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ADDITIONAL PROPERTIES OF THE MEASURE v_f

Sebastian Lindner

ABSTRACT. In the note [S. Lindner: *Generalization of the Banach indicatrix theorem*, Real Analysis Exchange **27**, (2001–2002)] the measure v_f on the interval $[0,1]$ connected with the variation of a given continuous function f has been defined. In this paper the relations between the function f and the shape of measure v_f are considered.

In the note [1] the following measure on the interval $[0,1]$ was defined: Let $f\colon [0,1] \to [0,1]$ be a given continuous function. For every closed interval $[a,b] \in [0,1]$ let $v^*\big([a,b]\big) = \bigvee_a^b f$. (We allow $[a,b]$ to be a degenerate interval, (i.e., $a = b$). In this case $v^*\big([a,b]\big) = 0$.)

Now, for every set $A \subset [0,1]$ let

$$v_z(A) = \inf_{\{I_n\}} \left\{ \sum_{n\in\mathbb{N}} v^*(I_n) \colon A \subset \bigcup_{n\in\mathbb{N}} I_n \right\},$$

where $\{I_n\}$ denotes an arbitrary family of closed intervals covering A. At last, let $\mathcal{S}$ be a σ-algebra of sets satisfying the Carathéodory condition, and v_f be a restriction of v_z to $\mathcal{S}$.

Let $\mathfrak{M}$ denote the σ-algebra of sets measurable with respect to the Lebesgue measure μ. How does the shape of the measure v_f and σ-algebra $\mathcal{S}$ depend on the function f? Let us consider some examples.

Example 1. Let $f\colon [0,1] \to [0,1]$ be a constant. Then $\bigvee_0^1 f = 0$ and the measure v_f is identically equal to 0. So, of course, $\mathcal{S} = 2^{[0,1]}$.

Example 2. Let $f\colon [0,1] \to [0,1]$, $f(x) = x$. In this case, for every $[a,b] \subset [0,1]$, we have $v^*\big([a,b]\big) = \bigvee_a^b f = b - a = \mu\big([a,b]\big)$. Hence $v_f = \mu$ and $\mathcal{S} = \mathfrak{M}$.

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Lemma 1. *If $f\colon [0,1] \to [0,1]$ is strictly increasing function, then for every $A \subset [0,1]$ it holds $v_z(A) = \mu_z\bigl(f(A)\bigr)$, and, consequently,*

$$\mathcal{S} = \bigl\{A \subset [0,1]\colon f(A) \in \mathfrak{M}\bigr\}.$$

P r o o f. In fact, we have that

$$A \subset \bigcup_{n\in\mathbb{N}} I_n \iff f(A) \subset \bigcup_{n\in\mathbb{N}} f(I_n).$$

Furthermore,

$$v_z(A) = \inf_{\{I_n\}}\left\{\sum_n v^*(I_n)\right\} = \inf_{\{I_n\}}\left\{\sum_n \bigl|f(I_n)\bigr|\right\} = \mu_z\bigl(f(A)\bigr).$$

$\square$

Example 3. Let $c\colon [0,1] \to [0,1]$ be the Cantor function and for $x \in X$ let

$$f(x) = \frac{c(x)+x}{2}.$$

In this case f is continuous and strictly increasing. So, in virtue of Lemma 1,

$$A \in \mathcal{S} \iff f(A) \in \mathfrak{M}.$$

Let C be the Cantor set. Since $\mu\bigl(f(C)\bigr) > 0$, there exists a nonmeasurable set $B \subset f(C)$. It implies that $f^{-1}(B) \in \mathfrak{M} \setminus \mathcal{S}$.

Theorem 1. *The following diagram shows the relation between the function f and a shape of the measure v_f.*

$$\boxed{\text{(1) The function } f \text{ is absolutely continuous.}}$$

$$\Updownarrow$$

$$\boxed{(2)\ \forall_{\varepsilon>0}\,\exists_{\delta>0}\,\forall_{A\in\mathfrak{M}}\quad \mu(A) < \delta \implies v_z(A) < \varepsilon.}$$

$$\Downarrow$$

$$\boxed{\text{(3) The measure } v_f \text{ is absolutely continuous with resp. to } \mu.}$$

$$\Downarrow$$

$$\boxed{\text{(4) The function } f \text{ satisfies the Lusin N condition.}}$$

$$\Downarrow$$

$$\boxed{(5)\ \mathfrak{M} \subset \mathcal{S}.}$$

P r o o f. (1) $\Rightarrow$ (2) Let $f\colon [0,1] \to [0,1]$ be absolutely continuous. Then the derivative f' exists almost everywhere and it is integrable. Moreover, $\bigvee_a^b f =$

$\int\limits_a^b |f'|\,d\mu$, (c.f. [2], Chap. 9).

