbb{N}$.

First let us introduce the following notion (cf. [2]). Let D be a given monounary algebra. Then $\mathcal{U}(D)$ is the class of all monounary algebras A such that

(i) the number of connected components of A is finite,

(ii) if E is a connected component of A, then E can be expressed as a direct product of a finite number of subalgebras $A_1,\dots,A_l$ of D such that no A_i $(i = 1,\dots,l)$ is a cycle.

Fact. $\mathcal{K}(D) \supseteq \mathcal{U}(D)$ for any monounary algebra D.

Proposition 2.1. *The cancellation law* (1) *in* $\mathcal{K}(\mathbb{Z})$ *does not hold in general.*

P r o o f . The assertion is a consequence of the fact that $\mathcal{K}(\mathbb{Z}) \supseteq \mathcal{U}(\mathbb{Z})$ and of 4.5 [2] by which (1) does not hold in general in the class $\mathcal{U}(\mathbb{Z})$. $\square$

Lemma 2.2. $\mathcal{K}(D_1) = \mathcal{U}(D_1)$.

P r o o f . We have $\mathcal{K}(D_1) \subseteq \mathcal{U}(D_1)$ according to the definition. Conversely, let $A \in \mathcal{K}(D_1)$. Then A is a disjoint union of algebras $E_1,\dots,E_k$. Next, each E_j (for $j = 1,\dots,k$) is a direct product of a finite number $A_{j1},\dots,A_{jm_j}$ of subalgebras of D_1 such that no A_{ji} is a cycle. If $i \in \{1,\dots,m_j\}$, then A_{ji} is connected and contains a one-element cycle; thus the product $A_{j1}A_{j2}\dots A_{jm_j}$ is connected (and contains a one-element cycle). Hence each E_j for $j \in \{1,\dots,k\}$

is a connected component of A and we have proved that $A \in \mathcal{U}(D_1)$. Therefore $\mathcal{K}(D_1) \subseteq \mathcal{U}(D_1)$. □

Proposition 2.3. *The cancellation law* (1) *is valid in* $\mathcal{K}(D_1)$.

Proof. This is Thm. 3.8 [2]. □

Now let $n \in \mathbb{N}$, $n > 1$. We are going to show that the cancellation law (1) holds in $\mathcal{K}(D_n)$. Here a method analogous to that used in [2] will be applied, we will deal with a system of polynomials with several indeterminates over the integrity domain $\mathbb{Z}$.

Let A, B, C be monounary algebras from the class $\mathcal{K}(D_n)$. There exists a system $\{Y_1, Y_2, \dots, Y_m\}$ of monounary algebras with the following properties:

(a) if $i, j \in \{1, \dots, m\}$, $i \neq j$, then $Y_i \not\cong Y_j$;

(b) if $i \in \{1, \dots, m\}$, then Y_i is a subalgebra of D_n and it is not a cycle;

(c) for each $A^* \in \{A, B, C\}$, if A^* is a disjoint union of a finite number of algebras $E_1, \dots, E_k$ and E_j ($j \in \{1, \dots, k\}$) is a direct product of subalgebras $X_{j1}, \dots, X_{jl_j}$ of D_n such that no X_{ji} ($i \in \{1, \dots, l_j\}$) is a cycle, then for each $i \in \{1, \dots, l_j\}$ there is $\iota \in \{1, \dots, m\}$ such that $X_{ji} \cong Y_\iota$.

The existence of the system $\{Y_1, \dots, Y_m\}$ follows from the definition of the class $\mathcal{K}(D_n)$.

Notation 2.4. If X is a monounary algebra, $k \in \mathbb{N}$, then we denote by X^0 a fixed one-element monounary algebra and

$$X^k = XX \dots X \quad (k\text{-times}).$$

If $k \in \mathbb{N}$ and $X_1, \dots, X_k$ are mutually disjoint monounary algebras, then $\sum_{i=1}^{k} X_i$ is a disjoint union of the given algebras. Further, if $X_1 \cong X_2 \cong \dots \cong X_k$, then we write also kX_1 instead of $\sum_{i=1}^{k} X_i$; thus kX_1 is an algebra consisting of k copies of X_1.

