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FORMS WITH A UNIQUE REPRESENTATION AS A SUM OF POWERS OF LINEAR FORMS

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ABSTRACT. We consider forms of degree $d \geqslant 3$ over an algebraically closed field of characteristic 0 which have a representation $f = l_1^d + \ldots + l_r^d$, where l_i are non--zero, projectively different linear forms in n variables. Using some properties of d-independence we show that such a representation is unique provided that f is non-degenerate and $d \geqslant 2\lceil r - \frac{n}{2} \rceil + 1$. We also show uniqueness of representation of a non-degenerate, indecomposable form of dimension $n \geqslant 4$ being a sum of $n+1$ powers of linear forms.

1. Introduction and terminology

Let f be a form (homogeneous polynomial) of degree $d \geqslant 3$ in n variables over a field K of characteristic 0. All such forms form a K-vector space which we denote $F_{n,d}(K)$. The basis of this space is given by the monomials $X_1^{i_1} \ldots X_n^{i_n}$, where $i_1 + \cdots + i_n = d$, $i_1, \ldots, i_n \in \mathbb{Z}_+$, thus $\dim\, F_{n,d}(K) = \binom{n+d-1}{d}$. Two forms $f, g \in F_{n,d}(K)$ are equivalent (notation $f \sim g$) if they differ only by a linear change of variables i.e., there exists an invertable $n \times n$ matrix (a_{ij}) with entries in K such that

$$f\left(\sum_{j=1}^{n} a_{1j} X_j, \ldots, \sum_{j=1}^{n} a_{nj} X_j\right) = g(X_1, \ldots, X_n).$$

If $f \in F_{n,d}(K)$ and $g \in F_{m,d}(K)$, then a form

$$(f \oplus g)(X_1, \ldots, X_{n+m}) := f(X_1, \ldots, X_n) + g(X_{n+1}, \ldots, X_{n+m})$$

is called a direct sum of forms f and g. A form $f \in F_{n,d}(K)$ is decomposable if there exist non-zero forms g, h of degree d such that $f \sim (g \oplus h)$. Otherwise f is indecomposable. We say that $f \in F_{n,d}(K)$ is non-degenerate if $0 = (0, \ldots, 0) \in K^n$ is the only common zero of all partial derivatives of f of order $d-1$. It is

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known that the space $F_{n,d}(K)$ is spanned by dth powers of linear forms in n variables (see [Rez 1]). Therefore any form $f \in F_{n,d}(K)$ over an algebraically closed field K can be written in the following way

$$f = l_1^d + \cdots + l_r^d\,, \qquad \text{where } l_j^d = (\alpha_{1j}X_1 + \cdots + \alpha_{nj}X_n)^d,\ \alpha_{ij} \in K\,.$$

For any form $f \in F_{n,d}(K)$ and $r \in \mathbb{Z}_+$ it is natural to ask

(1) Does f have a representation of the length r?

(2) Is this representation unique?

(3) Is this representation minimal?

Many authors tried to answer those questions but most of facts they proved referred to generic forms (see [Cle], [CM], [ER], [Hil], [Rei], [Syl 1], [Syl 2]). Unfortunately, little is known when special forms are considered (see [Kle], [Rez 3], [Sch]). In this paper we deal with special forms (especially non-degenerate) and we give some results connected with the question (2).

2. Some properties of d-independence

In this section we describe a few properties of d-independence we will need in the next part of this paper.

Definition 2.1. Let $d \in \mathbb{N}$ and $\alpha_1, \ldots, \alpha_m \in K^n$ be non-zero and projectively different. We say that the set $A = \{\alpha_1, \ldots, \alpha_m\} \subset K^n$ is *d-independent* if the forms $l_1^d, \ldots, l_m^d$ are linearly independent in $F_{n,d}(K)$, where

$$l_j^d(\boldsymbol{X}) = (\alpha_j \boldsymbol{X})^d = (\alpha_{1j}X_1 + \cdots + \alpha_{nj}X_n)^d \qquad \text{for } j = 1, \ldots, m\,.$$

Of course, it does not cause misunderstanding if we just say that the linear forms $l_1, \ldots, l_m$ are d-independent when the vectors of their coefficients are d-independent.