Fix $\varepsilon > 0$. Since $|f'|$ is integrable, there exists $\delta > 0$ such that for all $A \in \mathfrak{M}$, the condition $\mu(A) < \delta$ implies $\int\limits_A |f'(x)|dx \ < \varepsilon$.

Let $A \in \mathfrak{M}$, $\mu(A) < \delta$. Then there exists a family $(I_n)_{n\in\mathbb{N}}$ of closed intervals covering A such that $\sum\limits_n |I_n| < \delta$. We can assume that interiors of intervals I_n are disjoint. We obtain

$$\varepsilon > \int\limits_{\bigcup I_n} |f'(x)|dx = \sum_n \int\limits_{I_n} |f'(x)|dx = \sum_n \bigvee_{I_n} f \geq v_z(A).$$

$(2) \Rightarrow (3)$ The proof is easy and can be omitted.

$(3) \Rightarrow (4)$ On the contrary, suppose that f does not satisfy the Lusin condition N. So there exists a set $A \subset [0,1]$, $\mu(A) = 0$ such that $\mu_z\big(f(A)\big) > 0$. Let $(I_n)_{n\in\mathbb{N}}$ be a family of closed intervals covering A. Then

$$\begin{aligned}\sum_n \bigvee_{I_n} f \geq \sum_n \omega(f, I_n) = \sum_n \mu\big(f(I_n)\big) &\geq \mu\Big(\bigcup_n f(I_n)\Big) \\ &= \mu\Big(f\Big(\bigcup_n I_n\Big)\Big) \geq \mu_z\big(f(A)\big),\end{aligned}$$

where $\omega(f, I_n)$ denotes the oscillation of the function f on I_n. As (I_n) was arbitrary, we have that $v_z(A) > 0$, to the contrary to (3).

$(4) \Rightarrow (5)$ Our proof starts with an observation, that if the function $f\colon [0,1] \to [0,1]$ satisfies the Lusin condition, then

$$\mu(A) = 0 \implies \big(v_z(A) = 0 \vee v_z(A) = +\infty\big)$$

for every set $A \subset [0,1]$. To the contrary to our claim, suppose that $\mu(A) = 0$, but $v_z(A) = \alpha$, $0 < \alpha < \infty$. Then there exists such a sequence of intervals (I_n) covering A, that $\sum\limits_{n\in\mathbb{N}} \bigvee\limits_{I_n} f < \frac{3}{2}\alpha$.

Fix the interval I_{n_0}. The function f restricted to this interval is absolutely continuous, so, according to the first part of the proof, we obtain that $v_z(A \cap I_{n_0}) = 0$. Hence there exists such a sequence of intervals $(I_{n_0,k})$ that

$$A \cap I_{n_0} \subset \bigcup_k I_{n_0,k} \subset I_{n_0}$$

and

$$\sum_k \bigvee_{I_{n_0,k}} f < \frac{1}{2} \bigvee_{I_{n_0}} f.$$

Repeating this for every interval I_n we construct a new covering $(I_{n,k})$ of the set A, such that

$$\sum_{n,k\in\mathbb{N}} \bigvee_{I_{n,k}} f < \sum_{n\in\mathbb{N}} \frac{1}{2} \bigvee_{I_n} f < \frac{3}{4}\alpha.$$

This contradicts our assumption.

We show now, that for every set A the condition $\mu(A) = 0$ implies $A \in \mathcal{S}$. Let $A \subset [0,1]$ and $\mu(A) = 0$. Let $W \subset A$, $Z \subset [0,1] \setminus A$. By the previous part,

$$v_z(A) = 0 \vee v_z(A) = +\infty\,.$$

If $v_z(A) = 0$, of course, then $A \in \mathcal{S}$. If $v_z(A) = \infty$, then we have $v_z(W) = 0$ or $v_z(W) = +\infty$.

In the first case, obviously $v_z(W) + v_z(Z) \le v_z(W \cup Z)$. In the second one we obtain $v_z(W) + v_z(Z) = +\infty = v_z(W \cup Z)$.

In the both cases A satisfies the Carathéodory condition with respect to the measure v_z, hence it is v_f-measurable.