We denote $0X = \emptyset$ for each monounary algebra X.

Lemma 2.5. *Let* $t_1, \dots, t_m, s_1, \dots, s_m \in \mathbb{N} \cup \{0\}$. *Then* $Y_1^{t_1} Y_2^{t_2} \dots Y_m^{t_m} \cong Y_1^{s_1} Y_2^{s_2} \dots Y_m^{t_m}$ *if and only if* $t_1 = s_1, \dots, t_m = s_m$.

Proof. The assertion is a consequence of 1.6, since for the system $\{Y_1, \dots, Y_m\}$ (a) is valid. □

COROLLARY 2.6. *Let $A^* \in \{A, B, C\}$, A^* be a disjoint union of algebras $E_1, \dots, E_k$ and E_j for $j \in \{1, \dots, k\}$ be a direct product of subalgebras $X_{j1}, \dots, X_{jl_j}$ of D_n, where no X_{ji} $(i = 1, \dots, l_j)$ is a cycle. If $j \in \{1, \dots, k\}$, then $E_j \cong Y_1^{t_1} \dots Y_m^{t_m}$, where $t_1, \dots, t_m \in \mathbb{N} \cup \{0\}$; further, $t_1, \dots, t_m$ are uniquely determined.*

NOTATION 2.7. The system of all polynomials with indeterminates $y_1, \dots, y_m$ over $\mathbb{Z}$ will be denoted $\mathbb{Z}[y_1, \dots, y_m]$. Let $f(y_1, \dots, y_n) \in \mathbb{Z}[y_1, \dots, y_n]$ be a polynomial with non-negative coefficients. Then it can be written in the form

$$f(y_1, \dots, y_n) = \sum_{i=1}^{r} p_i y_1^{t_{i1}} y_2^{t_{i2}} \dots y_m^{t_{im}}$$

such that

(i) $p_i \geq 0$ for each $i \in \{1, \dots, r\}$,

(ii) if j, k are distinct elements of $\{1, \dots, r\}$, then

$$y_1^{t_{j1}} y_2^{t_{j2}} \dots y_m^{t_{jm}} \neq y_1^{t_{k1}} y_2^{t_{k2}} \dots y_m^{t_{km}} .$$

We will say that $f(y_1, \dots, y_m)$ is written in a normal form. By $f(Y_1, \dots, Y_m)$ we denote

$$\sum_{i=1}^{r} p_i Y_1^{t_{i1}} Y_2^{t_{i2}} \dots Y_m^{t_{im}} .$$

Analogously as in [2] we have 2.8 – 2.10:

LEMMA 2.8. *Let $f(y_1, \dots, y_n)$, $g(y_1, \dots, y_m) \in \mathbb{Z}[y_1, \dots, y_m]$ be polynomials with non-negative coefficients which are written in a normal form. Then $f(Y_1, \dots, Y_m) = g(Y_1, \dots, Y_m)$ if and only if $f(y_1, \dots, y_m) = g(y_1, \dots, y_m)$.*

COROLLARY 2.9. *There are uniquely determined polynomials $f_A(y_1, \dots, y_m)$, $f_B(y_2, \dots, y_m)$, $f_C(y_1, \dots, y_m)$ with non-negative coefficients such that $A \cong f_A(Y_1, \dots, Y_m)$, $B \cong f_B(Y_1, \dots, Y_m)$, $C \cong f_C(Y_1, \dots, Y_m)$.*

COROLLARY 2.10. *Let $(f_A \cdot f_B)(y_1, \dots, y_m) = f_A(y_1, \dots, y_m) \cdot f_B(y_1, \dots, y_m)$. Then $AB \cong (f_A \cdot f_B)(Y_1, \dots, Y_m)$.*

THEOREM 2.11. *The cancellation law (1) is valid in $\mathcal{K}(D_n)$.*

P r o o f. Suppose that $AB \cong AC$. By 2.10,

$$AB \cong (f_A \cdot f_B)(Y_1, \dots, Y_m),$$
$$AC \cong (f_A \cdot f_C)(Y_1, \dots, Y_m).$$

