Tatra Mt. Math. Publ. **32** (2005), 53–61

TATRA MOUNTAINS
Mathematical Publications

INTEGRAL EQUIVALENCE OF REAL ALGEBRAIC FUNCTION FIELDS

Przemysław Koprowski

ABSTRACT. In this paper we deal with a quaternion-symbol equivalence of two formally real algebraic function fields over a common real closed field. We investigate the conditions for it to map Witt ring of one Dedekind domain onto the Witt ring of another Dedekind domain. We find a necessary and sufficient condition for this property. It turns out to be analogous to S-integral equivalence known in the realms of algebraic number theory. In addition, we strengthen the results concerning tame equivalence. In particular, we show that any quaternion-symbol equivalence is tame at every real point at which it is defined.

1. Introduction

The notion of Witt ring plays a central role in the algebraic theory of quadratic forms. Two fields K and L are said to be Witt equivalent when their Witt rings are isomorphic. The classification of fields with respect to Witt equivalence has been studied by several authors (cf. [2, 7, 3]). In [3] we found criteria for two formally real algebraic function fields of one variable with real closed fields of constants to be Witt equivalent. The tool used there was a quaternion-symbol equivalence, which is an off-spring of a Hilbert-symbol equivalence used for determining Witt equivalence of global fields (see, e.g., [2]).

In a given function field we have a number of Dedekind domains obtained by intersecting valuation rings of almost all (i.e., all but finitely many) points. With every such a Dedekind ring $\mathcal{O}_S$ we may associate its Witt ring $W\mathcal{O}_S$. By the means of Knebusch–Milnor exact sequence (c.f. [6, Corollary 3.3]) there is a natural injection of $W\mathcal{O}_S$ into the Witt ring of the field. Thus, it is natural to ask whether a given isomorphism of Witt rings of fields preserves Witt rings of Dedekind domains. This is the central theme of this paper. The similar question was investigated in [2] for global fields. In the Theorem 4.5 we give a necessary and sufficient condition for a given quaternion-symbol equivalence to induce an

2000 Mathematics Subject Classification: Primary 11E81; Secondary 11E10, 14H05, 14P05.
Keywords: Witt equivalence, tame equivalence, real algebraic curves.

isomorphism of Witt rings of Dedekind domains. This result is similar to [2, Theorem 5.4]. This shows that, although tools and methods differ, the resulting theory for function fields over real closed fields is similar to the one for global fields.

The side-effect of this paper is also to strengthen the results of [3] concerning tame equivalence. Namely, we show that every quaternion-symbol equivalence is tame at each point in which it is defined (cf. Theorem 3.1). We discuss also the tameness of quaternion-symbol equivalence at complex points (see Lemma 3.3.)

2. Notation

In this paper we deal with formally real fields only. Hence, whenever we use the term *function field* it actually means 'formally real algebraic function field of one variable with a real closed field of constants $\Bbbk$'. Throughout this paper letters K and L denote two function fields over *the same* real closed field $\Bbbk$. The field $\Bbbk$ is fixed once and for all. We use symbol $\Omega(K)$ to denote the set of all points of K trivial on its field of constants. The completion of the field K with respect to the point $\mathfrak{p} \in \Omega(K)$ is denoted by $K_{\mathfrak{p}}$ while its residue field by $K(\mathfrak{p})$. Further, $\operatorname{ord}_{\mathfrak{p}}$ is the associated valuation.

The residue field $K(\mathfrak{p})$ of a point $\mathfrak{p} \in \Omega(K)$ is isomorphic either to $\Bbbk$ or to its algebraic closure $\Bbbk(\sqrt{-1})$. Following [4], we call a point of the first kind *real* and one of the second kind *complex*. The set of all real points is denoted by $\gamma(K)$.