(2) $\Rightarrow$ (1) We have already established, that (2) $\Rightarrow$ (4). Only the condition that f has a bounded variation on $[0,1]$ has left.

On the contrary, suppose that $\bigvee_0^1 f = +\infty$. Then at least one of the numbers $\bigvee_0^{1/2} f$ or $\bigvee_{1/2}^1 f$ is infinite. Continuing this process we obtain the sequence $[0,1] \supset I_1 \supset I_2 \ldots$ of intervals such that for every n it holds $\bigvee_{I_n} f = +\infty$, and $|I_n| = 2^{-n}$. So there exists such a point $x_0 = \bigcap I_n$ that $\bigvee_J f = +\infty$ for any open interval J containing x_0. This contradicts (2). $\square$

Counter-examples

The following example shows that the conditions (2) and (3) are not equivalent for an arbitrary function f.

EXAMPLE 4. Let $f\colon [0,1] \to [0,1]$ be defined by the following formula:

$$f(x) = \begin{cases} \frac{1}{2} x \sin \frac{1}{x} + \frac{1}{2} & \text{if } x \in (0,1]\,, \\ \frac{1}{2} & \text{if } x = 0\,. \end{cases}$$

Then for every $a > 0$ we have that $v_f\big([0,a]\big) = \bigvee_0^a f = +\infty$, so obviously the measure v_z does not satisfy (2). However, v_f satisfies the condition (3).

In fact: let $A \in \mathfrak{M}$ such that $\mu(A) = 0$. Then

$$A = A \cap \{0\} \cup \bigcup_{n=1}^{\infty} A \cap \left[\frac{1}{n+1}, \frac{1}{n}\right].$$

Since for every $n \in \mathbb{N}$ the function f is absolutely continuous on the interval $\left[\frac{1}{n+1}, \frac{1}{n}\right]$, we obtain $v_f\left(A \cap \left[\frac{1}{n+1}, \frac{1}{n}\right]\right) = 0$. Of course, $v_f(\{0\}) = 0$, so $v_f(A) = 0$.

In the following example we show that (4) does not always imply (3). According to this, we construct the function f such that

(a) f satisfies the Lusin condition N.

(b) f has an unbounded variation on every non-degenerate subinterval of $[0,1]$.

Example 5. Let the sequence $\left((a_n, b_n)\right)$ be the basis of the natural topology on $(0,1)$. Let (r_n) be such a sequence of positive numbers that $\sum\limits_{n=1}^{\infty} r_n < +\infty$.

Now by induction we simultaneously construct the sequence f_n of continuous functions and the sequence of intervals $\left((c_n, d_n)\right)$ in the following way.

Let $f_0(x) = x$.

Let us assume that $f_{n-1} \colon [0,1] \to \mathbb{R}$ is defined, and that its plot is a broken line. Let $(c_n, d_n) \subset (a_n, b_n)$ be an interval such that

$(*)$ f_{n-1} restricted to (c_n, d_n) is linear, and

$(**)$ $\mu\left(f_{n-1}\left((c_n, d_n)\right)\right) < r_n$.

Defining f_n we will modify the function f_{n-1} on (c_n, d_n) to meet the following conditions:

(i) The graph of f_n is a broken line.

(ii) $\mu\left(f_n\left((c_n, d_n)\right)\right) < r_n$,

(iii) $\bigvee\limits_{c_n}^{d_n} f_n > n$.

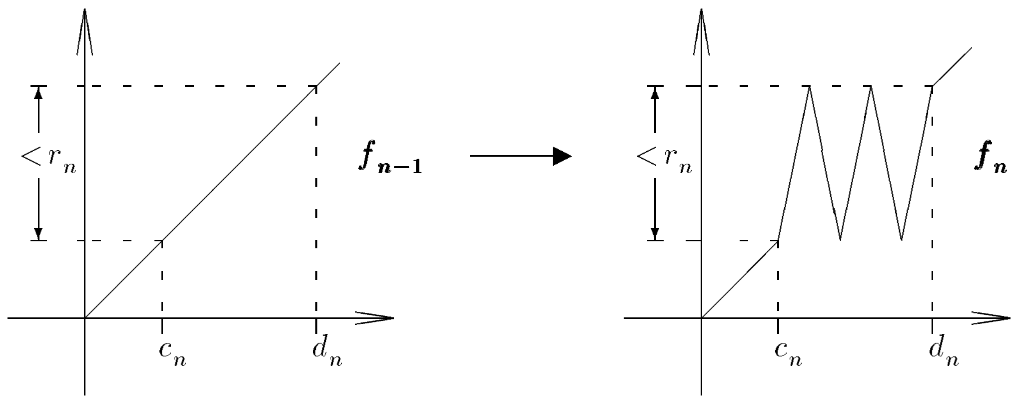

Figure 1. One step of the construction of the function f_n.