Thus 2.8 implies $f_A(y_1,\dots,y_m)\cdot f_B(y_1,\dots,y_m) = f_A(y_1,\dots,f_m)\cdot f_C(y_1,\dots,y_m)$. The polynomial $f_A(y_1,\dots,y_m)$ is non-zero, thus we can use the cancellation law in the integrity domain $\mathbb{Z}[y_1,\dots,y_m]$, which implies $f_B(y_1,\dots,y_m) = f_C(y_1,\dots,y_m)$. According to 2.8, $B \cong f_B(Y_1,\dots,Y_m) = f_C(Y_1,\dots,Y_m) \cong C$. $\square$

Now we will deal with the implication (2).

PROPOSITION 2.12. *The implication* (2) *in* $\mathcal{K}(\mathbb{Z})$ *does not hold in general.*

Proof. Consider the monounary algebras $A = \mathbb{Z}$ and $B = \mathbb{Z}+\mathbb{Z}$. Then A and B belong to $\mathcal{K}(\mathbb{Z})$ and we get

$$\mathbb{Z}^2 \cong \aleph_0\mathbb{Z} \cong (\mathbb{Z}+\mathbb{Z})^2,$$

while $\mathbb{Z}$ and $\mathbb{Z}+\mathbb{Z}$ are not isomorphic. $\square$

THEOREM 2.13. *The implication* (2) *is valid for each* $k \in \mathbb{N}$ *in the class* $\mathcal{K}(D_n)$, $n > 1$.

Proof. Let $A, B \in \mathcal{K}(D_n)$, $n > 1$. Further, let $\{Y_1,\dots,Y_m\}$ be as above with the distinction that instead of the triple A, B, C we consider the pair A, B. Suppose that $k \in \mathbb{N}$ and $A^k \cong B^k$. By induction, 2.10 yields

$$A^k \cong (f_A)^k(Y_1,\dots,Y_m),$$
$$B^k \cong (f_B)^k(Y_1,\dots,Y_m).$$

Then 2.8 implies

$$(f_A)^k(y_1,\dots,y_m) = (f_B)^k(y_1,\dots,y_m),$$

i.e., the polynomial $f_A^k - f_B^k$ is a zero-polynomial. We get

$$0 = f_A^k - f_B^k = (f_A - f_B)\cdot\left(f_A^{k-1} + f_A^{k-2}\cdot f_B + \dots + f_B^{k-1}\right).$$

By the cancellation in the integrity domain $\mathbb{Z}[y_1,\dots,y_m]$, either $f_A - f_B = 0$ or $f_A^{k-1} + f_A^{k-2}\cdot f_B + \dots + f_B^{k-1} = 0$. The second case is impossible, since both f_A and f_B are non-zero polynomials and have non-negative coefficients. Thus $f_A = f_B$ and then $A \cong f_A(Y_1,\dots,Y_m) = f_B(Y_1,\dots,Y_m) \cong B$. $\square$

THEOREM 2.14. *The implication* (2) *is valid for each* $k \in \mathbb{N}$ *in the class* $\mathcal{K}(D_1)$.

Proof. Let us use the fact that $\mathcal{K}(D_1) = \mathcal{U}(D_1)$. Let $k \in \mathbb{N}$, $A, B \in \mathcal{U}(D_1)$. Using the notation from [2], there exists a system $\{Y_1,\dots,Y_m\}$ such that

(a) if $i, j \in \{1,\dots,m\}$, $i \neq j$, then $Y_i \not\cong Y_j$;

(b) if $i \in \{1, \dots, m\}$, then $|Y_i| > 1$;

(c) if $E \in \{A, B\}$, F is a connected component of E and $F = X_1 X_2 \dots X_l$, where $X_1, \dots, X_l$ are subalgebras of D_1 which are not cycles, then for each $i \in \{1, \dots, l\}$ there is $\iota \in \{1, \dots, m\}$ such that $X_i \cong Y_\iota$. It was proved in 3.6 [2] that

$$A \cong f_A(Y_1, \dots, Y_m),$$
$$B \cong f_B(Y_1, \dots, Y_m)$$

for uniquely determined polynomials f_A, f_B of $\mathbb{Z}[y_1, \dots, y_m]$.
Then we can repeat steps of the proof 2.13 to show that if $A^k \cong B^k$, then $A \cong B$. □

3. Concluding remarks

The definition of $\mathcal{K}(D)$ and $\mathcal{U}(D)$ immediately implies

$$\mathcal{K}(D) \supseteq \mathcal{U}(D)$$

for any monounary algebra D. Let us show that, in general, $\mathcal{K}(D)$ does not coincide with $\mathcal{U}(D)$.