By Harrison's criterion (cf. [7, p. 370]) two fields K and L are Witt equivalent if and only if there exists an isomorphism $t\colon K^*/K^{*2} \to L^*/L^{*2}$ of square-class groups sending -1 to -1 and such that a binary form $\langle a, b\rangle$ represents a square over K if and only if $\langle ta, tb\rangle$ represents a square over L. Such a map is called *Harrison isomorphism.* The mapping $\langle f_1, \dots, f_n\rangle \mapsto \langle tf_1, \dots, tf_n\rangle$ is then an isomorphism of Witt rings. It is called the *strong* Witt isomorphism associated with Harrison isomorphism t.

Let $A \subseteq \Omega(K)$ and $B \subseteq \Omega(L)$ be two non-empty subsets of sets of points of K and L. The pair (T, t) of maps in which $T\colon A \to B$ is a bijection and $t\colon K^*/K^{*2} \to L^*/L^{*2}$ is an isomorphism of square-class groups and is called a *quaternion-symbol equivalence* (see [3, Definition 2.3]) of K, L with respect to the pair (A, B) if and only if it preserves the splitting of local quaternion symbols in the sense that

$$\left(\frac{f, g}{K_{\mathfrak{p}}}\right) \text{ splits } \iff \left(\frac{tf, tg}{L_{T\mathfrak{p}}}\right) \text{ splits}$$

for all square classes $f, g \in K^*/K^{*2}$ and all points $\mathfrak{p} \in A$. A quaternion-symbol equivalence is said to be tame at $\mathfrak{p} \in \Omega(K)$ (cf. [3, p. 335]) if and only if

$$\operatorname{ord}_{\mathfrak{p}} f \equiv \operatorname{ord}_{T\mathfrak{p}} tf \pmod 2$$

for all square-classes $f \in K^*/K^{*2}$.

The symbol $S(K)$ (resp. $S(L)$) will always denote a finite set of points of K (resp. L). It was proved in [3, Section 4] that the existence of quaternion-symbol equivalence with respect to either $\big(\gamma(K)\setminus S(K), \gamma(L)\setminus S(L)\big)$ or $\big(\Omega(K)\setminus S(K), \Omega(L)\setminus S(L)\big)$ implies that K and L are Witt equivalent. The intersection of valuation rings associated with points not in $S(K)$ is denoted by $\mathcal{O}_{S(K)}$:

$$\mathcal{O}_{S(K)} = \bigcap_{\mathfrak{p}\in\Omega(K)\setminus S(K)} \mathcal{O}_{\mathfrak{p}} = \big\{f \in K\colon \operatorname{ord}_{\mathfrak{p}} f \geqslant 0 \text{ for all } \mathfrak{p} \notin S(K)\big\}.$$

We will often refer to the following Knebusch–Milnor exact sequence:

$$0 \to W\mathcal{O}_{S(K)} \xrightarrow{i} WK \xrightarrow{\partial_{S(K)}} \coprod_{\mathfrak{p}\in\Omega(K)\setminus S(K)} WK(\mathfrak{p}).$$

Here i is the canonical injection and the additive group homomorphism $\partial_{S(K)}$ is a direct sum (over $\mathfrak{p} \notin S(K)$) of the second residue homomorphisms $\partial_{\mathfrak{p}}\colon WK \to WK(\mathfrak{p})$. We will identify the ring $W\mathcal{O}_S(K)$ with its image $i\big(W\mathcal{O}_{S(K)}\big) = \ker \partial_{S(K)}$. By abuse of notation, we use the same symbol f to denote both an element of a field and a square class containing this element. Analogously, $\langle f_1, \dots, f_n\rangle$ is (depending on a context) either a quadratic form or its class in Witt ring.

3. Tame equivalence

In this section we investigate tame quaternion-symbol equivalence. It was proved in [3, Section 4] that for any two real algebraic function fields K, L over a fixed real closed field there is a tame quaternion-symbol equivalence of K and L with respect to a pair $\big(\gamma(K)\setminus S(K), \gamma(L)\setminus S(L)\big)$ for some finite sets $S(K)$, $S(L)$. It was also remarked that this equivalence cannot a priori be extended to the whole $\big(\gamma(K), \gamma(L)\big)$. We now strengthen this result showing that this is actually the only possibility, namely every quaternion-symbol equivalence preserving -1 must be tame wherever it is defined. In other words at any real point an equivalence is either tame or undefined. This property nicely corresponds to the global case (cf. [1, Lemma 1.18]).