The following useful criterion for d-independence was proved by P. Serret over 100 years ago (see [Rez 2]). Actually the original Serret Theorem refers to the field $\mathbb{R}$ but one can check that it holds true for any field K.

Theorem 2.2. (Serret Theorem) *A subset $A = \{\alpha_1, \ldots, \alpha_m\}$ of K^n is d-independent if and only if there exist "dual forms" $h_1, \ldots, h_m \in F_{n,d}(K)$ such that $h_i(\alpha_j) = \delta_{ij}$ for $i, j = 1, \ldots, m$ (where δ_{ij} denotes the Kronecker delta).*

Corollary 2.3. *If $A = \{\alpha_1, \dots, \alpha_m\} \subset K^n$ is d-independent, then A is $(d+1)$-independent.*

P r o o f . By Serret's Theorem there exist forms $h_1, \dots, h_m \in F_{n,d}(K)$ such that $h_i(\alpha_j) = \delta_{ij}$. To show $(d+1)$-independence of A it suffices to take "dual forms"

$$g_i = \big(\phi(\alpha_i)\big)^{-1} \phi h_i \,,$$

where ϕ is a linear form satisfying $\phi(\alpha_i) \neq 0$ for $i = 1, \dots, m$. □

Lemma 2.4. *Let $A = \{\alpha_1, \dots, \alpha_m\} \subset K^n$, where α_i are non-zero and projectively different. Suppose there exists a number k and subsets $A_{i1}, \dots, A_{ik}$ of A for $i = 1, \dots, m$ such that*

(1) $A = \bigcup_{j=1}^{k} A_{ij}\,,\ i = 1, \dots, m\,,$

(2) $\alpha_i \notin \operatorname{lin}\big(A_{ij} \setminus \{\alpha_i\}\big)\,,\ i = 1, \dots, m\,,\ j = 1, \dots, k\,.$

Then A is k-independent.

P r o o f . For any $i = 1, \dots, m$ and any $j = 1, \dots, k$ there exists a linear form $f_{ij} \colon K^n \to K$ such that

$$f_{ij}(\alpha_i) \neq 0 \quad \text{and} \quad f_{ij}(\alpha) = 0 \ \text{ for } \ \alpha \in \operatorname{lin}\big(A_{ij} \setminus \{\alpha_i\}\big)\,.$$

Now we can define "dual forms" h_i by formula

$$h_i = a_i \prod_{j=1}^{k} f_{ij}, \qquad \text{where } a_i = \left(\prod_{j=1}^{k} f_{ij}(\alpha_i)\right)^{-1}, \ \ i = 1, \dots, m\,.$$

It follows from Serret's Theorem that A is k-independent. □

Theorem 2.5. *Suppose $A = \{\alpha_1, \dots, \alpha_m\} \subset K^n$, where α_i are non-zero and projectively different. If A can be decomposed into a union of s subsets, which are disjoint and each of them is linearly independent, then A is $(2s-1)$-independent.*

P r o o f . Assume that $B_1 \cup \dots \cup B_s$ is a decomposition of A into a union of subsets, which are disjoint and each of them is linearly independent. Let us choose $\alpha_i \in A$. We construct sets $A_{i1}, \dots, A_{ik}$ with $k \leqslant 2s-1$, satisfying conditions of Lemma 2.4. If B_j is a subset of A such that $\alpha_i \in B_j$, then we put $A_{i1} = B_j$. Taking $B_k \neq B_j$ we have to consider two cases. If $\alpha_i \notin \operatorname{lin}(B_k)$, then we choose A_{i2} to be B_k. If $\alpha_i \in \operatorname{lin}(B_k)$, then by Steinitz Exchange Theorem there exists $\alpha_l \in B_k$ such that $\alpha_i \notin \operatorname{lin}(B_k \setminus \{\alpha_l\})$. In this case we take $A_{i2} = B_k \setminus \{\alpha_l\}$ and $A_{i3} = \{\alpha_l\}$. Continuing in this way, for each i we obtain at most $2s-1$ sets which fulfil assumptions of the previous Lemma. Thus A is $(2s-1)$-independent. □

3. Forms with a unique representation

In this section we observe how the degree of a form, the number of its variables and the number of summands in its representation as a sum of powers of linear forms influence uniqueness of such a representation. From now on we assume K is an algebraically closed field.