We have $\left|f_{n-1} - f_n\right| < r_n$, so the sequence f_n is uniformly convergent to the continuous function f.

Let us remark that if for some $n \in \mathbb{N}$ the point (x_0, y_0) is an end point of a segment of the plot of f_n, then $f_{n+1}(x_0) = f_{n+2}(x_0) = \cdots = f(x_0) = y_0$.

Let $[a,b] \subset [0,1]$, $a < b$. The inclusion $(a_n, b_n) \subset [a,b]$ holds for infinitely many $n \in \mathbb{N}$. In virtue of the previous remark we have, that $\bigvee_a^b f > \bigvee_{c_n}^{d_n} f \geq \bigvee_{c_n}^{d_n} f_n > n$ for infinitely many n. Therefore $\bigvee_a^b f = +\infty$.

Now, we show that the function f satisfies the Lusin condition N. Let

$$E_0 = [0,1] \setminus \bigcup_{i=1}^{\infty} (c_i, d_i),$$

for every $k \in \mathbb{N}$

$$E_k = (c_k, d_k) \setminus \bigcup_{i=k+1}^{\infty} (c_i, d_i),$$

and

$$E_\infty = \bigcap_{k=1}^{\infty} \bigcup_{i=k}^{\infty} (c_i, d_i).$$

It is clear that $[0,1] = E_\infty \cup \bigcup\limits_{k=0}^{\infty} E_k$.

Let $A \subset [0,1]$, $\mu(A) = 0$. We have

$$A = (A \cap E_\infty) \cup \bigcup_{k=0}^{\infty} A \cap E_k.$$

On every set E_k, the function f is equal to f_k, which, of course, satisfies the Lusin condition. So,

$$\mu\big(f(A \cap E_k)\big) = \mu\big(f_k(A \cap E_k)\big) = 0.$$

Simultaneously,

$$f(E_\infty) = f\left(\bigcap_{k=1}^{\infty} \bigcup_{i=k}^{\infty} (c_i, d_i)\right) \subset \bigcap_{k=1}^{\infty} \bigcup_{i=k}^{\infty} f\big((c_i, d_i)\big).$$

For every $k \in \mathbb{N}$,

$$\mu\left(\bigcup_{i=k}^{\infty} f\big((c_i, d_i)\big)\right) \leq \sum_{i=k}^{\infty} \mu\Big(f\big((c_i, d_i)\big)\Big) < \sum_{i=k}^{\infty} r_i.$$

So, $f(E_\infty) = 0$.

The function f at the same time does not satisfy the condition (3) as we have $v_f(A) = +\infty$ for every uncountable set $A \subset [0,1]$.

EXAMPLE 6. The last example shows that it may happen that the condition (5) is satisfied meanwhile the condition (4) does not hold.

We construct the function f such that:

(a) f does not satisfy the Lusin condition N.

(b) f has an unbounded variation on every nondegenerate subinterval of $[0,1]$.

Let I_n denote the nth open interval deleted during construction of the Cantor set C. Let h_n be such a continuous function, that

$(*)$ $h_n(x) = 0$ for every $x \notin I_n$,

$(**)$ $\bigvee_a^b h_n = +\infty$ for every $(a,b) \subset I_n$, $a < b$,

$(***)$ $\sup_{x\in[0,1]} \left|h_n(x)\right| < \frac{1}{2^n}$.

Let $f(x) = \sum_{n=1}^{\infty} h_n(x) + c(x)$, where $c(x)$ denotes the Cantor function. Then f is continuous as an uniformly convergent sum of continuous functions.

Of course, $\mu\big(f(C)\big) = 1$. So, f does not satisfy the Lusin condition N.

The same time for every $a < b$ the interval (a,b) contains some subinterval contained in one of the intervals I_n, so

$$\bigvee_a^b f(x) = +\infty\,.$$

This is easy to see that in this case for every set $A \subset [0,1]$ we have $v_f(A) \in \{0,+\infty\}$, therefore $\mathcal{S} = 2^{[0,1]}$.

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Faculty of Mathematics
University of Łódź
Banacha 22
PL–90-238 Łódź
POLAND
E-mail: lindner@imul.uni.lodz.pl