Theorem 3.1. *Let (T,t) be a quaternion-symbol equivalence of algebraic function fields K, L with respect to the pair $\bigl(\gamma(K)\setminus S(K), \gamma(L)\setminus S(L)\bigr)$ preserving -1 (i.e., $t(-1)=-1$). Then (T,t) is tame.*

P r o o f. Fix a point $\mathfrak{p}\in\gamma(K)\setminus S(K)$ and take a square-class $f\in K^*/K^{*2}$. The following equivalences hold:

$$f\in K_{\mathfrak{p}}^{*2} \iff \left(\frac{f,-1}{K_{\mathfrak{p}}}\right)=1 \iff \left(\frac{tf,-1}{L_{T_{\mathfrak{p}}}}\right)=1 \iff tf\in L_{T_{\mathfrak{p}}}^{*2}\,.$$

Here, the middle equivalence follows from (T,t) being a quaternion-symbol equivalence while the others from [3, Lemma 2.4]. Thus f is a local square if and only if tf is one and this remains true for any real $\mathfrak{p}$ not in $S(K)$. Now [3, Proposition 2.11] implies[1] that there are local Harrison maps $t_{\mathfrak{p}}\colon K_{\mathfrak{p}}^*/K_{\mathfrak{p}}^{*2}\to L_{T_{\mathfrak{p}}}^*/L_{T_{\mathfrak{p}}}^{*2}$ and associated local strong Witt isomorphisms $i_{t_{\mathfrak{p}}}\colon WK_{\mathfrak{p}}\to WL_{T\mathfrak{p}}$ such that the following two diagrams commute:

$$\begin{array}{ccc} WK & \xrightarrow{\;i\;} & WL \\ {\scriptstyle\Theta_{\mathfrak{p}}}\big\downarrow & & \big\downarrow{\scriptstyle\Theta_{T_{\mathfrak{p}}}} \\ WK_{\mathfrak{p}} & \xrightarrow{\;i_{\mathfrak{p}}\;} & WL_{T_{\mathfrak{p}}} \end{array} \qquad \begin{array}{ccc} K^*/K^{*2} & \xrightarrow{\;t\;} & L^*/L^{*2} \\ {\scriptstyle\vartheta_{\mathfrak{p}}}\big\downarrow & & \big\downarrow{\scriptstyle\vartheta_{T_{\mathfrak{p}}}} \\ K_{\mathfrak{p}}^*/K_{\mathfrak{p}}^{*2} & \xrightarrow{\;t_{\mathfrak{p}}\;} & L_{T_{\mathfrak{p}}}^*/L_{T_{\mathfrak{p}}}^{*2} \end{array}$$

On the other hand $K_{\mathfrak{p}}^*/K_{\mathfrak{p}}^{*2}=\{\pm1,\pm p\}\cong\{\pm1,\pm q\}=L_{T_{\mathfrak{p}}}^*/L_{T_{\mathfrak{p}}}^{*2}$, for some uniformizing elements p, q. Therefore $t_{\mathfrak{p}}(\pm1)=\pm1$ and so also $t_{\mathfrak{p}}(\pm p)=\pm q$. Hence $\operatorname{ord}_{\mathfrak{p}} f\equiv \operatorname{ord}_{T\mathfrak{p}} tf \pmod 2$ which means that (T,t) is tame. □

The following proposition, in conjunction with the above theorem, is used in the next section.

