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ON THE EXISTENCE OF DISCONTINUOUS DYNAMICAL SYSTEMS INDUCED BY FILIPPOV SOLUTION

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ABSTRACT. Generalized solutions of differential equations are presently utilized to analyse discontinuous mathematical models in many tasks of modern control engineering. In this paper, some conditions under which Filippov solutions create the dynamical system (flow) are shown. They constitute the basis of qualitative conclusions about the evolution of the state of control systems used in contemporary applications of the differential equations theory.

1. Introduction

The main role in contemporary control engineering plays a notion of a dynamical system. It arises from the use of different mathematical models dependent on various aspects of their behaviour being considered (problems of sensitivity analysis, optimal control synthesis, etc.). Thus, for example, variable structure systems with sliding mode and other nonlinear phenomena may be worked out. However, possible discontinuities of differential equation's right-hand side make it impossible to apply the classical concept of a dynamical system (flow). Its definition is based on the classical solution of differential equations, i.e., a function which is differentiable and fulfils an equation at every point of its domain. However, in many modern engineering applications, the classical solutions do not exist in all points of a domain. A well-known example [5] is the following equation which models a sliding process on the axis t or a bang-bang control

$$\dot{x}(t) = 1 - 2\,\mathrm{sgn}\big(x(t)\big)\,. \tag{1.1}$$

The solution exists only until it reaches the axis t.

The generalization of the classical approach for ordinary differential equations with discontinuous right-hand sides (called discontinuous dynamical systems hereinafter) constitutes Carathéodory and Filippov solutions [4], [6]. Such

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qualitative methods use the sets of mappings of the phase space into the phase space. Because of that, for the generalized solutions, the idea of a dynamical system exists in the qualitative theory of discontinuous differential equations.

In this paper, some conditions under which the Filippov solutions create the discontinuous dynamical system are shown. Thus, qualitative analysis of differential equations may be done with the use of the generalized solutions in Filippov sense.

2. Preliminaries

Let x denote a point of the space $\mathbb{R}^n$, (t,x) mean a point of the space $\mathbb{R}\times\mathbb{R}^n$, as well as introduce the following nomenclature: the (t,x)-space for the space $\mathbb{R}\times\mathbb{R}^n$, and the x-space for the space $\mathbb{R}^n$. The variable t belongs to the interval $I\subset\mathbb{R}$; this interval is non-degenerated, i.e., contains more than one point. The symbol G denotes a subset of the space $\mathbb{R}\times\mathbb{R}^n$; the point (t,x) belongs, therefore, to $G\subset I\times\mathbb{R}^n$. For details see [1], [4], [7], [8].

Definition 1. The function $f\colon G\to\mathbb{R}^n$ is called *piecewise continuous* if it is continuous throughout G except the set M such that $M\subset G$, of a zero measure, at which this function tends to finite limits.

Consider the autonomous (or non-autonomous) differential equation

$$\dot{x}(t)=f\big(x(t)\big)\quad \big(x(t)=\dot{f}\big(t,x(t)\big)\big)\,,\qquad x(t_0)=x_0\,, \tag{2.1}$$

where $t_0\in I$ and $f\colon G\to\mathbb{R}^n$ may have discontinuities with respect to the variable t and also to the phase variable x; namely, the function f is piecewise continuous on the domain G.

Definition 2. ([4]) Let the function $x\colon I\to\mathbb{R}^n$ be absolute continuous on every compact subinterval of I. Then x is called a *Filippov solution* (F-solution) of the differential equation (2.1) if

$$\dot{x}(t)\in\mathcal{F}f\big(t,x(t)\big)\quad \text{a.e. in } I\,, \tag{2.2}$$

where the Filippov operator $\mathcal{F}$ is defined as

$$\mathcal{F}f\big(t,x(t)\big)=\bigcap_{\varepsilon>0}\ \bigcap_{m(M)=0}\operatorname{conv}\big[f\big(t,(x(t)+\varepsilon B)\backslash M\big)\big]\,, \tag{2.3}$$

while $m(M)$ is the Lebesgue measure of the set M, $\operatorname{conv}[A]$ means a closure of the convex hull of the set A, and the open ε-ball in the space $\mathbb{R}^n$, centered at the point x is denoted by $x+\varepsilon B$.

The Filippov solution constitutes the most frequently used concept of solution of discontinuous differential equation. The Carathéodory solution is not adequate for discontinuities with respect to the phase variable x. The Krasowski and Hermes solutions constitute in practice an excessively broad class (for details see [4], [6]).

The existence problems and the problems on continuation of Filippov solutions of equation (2.1) were treated in many ways in order to establish sufficient conditions for some particular tasks; for example in [4], [5], [6], [3].

The existence problem for a Filippov solution of autonomous equation (2.1) was treated, particularly, in [3]; the existence problem for periodic Filippov solutions of equations like these, but with T-periodic right-hand side, was also investigated in [5].

Below, there is only a small collection of definition, theorems and remark, which establish some sufficient conditions for these problems; for details see [4], [5], [6], [3].

Let in the collection only, the right-hand side of inclusion (2.2) be denoted as $F\big(t, x(t)\big)$.

Definition 3. ([5]) A set-valued function $F\big(t, x(t)\big)$ is called to *satisfy the basic conditions* in a domain G, if for all $(t, x) \in G$ the set $F\big(t, x(t)\big)$ is nonempty, bounded and closed, convex, and the function F is upper semicontinuous in t, x.