For a given $f \in F_{n,d}(K)$ and $r \in \mathbb{N}$ we define $S_r(f)$ as the family of sets $\{l_1^d, \ldots, l_r^d\}$ such that $l_1, \ldots, l_r$ are non-zero, projectively different linear forms in the same n variables as f and $f = \sum\limits_{i=1}^{r} l_i^d$. We refer to the elements of $S_r(f)$ as representations of f of the length r. We denote by $N_r(f)$ the cardinality of $S_r(f)$. We say that the representation of a form f of the length r is unique if $N_r(f) = 1$, that is if the linear forms $l_1, \ldots, l_r$ in $f = \sum\limits_{i=1}^{r} l_i^d$ are unique up to reordering and multiplying by d-th roots of unity.

Obviously $\dim \operatorname{lin}(l_1, \ldots, l_r) \leqslant \min\{n, r\}$ and in particular for a non-degenerate form we get the equality $\dim \operatorname{lin}(l_1, \ldots, l_r) = n$. Hence non-degenerate forms have representations of the length $r \geqslant n$.

Theorem 3.1. *If $f \in F_{n,d}(K)$ is non-degenerate and $S_n(f) \neq \emptyset$, then $N_n(f) = 1$.*

P r o o f. Let $\{l_1^d, \ldots, l_n^d\}$, $\{l_1'^d, \ldots, l_n'^d\} \in S_n(f)$. Suppose these representations differ in k elements. To simplify we can take $l_i^d \neq l_j'^d$ for $i, j = 1, \ldots, k$ and hence we obtain

$$l_1^d + \cdots + l_k^d = l_1'^d + \cdots + l_k'^d .$$

Without loss of generality we can assume that $l_i'^d = a_i l_i^d$ for $i = 1, \ldots, m$, where $a_i \neq 1$, $m \leqslant k$ and $l_1, \ldots, l_k, l'_{m+1}, \ldots, l'_k$ are projectively different. Then we get

$$(1-a_1)l_1^d + \cdots + (1-a_m)l_m^d + l_{m+1}^d + \cdots + l_k^d - l'^d_{m+1} - \cdots - l'^d_k = 0 .$$

Thus the forms $l_1^d, \ldots, l_k^d, {l'_{m+1}}^d, \ldots, l'^d_k$ are linearly dependent.

On the other hand, the set $\{l_1, \ldots, l_k, l'_{m+1}, \ldots, l'_k\}$ can be decomposed into a union of subsets $\{l_1, \ldots, l_k\}$ and $\{l'_{m+1}, \ldots, l'_k\}$. These subsets are disjoint and each of them is linearly independent, so by Theorem 2.5 and Corollary 2.3 the forms $l_1^d, \ldots, l_k^d$, $l'^d_{m+1}, \ldots, l'^d_k$ are linearly independent for $d \geqslant 3$. A contradiction. ☐

Another proof of Theorem 3.1 can be found in [CM].