Proposition 3.2. *Let (T,t) be a tame quaternion-symbol equivalence of K and L with respect to the pair $\bigl(\gamma(K)\setminus S(K),\gamma(L)\setminus S(L)\bigr)$. Denote by $i_t\colon WK\to WL$ the associated strong Witt isomorphism. Let further $\mathfrak{p}$ be a real point not in $S(K)$ and φ any quadratic form. Then the second residue homomorphism $\partial_{\mathfrak{p}}\colon WK\to WK(\mathfrak{p})$ associated with the point $\mathfrak{p}$ vanishes on φ if and only if the second residue homomorphism $\partial_{T\mathfrak{p}}\colon WL\to WL(T\mathfrak{p})$ vanishes on $i_t\varphi$.*

P r o o f. Fix a quadratic form φ and a real point $\mathfrak{p}\in\gamma(K)\setminus S(K)$ and let p denote a uniformizer of $\mathfrak{p}$. We may decompose φ to

$$\varphi=\varphi_1\perp\langle p\rangle\otimes\varphi_2\,,$$

[1]The statement of [3, Proposition 2.11] requires the above condition to be true for *all* real points, but it is easy to check that the proof of implication we use here holds word for word also in our case.

where φ_1 and φ_2 have diagonalizations consisting only of $\mathfrak{p}$-adic units, i.e., $\varphi_1 = \langle f_1, \dots, f_k \rangle$, $\varphi_2 = \langle f_{k+1}, \dots, f_n \rangle$ with $\mathrm{ord}_{\mathfrak{p}} f_i \equiv 0 \pmod 2$. Since the equivalence (T,t) is assumed to be tame, then also $\mathrm{ord}_{T\mathfrak{p}} tf_i \equiv 0 \pmod 2$ and $\mathrm{ord}_{T\mathfrak{p}} tp \equiv 1 \pmod 2$. Hence, the square class tp contains a $T\mathfrak{p}$-uniformizer.

The point $T\mathfrak{p}$ is real because $\mathfrak{p}$ is real, thus $WK(\mathfrak{p}) \cong \mathbb{Z} \cong WL(T\mathfrak{p})$ and $\partial_{\mathfrak{p}}\varphi$, $\partial_{T\mathfrak{p}}\varphi$ are equal to the local signatures $\tau_{\mathfrak{p}}\varphi_2$ and $\tau_{T\mathfrak{p}} i_t \varphi_2$. Now [3, Lemma 3.3] implies[2] that

$$\tau_{\mathfrak{p}}\varphi_2 = \sum_{i=k+1}^{n} \tau_{\mathfrak{p}} f_i = \sum_{i=k+1}^{n} \tau_{T\mathfrak{p}} t f_i = \tau_{T\mathfrak{p}} i_t \varphi_2 \,.$$

Consequently, $0 = \partial_{\mathfrak{p}}\varphi = \tau_{\mathfrak{p}}\varphi_2 = \tau_{T\mathfrak{p}} i_t \varphi_2 = \partial_{T\mathfrak{p}} i_t \varphi_2$ and this proves one implication. To prove the other implication one relays on the symmetry of conditions substituting (T,t) by (T^{-1}, t^{-1}). □

Now it is time to turn our attention to complex points. In this case we show similar results. Consider a quaternion-symbol equivalence (T,t) of K and L with respect to the pair $\big(\Omega(K) \setminus S(K),\, \Omega(L) \setminus S(L)\big)$ and a fixed complex point $\mathfrak{p} \in \Omega(K) \setminus S(K)$.

Lemma 3.3. *The following two conditions are equivalent*

(1) *for every square-class $f \in K^*/K^{*2}$ one has*

$$f \in K_{\mathfrak{p}}^{*2} \iff tf \in L_{T_{\mathfrak{p}}}^{*2}\,;$$

(2) *the equivalence (T,t) is tame at $\mathfrak{p}$.*

P r o o f. Let $\theta_{\mathfrak{p}} \colon K^*/K^{*2} \to K_{\mathfrak{p}}^*/K_{\mathfrak{p}}^{*2}$ (resp. $\theta_{T\mathfrak{p}} \colon L^*/L^{*2} \to L_{T_{\mathfrak{p}}}^*/L_{T_{\mathfrak{p}}}^{*2}$) denote the canonical epimorphism of square-class groups. Define a map $t_{\mathfrak{p}}\colon K_{\mathfrak{p}}^*/K_{\mathfrak{p}}^{*2} \to L_{T_{\mathfrak{p}}}^*/L_{T_{\mathfrak{p}}}^{*2}$ by the formula

$$t_{\mathfrak{p}}\big(\theta_{\mathfrak{p}}(f)\big) := \theta_{T\mathfrak{p}}(tf)\,.$$