Example 1. Let D be a subalgebra of D_2 such that $D = \{0_2, 1_2, 1\}$. Put $A = D \times D$. Then $A \in \mathcal{K}(D)$. Further, A has the following components

$$A_1 = \{(0_2, 0_2), (1_2, 1_2), (1, 1_2), (1_2, 1), (1, 1)\},$$
$$A_2 = \{(0_2, 1_2), (1_2, 0_2), (1, 0_2), (0_2, 1)\}.$$

Neither A_1 nor A_2 is a subalgebra of D, therefore $A \notin \mathcal{U}(D)$.

Similarly as in 2.2 we can use the fact that a product of connected monounary algebras with a one-element cycle is a connected monounary algebra with a one-element cycle as well; this fact yields, for a connected monounary algebra D, that

- if D possesses a one-element cycle, then $\mathcal{K}(D) = \mathcal{U}(D)$.

Further, let D be a connected monounary algebra having no cyclic elements. The algebra $D \times D$ belongs to $\mathcal{K}(D)$ but is does not belong to $\mathcal{U}(D)$, since it consists of infinitely many connected components. Therefore, for a connected monounary algebra D,

•• if D possesses no cyclic elements, then $\mathcal{K}(D) \neq \mathcal{U}(D)$.

Finally, consider a connected monounary algebra having a cycle with more than one element. The example above shows that $\mathcal{K}(D) = \mathcal{U}(D)$ does not necessarily hold. We will give an example of a connected monounary algebra D with a cycle with more than one element and such that $\mathcal{K}(D) = \mathcal{U}(D)$.

EXAMPLE 2. Let $n \in \mathbb{N}$, $n > 1$. Denote by D the set of all sequences $(k, i_1, i_2, \dots) \in \mathbb{Z}_n \times (\mathbb{N} \cup \{0\}) \times (\mathbb{N} \cup \{0\}) \times \dots$ such that $\{i_t \colon t \in \mathbb{N},\ i_t \neq D\}$ is finite. Define a unary operation f on D by putting, for $i_1 \neq 0$,

$$f\big((k, i_1, i_2, \dots, i_t, 0, 0, \dots)\big) = (k, i_1, \dots, i_{t-1}, 0, 0, \dots),$$
$$f\big((k, 0, 0, \dots)\big) = (k + 1_n, 0, 0, \dots).$$

Then $D = (D, f)$ is a connected monounary algebra possessing an n-element cycle, $|f^{-1}(x)| = \aleph_0$ for each $x \in D$ and the degree $s(x) = \infty$ for each $x \in D$. Notice that

(†) if B is any connected monounary algebra with an n-element cycle and such that $|f^{-1}(b)| \leq \aleph_0$ for each $b \in B$, then B is isomorphic to some subalgebra of D.

We will show that $\mathcal{K}(D) = \mathcal{U}(D)$. Assume that $A \in \mathcal{K}(D)$. By the definition, A is a disjoint union of a finite number of algebras $E_1, \dots, E_k$, where E_j ($j = 1, \dots, k$) is a direct product of a finite number of subalgebras $A_1, \dots, A_m$ of D. Since $A_1, \dots, A_m$ contain an n-element cycle, E_j consists of n^m connected components, therefore A has finitely many connected components. Let E be a connected component of A (i.e., of some E_j, $j \in \{1, \dots, k\}$). Then E contains a cycle with n elements and $|f^{-1}(e)| \leq \aleph_0$ for each $e \in E$. According to (†), D is isomorphic to a subalgebra of D. Therefore $A \in \mathcal{U}(D)$ and hence $\mathcal{K}(D) = \mathcal{U}(D)$.

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