Lemma 3.2. *If $q = l_1^2 + \cdots + l_r^2$, where $l_1, \ldots, l_r$ are linear forms, then*

$$\operatorname{rank}(q) \geqslant 2 \dim \operatorname{lin}(l_1, \ldots, l_r) - r .$$

P r o o f . Assume $\dim \operatorname{lin}(l_1, \dots, l_r) = k$. Then q is equivalent to the form

$$\bar{q} = X_1^2 + \cdots + X_k^2 + L_{k+1}^2 + \cdots + L_r^2,$$

where L_i are linear forms in variables $X_1, \dots, X_k$. Hence $\operatorname{rank}(q) = \operatorname{rank}(I_k + B)$, where B is the matrix of a form $L_{k+1}^2 + \cdots + L_r^2$. Thus

$$k = \operatorname{rank}(I_k + B - B) \leqslant \operatorname{rank}(I_k + B) + \operatorname{rank}(B) \leqslant \operatorname{rank}(q) + r - k.$$

□

Lemma 3.3. *Suppose $q = a_1 l_1^2 + \cdots + a_{n+1} l_{n+1}^2$, where $a_i \in K$, $l_i \in F_{n,1}(K)$ are linear forms, $n \geqslant 4$ and each n-element subset of $\{l_1, \dots, l_{n+1}\}$ is linearly independent. Then $\operatorname{rank}(q) = 2$ if and only if exactly two of a_i's are non-zero.*

P r o o f . The implication $\Leftarrow$ is obvious so it suffices to show the opposite one. Assume $\operatorname{rank}(q) = 2$ and m denotes the number of non-zero a_i's in the sum $q = a_1 l_1^2 + \cdots + a_{n+1} l_{n+1}^2$. Note that $m \leqslant n$, because if $m = n+1$, then by Lemma 3.2 we get $n \leqslant 3$ and a contradiction. Therefore q is equivalent to the form $\bar{q} = X_1^2 + \cdots + X_m^2$. Hence $m = 2$. □

Theorem 3.4. *Let $f \in F_{n,d}(K)$ be a non-degenerate, indecomposable form and $n \geqslant 4$. If $S_{n+1}(f) \neq \emptyset$, then $N_{n+1}(f) = 1$.*

P r o o f . Since f is equivalent to the form

$$g = X_1^d + \cdots + X_n^d + (a_1 X_1 + \cdots + a_n X_n)^d, \quad a_i \neq 0,\ i = 1, \dots, n, \tag{1}$$

it suffices to prove $N_{n+1}(g) = 1$.

Suppose

$$g = l_1^d + \cdots + l_{n+1}^d, \tag{2}$$

where $l_j = \alpha_{1j} X_1 + \cdots + \alpha_{nj} X_n$ are non-zero and projectively different.

Let (α_{ij}) denote the matrix which columns are the coefficients of forms l_j. Compare the partial derivatives $\frac{\partial^{d-2} g}{\partial X_i^{d-2}}$ of both sides in (2). It follows from Lemma 3.3 that for any $i = 1, \dots, n$ exactly two of $\alpha_{i1}, \dots, \alpha_{i(n+1)}$ are non-zero. So $n(n-1)$ entries in the matrix (α_{ij}) are equal to zero. Hence at least two columns of (α_{ij}) contain exactly one non-zero element and thus at least two forms l_j are of the form $l_j = \alpha_{ij} X_i$. We can assume $l_1 = \alpha_{11} X_1$, $l_2 = \alpha_{22} X_2$. Then from (1) and (2) we have

$$(1-\alpha_{11}^d) X_1^d + (1-\alpha_{22}^d) X_2^d + X_3^d + \cdots + X_n^d + (a_1 X_1 + \cdots + a_n X_n)^d = l_3^d + \cdots + l_{n+1}^d. \tag{3}$$

It is clear that $\alpha_{11}^d = 1$ or $\alpha_{22}^d = 1$. Otherwise the left hand side of (3) would be a non-degenerate form, whereas the right hand side is a sum of $n-2$ powers of linear forms. Putting $\alpha_{11}^d = 1$ we can rewrite (3) in the following way

$$X_2^d + \cdots + X_n^d + (a_1 X_1 + \cdots + a_n X_n)^d = (\alpha_{22} X_2)^d + l_3^d + \cdots + l_{n+1}^d. \tag{4}$$

Let h denote the left hand side of (4). By Theorem 3.1 h has a unique representation, so $g = X_1^d + h$ has also a unique representation. □