It follows from the first condition that $t_{\mathfrak{p}}$ is a well defined isomorphism. Now since $K_{\mathfrak{p}}^*/K_{\mathfrak{p}}^{*2}$ is the 2-element group we arrive at the other condition.

Now assume that second condition holds. For any $f \in K^*/K^{*2}$ we have

$$f \in K_{\mathfrak{p}}^{*2} \iff \mathrm{ord}_{\mathfrak{p}} f \equiv 0 \pmod 2 \iff \mathrm{ord}_{T\mathfrak{p}} tf \equiv 0 \pmod 2 \iff tf \in L_{T_{\mathfrak{p}}}^{*2}\,.$$

The middle equivalence is due to the fact that (T,t) was assumed to be tame. □

[2]The statement of [3, Lemma 3.3] is for *all* real points, here we use it for *almost all*, but again it is straightforward to verify that the proof of the implication we need holds also in this more general case.

Take now two square classes $f, g \in K^*/K^{*2}$ that give rise to the same square class in a completion $K_{\mathfrak{p}}$, i.e., $\theta_{\mathfrak{p}} f = \theta_{\mathfrak{p}} g$. The fact that they are glued together in $K_{\mathfrak{p}}^*/K_{\mathfrak{p}}^{*2}$ is equivalent to $fg \in K_{\mathfrak{p}}^{*2}$. Hence the above lemma shows that a quaternion-symbol equivalence wild at a complex point $\mathfrak{p}$ cannot be extended to local fields $K_{\mathfrak{p}}, L_{T\mathfrak{p}}$. Now, we show a complex analogon of Proposition 3.2.

Proposition 3.4. *Let (T, t) be a tame quaternion-symbol equivalence of K and L with respect to the pair $\big(\Omega(K) \setminus S(K), \Omega(L) \setminus S(L)\big)$. Denote by $i_t\colon WK \to WL$ the associated strong Witt isomorphism. Let further $\mathfrak{p}$ be a complex point not in $S(K)$ and φ any quadratic form. Then the second residue homomorphism $\partial_{\mathfrak{p}}\colon WK \to WK(\mathfrak{p})$ associated with the point $\mathfrak{p}$ vanishes on φ if and only if the second residue homomorphism $\partial_{T\mathfrak{p}}\colon WL \to WL(T\mathfrak{p})$ vanishes on $i_t\varphi$.*

P r o o f. As in the Proposition 3.2, it is enough to prove one implication. Again let p denote a uniformizer of the point $\mathfrak{p}$ and decompose φ to $\varphi = \varphi_1 \perp \langle p\rangle \otimes \varphi_2$, with $\varphi_1 = \langle f_1, \dots, f_k\rangle$, $\varphi_2 = \langle f_{k+1}, \cdots, f_n\rangle$ and $\operatorname{ord}_{\mathfrak{p}} f_i \equiv 0 \pmod 2$. Hence, we may write $i_t\varphi$ in the form

$$i_t\varphi = i_t\varphi_1 \perp \langle tp\rangle \otimes i_t\varphi_2 = \langle tf_1, \dots, tf_k\rangle \perp \langle tp\rangle \otimes \langle tf_{k+1}, \dots, tf_n\rangle .$$

The equivalence (T, t) is tame, hence also $\operatorname{ord}_{T\mathfrak{p}} tf_i \equiv 0 \pmod 2$ and $\operatorname{ord}_{T\mathfrak{p}} tp \equiv 1 \pmod 2$. Notice that $WK(\mathfrak{p}) \cong \mathbb{Z}_2 \cong WL(T\mathfrak{p})$ and so $\partial_{\mathfrak{p}}\varphi = 0$ if and only if $\dim \varphi_2 \equiv 0 \pmod 2$. On the other hand $\dim\varphi_2 \equiv \dim i_t\varphi_2 \pmod 2$ and so $\partial_{T\mathfrak{p}} i_t\varphi \equiv 0 \pmod 2$. □

4. $\big(S(K), S(L)\big)$-integral equivalence

In this section we develop the main result of our note — the criterion for an isomorphism of Witt rings of fields to preserve Witt rings of rings. Define a subgroup $\mathbb{E}_S$ of the square-class group K^*/K^{*2} as follows

$$\mathbb{E}_S := \mathbb{E}_S(K) := \Big\{ f \in K^*/K^{*2}\colon \bigwedge_{\mathfrak{p}\in\Omega(K)\setminus S} \operatorname{ord}_{\mathfrak{p}} \equiv 0 \pmod 2 \Big\} .$$

We begin by a lemma which was proved in [2] for global fields. Since we have a different setup here, for reader's convenience, we present a sketch of a proof.

Lemma 4.1. ([2, Proposition 3.1]) *Let $f \in K^*/K^{*2}$ be any square-class. Then f belongs to $\mathbb{E}_S$ if and only if $\langle f\rangle$ belongs to the Witt ring $W\mathcal{O}_S$ of the ring $\mathcal{O}_S$.*

P r o o f. Fix any square-class $f \in K^*/K^{*2}$. Assume first, that $f \in \mathbb{E}_S$ and take a point $\mathfrak{p} \in \Omega(K) \setminus S$. By the very definition of $\mathbb{E}_S$ we have $\operatorname{ord}_{\mathfrak{p}} f \equiv 0$

(mod 2). Hence, we may find a $\mathfrak{p}$-adic unit $g \in \mathcal{O}_S \setminus \mathfrak{p}$ in the square-class f. Thus elements $\langle f\rangle$ and $\langle g\rangle$ of the Witt ring $W\mathcal{O}_S$ are equal, and so $\langle f\rangle \in W\mathcal{O}_S$.

Conversely, assume that $\langle f\rangle \in W\mathcal{O}_S$ and again take a point $\mathfrak{p}$ not in S. Knebuch–Milnor exact sequence (c.f. [6, Corollary 3.3]) implies that the second residue homomorphism $\partial_{\mathfrak{p}}$ vanishes on $\langle f\rangle$. In consequence $\operatorname{ord}_{\mathfrak{p}} f$ is even. □

The above lemma allows us to associate the behaviour of a strong Witt isomorphism i_t on a ring $W\mathcal{O}_S$ with a behaviour of the associated Harrison map t on a group $\mathbb{E}_S$. The following corollary is based upon [1, Theorem 4.8].

Corollary 4.2. *Let (T,t) be a quaternion-symbol equivalence of K and L with respect to $\big(\Omega(K)\setminus S(K),\, \Omega(L)\setminus S(L)\big)$ and let $i_t\colon WK \to WL$ be the associated strong Witt isomorphism. If i_t maps $W\mathcal{O}_{S(K)}$ onto $W\mathcal{O}_{S(L)}$, then t maps $\mathbb{E}_{S(K)}$ onto $\mathbb{E}_{S(L)}$*

Proof. Fix $f \in \mathbb{E}_{S(K)}$. By the previous lemma we have $\langle f\rangle \in W\mathcal{O}_{S(K)}$. Consequently $\langle tf\rangle = i_t\langle f\rangle \in W\mathcal{O}_{S(L)}$. Using again the previous lemma we have $tf \in \mathbb{E}_{S(L)}$. This shows that $t\big(\mathbb{E}_{S(K)}\big) \subseteq \mathbb{E}_{S(L)}$. Changing the roles of K, L and substituting (T,t) by (T^{-1},t^{-1}) one proves the other inclusion. □