Theorem 3.5. *Suppose $f \in F_{n,d}(K)$, $\{l_1^d, \dots, l_r^d\}$, $\{l_1'^d, \dots, l_r'^d\} \in S_r(f)$ and $k := \max\{\dim \operatorname{lin}(l_1, \dots, l_r), \dim \operatorname{lin}(l_1', \dots, l_r')\}$. If $d \geqslant 2\lceil r - \frac{k}{2}\rceil + 1$, then $(l_1^d, \dots, l_r^d)$ is a permutation of $(l_1'^d, \dots, l_r'^d)$.*

Proof. Assume $f = l_1^d + \dots + l_r^d = l_1'^d + \dots + l_r^d$ and

$$k = \max\{\dim \operatorname{lin}(l_1, \dots, l_r), \dim \operatorname{lin}(l_1', \dots, l_r')\}.$$

We proceed by induction on r. For $r = 1$ the statement is obvious. Suppose $r \geqslant 2$ and the statement is valid for forms with a representation of the length less than r. We consider two cases.

I. Assume there exist $i, j = 1, \dots, r$ such that $l_i^d = l_j'^d$, say $l_r^d = l_r'^d$. Since

$$m := \max\{\dim \operatorname{lin}(l_1, \dots, l_{r-1}), \dim \operatorname{lin}(l_1', \dots, l_{r-1}')\} \in \{k-1, k\},$$

then the claim follows from the induction hypothesis, because $d \geqslant 2\lceil r - \frac{k}{2}\rceil + 1 \geqslant 2\lceil r - 1 - \frac{m}{2}\rceil + 1$.

II. Suppose $l_i^d \neq l_j'^d$ for $i, j = 1, \dots, r$ and $k = \dim \operatorname{lin}(l_1, \dots, l_r)$. It is possible that some forms l_i are proportional to some forms l_j'. Without loss of generality we can assume that $l_i'^d = a_i l_i^d$ for $i = 1, \dots, t$, where $a_i \neq 1$ and $t \leqslant r$, and that $l_1, \dots, l_r, l_{t+1}', \dots, l_r'$ are projectively different. Then

$$l_1^d + \dots + l_r^d = a_1 l_1^d + \dots + a_t l_t^d + l_{t+1}'^d + \dots + l_r'^d$$

and thus

$$(1 - a_1) l_1^d + \dots + (1 - a_t) l_t^d + l_{t+1}^d + \dots + l_r^d - l_{t+1}'^d - \dots - l_r'^d = 0.$$

Hence the forms $l_1^d, \dots, l_r^d, l_{t+1}'^d, \dots, l_r'^d$ are linearly dependent.

On the other hand, a set $A = \{l_1, \dots, l_r, l_{t+1}', \dots, l_r'\}$ can be decomposed into the union of $1 + \lceil \frac{2r-k-t}{2} \rceil$ disjoint, linearly independent subsets obtained in the following way. The first subset consists of k forms l_i which are linearly independent. The rest $2r - k - t$ elements of A are not pairwise proportional so they give $\lceil \frac{2r-k-t}{2} \rceil$ linearly independent subsets. Therefore by Theorem 2.5 and Corollary 2.3 the forms $l_1^d, \dots, l_r^d, l_{t+1}'^d, \dots, l_r'^d$ are linearly independent if $d \geqslant 2\lceil \frac{2r-k-t}{2} \rceil + 1$. Since $2\lceil r - \frac{k}{2}\rceil + 1 \geqslant 2\lceil \frac{2r-k-t}{2} \rceil + 1$, we get a contradiction. ☐

Corollary 3.6. *Let $f \in F_{n,d}(K)$ be a non-degenerate form and $S_r(f) \neq \emptyset$. If $d \geqslant 2\lceil r - \frac{n}{2}\rceil + 1$, then $N_r(f) = 1$.*

Corollary 3.7. *Let $f \in F_{n,d}(K)$ and $S_r(f) \neq \emptyset$. If $d \geqslant 2r - 1$, then $N_r(f) = 1$.*

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