A quaternion-symbol equivalence (T,t) (of function fields K and L with respect to the pair $\big(\Omega(K)\setminus S(K),\, \Omega(L)\setminus S(L)\big)$) such that $t\big(\mathbb{E}_{S(K)} = \mathbb{E}_{S(L)}\big)$ will be, following [2], called $\big(S(K),S(L)\big)$-*integral*. Our main concern now is to prove the converse of the above corollary, i.e., to show that $\big(S(K),S(L)\big)$-integral equivalence preserves Witt rings of Dedekind domains. It is particularly easy in case of tame equivalence.

Proposition 4.3. *Let (T,t) be a tame quaternion-symbol equivalence of K and L with respect to the pair $\big(\Omega(K)\setminus S(K),\, \Omega(L)\setminus S(L)\big)$. If t maps $\mathbb{E}_{S(K)}$ onto $\mathbb{E}_{S(L)}$ then i_t maps $W\mathcal{O}_{S(K)}$ onto $W\mathcal{O}_{S(L)}$.*

Proof. Fix any $\varphi \in W\mathcal{O}_{S(K)} = \ker \partial_{S(K)}$. For any point $\mathfrak{p}$ not in $S(K)$ we have $\partial_{\mathfrak{p}}\varphi = 0$. Using Proposition 3.2 when $\mathfrak{p}$ is real and Proposition 3.4 when $\mathfrak{p}$ is complex we see that $\partial_{T\mathfrak{p}} i_t\varphi = 0$. The map T is a bijection, thus $\partial_{\mathfrak{q}} i_t\varphi = 0$ for all points $\mathfrak{q}$ not in $S(L)$. Consequently, $i_t\varphi \in \ker \partial_{S(L)} = W\mathcal{O}_{S(L)}$. Changing roles of K and L and substituting (T,t) with (T^{-1},t^{-1}) we arrive at the second implication. □

Now, when we know that tame equivalence suffices to show the converse of Corollary 4.2, we may apply our studies of tame equivalence in the previous section.

Lemma 4.4. *Let (T,t) be a quaternion-symbol equivalence of K and L with respect to the pair $\big(\Omega(K)\setminus S(K), \Omega(L)\setminus S(L)\big)$ preserving -1 (i.e., $t(-1) = -1$). If t maps $\mathbb{E}_{S(K)}$ onto $\mathbb{E}_{S(L)}$ then i_t maps $W\mathcal{O}_{S(K)}$ onto $W\mathcal{O}_{S(L)}$.*

Proof. Fix a quadratic form φ over K and assume that $\varphi \in W\mathcal{O}_{S(K)} \subset WK$. It follows from Knebusch–Milnor exact sequence (see [6, Corollary 3.3]) that $\partial_{\mathfrak{p}}\varphi = 0$ for any point $\mathfrak{p} \in \Omega(K)$ not in $S(K)$. We shall prove that $\partial_{\mathfrak{q}} i_t \varphi = 0$ for any $\mathfrak{q}$ not in $S(L)$. Take any point $\mathfrak{q} \in \Omega(L) \setminus S(L)$ and let $\mathfrak{p} := T^{-1}\mathfrak{q}$. We have $\partial_{\mathfrak{p}}\varphi = 0$. Of course $\mathfrak{p}$ is real if and only if $\mathfrak{q}$ is real. We have to consider two cases.

First assume that $\mathfrak{p}$ and $\mathfrak{q}$ are real. Theorem 3.1 implies that $\left(T|_{\gamma(K)\setminus S(K)}, t\right)$ is a tame quaternion-symbol equivalence with respect to the pair $\big(\gamma(K) \setminus S(K), \gamma(L) \setminus S(L)\big)$. Hence, Proposition 3.2 shows that $\partial_{\mathfrak{q}} i_t \varphi = 0$.

Next, suppose that $\mathfrak{p}, \mathfrak{q}$ are complex. Decompose again φ into $\varphi = \varphi_1 \perp \langle p \rangle \otimes \varphi_2$ with $\operatorname{ord}_{\mathfrak{p}} p \equiv 1 \pmod 2$ and φ_1, φ_2 having diagonalizations consisting of $\mathfrak{p}$-adic units. Now $\partial_{\mathfrak{p}}\varphi = 0$ implies $\dim \varphi_2 \equiv 0 \pmod 2$ and so $\operatorname{disc}\varphi \in \mathbb{E}_{S(K)}$. Thus, $\operatorname{disc} i_t \varphi = t(\operatorname{disc}\varphi) \in E_{S(L)}$. Observe that

$$WK_{\mathfrak{p}} = \{0, \langle 1 \rangle, \langle p \rangle, \langle 1, p \rangle\}, \qquad WL_{\mathfrak{q}} = \{0, \langle 1 \rangle, \langle p \rangle, \langle 1, p \rangle\},$$

where $\operatorname{ord}_{\mathfrak{p}} p \equiv \operatorname{ord}_{\mathfrak{q}} q \equiv 1 \pmod 2$. Consequently, $\operatorname{disc} i_t \varphi \in \mathbb{E}_{S(L)}$ implies that $\Theta_{\mathfrak{q}} i_t \varphi \in \{0, \langle 1 \rangle\}$. Therefore, $\partial_{\mathfrak{q}} i_t \varphi = 0$. □

Finally, Corollary 4.2 together with the above lemma give raise to our main result. The following theorem gives a necessary and sufficient condition for a strong Witt isomorphism (associated with the given quaternion-symbol equivalence) to map Witt ring of the ring $\mathcal{O}_{S(K)}$ onto the Witt ring of $\mathcal{O}_{S(L)}$.

THEOREM 4.5. *Let K, L be two real algebraic function fields with a common real closed field of constants. Let further (T, t) be a quaternion-symbol equivalence of K and L with respect to the pair $\big(\Omega(K) \setminus S(K), \Omega(L) \setminus S(L)\big)$ preserving -1. The following two conditions are equivalent:*

- *t maps $\mathbb{E}_{S(K)}$ onto $\mathbb{E}_{S(L)}$;*
- *i_t maps $W\mathcal{O}_{S(K)}$ onto $W\mathcal{O}_{S(L)}$ (in particular rings $\mathcal{O}_{S(K)}$ and $\mathcal{O}_{S(L)}$ are Witt equivalent).*

REFERENCES

[1] CZOGALA, A.: *Hilbert equivalence of global fields*, Równoważność Hilberta ciał globalnych, Pr. Nauk. Uniw. Śląsk., 1969, (2001). (In Polish)

[2] CZOGALA, A.: *Witt rings of Hasse domains of global fields*, J. Algebra **244** (2001), 604–630.

[3] KOPROWSKI, P.: *Witt equivalence of algebraic function fields over real closed fields*, Math. Z. **242** (2002), 323–345.

[4] KNEBUSCH, M.: *On algebraic curves over real closed fields. I.*, Math. Z. **150** (1976), 49–70.

[5] KNEBUSCH, M.: *On algebraic curves over real closed fields. II.*, Math. Z. **151** (1976), 189–205.

[6] MILNOR, J.—HUSEMOLLER, D.: *Symmetric Bilinear Forms*, Springer-Verlag, Berlin, 1973.

[7] PERLIS, R.—SZYMICZEK, K.—CONNER, P. E.—LITHERLAND, R.: *Matching Witts with global fields*, Contemp. Math. **155** (1994), 365–387.

[8] SZYMICZEK, K.: *Bilinear Algebra. An Introduction to the Algebraic Theory of Quadratic Forms*, Algebra Logic Appl., Vol. 7, Gordon and Breach Science Publishers, New York, 1997.

Received December 8, 2004

Institute of Mathematics
University of Silesia
ul. Bankova 14
PL–40-007 Katowice
POLAND

E-mail: pkoprowski@member.ams.org