Theorem 1. ([5]) *Let $F\big(t, x(t)\big)$ satisfy the basic conditions in the domain G. Then for any point $(t_0, x_0) \in G$ there exists a solution of the problem $\dot{x}(t) \in F\big(t, x(t)\big)$, $x(t_0) = x_0$. If the domain G contains a cylinder $Z\big(t_0 \le t \le t_0 + a, |x - x_0| \le b\big)$, the solution exists at least on the interval $t_0 \le t \le t_0 + d$, $d = \min\big\{a, \frac{b}{m}\big\}$, $m = \sup\limits_{Z}\big|F\big(t, x(t)\big)\big|$.*

Remark 1. ([5]) If the domain G contains a cylinder $Z'\big(t_0 - a \le t \le t_0$, $|x - x_0| \le b\big)$, the solution exists for $t_0 - d' \le t \le t_0$, where d' is expressed as in Theorem 1, in terms of a, b, and $m' = \sup\big|F\big(t, x(t)\big)\big|$ in Z'.

Definition 4. ([3]) The right-hand side function in inclusion $\dot{x}(t) \in F\big(t, x(t)\big)$ satisfies a *strengthened one-sided Lipschitz condition* with one-side Lipschitz condition constant $(\lambda_1, \lambda_2, \dots, \lambda_n)$ if the implication $x_i' > x_i'' \implies \zeta_i' \le \zeta_i'' + \lambda_i \|x' - x''\|$ is true for all $x', x'' \in \mathbb{R}^n$, $\zeta_i' \in F(x')$, $\zeta_i'' \in F(x'')$ and all components $i = 1, 2, \dots, n$.

Theorem 2. ([3]) *For the initial value problem $\dot{x}(t) \in F\big(t, x(t)\big)$, $x(t_0) = x_0$, a.e. in I, with F satisfying strengthened one-sided Lipschitz condition there exists at most one solution.*

Theorem 3. ([5]) *Let $F(t, x(t))$ satisfy the basic conditions in a closed bounded domain D. Then each solution of the inclusion $\dot{x}(t) \in F(t, x(t))$ lying within D can be continued on both sides of the domain D.*

To get the uniqueness of the Cauchy problem (2.1) the following idea can be used.

Let the right-hand side of the inclusion (2.2) satisfy the basic conditions, then by virtue of Theorem 1 and Remark 1 there exist solutions of two Cauchy problems on the intervals $t_0 \leq t \leq t_0 + d$, and $t_0 - d' \leq t \leq t_0$. Some closed interval, for example, $[\alpha, \beta]$, may be found such that $[\alpha, \beta] \supset (t_0 - d', t_0 + d)$. Each of these Cauchy problems has at most one solution on its interval. To prove this, one can use Theorem 2. By virtue of Theorem 3 these solution can be continued till the boundary of the interval $[\alpha, \beta]$. By Theorem 2 these solutions equal each other on the interval. Therefore, there is only one solution on the interval $[\alpha, \beta]$ for $t_0 \in [\alpha, \beta]$. Thus, the solution is unique.

The following theorem is the existence problem for periodic Filippov solutions.

Theorem 4. ([5]) *Let W be a bounded closed convex domain in $\mathbb{R}^n$, $\varphi(x) \in C^1$, $\varphi(x) \leq 0$ in domain W, $\varphi(x) = 0$ on the boundary ∂W of this domain. Let a set-valued function $F(t, x)$ satisfy the basic condition given in Definition 3, and*

$$F(t + l, x) \equiv F(t, x), \qquad x \in W, \tag{2.4}$$

$$\operatorname{grad} \varphi(x) \neq 0, \qquad \operatorname{grad} \varphi(x) \cdot y \leq 0, \tag{2.5}$$

for all $x \in \partial W$, $y \in F(t, x)$.

Then the inclusion $\dot{x}(t) \in F(t, x)$ in the domain W has a solution with a period l.

The notion of dynamical system (both continuous-time and discrete-time dynamical systems), which is the subject of investigations presented in this paper, will be defined below.

Definition 5. Let X be a linear space, $(A, +)$ be an Abelian semi-group, and let $\pi \colon A \times X \to X$. The triplet (X, A, π) is called a *semi-dynamical system* (a *semiflow*) if

$$\pi(0, x) = x \qquad \text{for } x \in X, \tag{2.6}$$

$$\pi(t, \pi(s, x)) = \pi(t + s, x) \qquad \text{for } x \in X,\ t, s \in A. \tag{2.7}$$

If $(A, +)$ is an Abelian group, then such triplet is called a *dynamical system* (a *flow*). If $A = \mathbb{Z}+ = \{0, 1, 2, \dots\}$, then such triplet is called a *discrete semi-dynamical system* (a *semi-cascade*). If $A = \mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$, then such triplet is called a *discrete dynamical system* (a *cascade*).

The notion of dynamical system has been generalized in Definition 5 to the case when the mapping π can also be discontinuous. Because of that, a dynamical system can model a broad class of physical objects in the phase space; for example, piecewise-continuous, piecewise-smooth, piecewise-linear physical devices or control systems. The property of π to be discontinuous is useful from the point of view of modern control engineering.

3. Main results

Theorem 5, which was introduced and proved below, establishes conditions for the existence of dynamical and semi-dynamical systems, when they are defined by the Filippov solutions of the autonomous differential equation (2.1) which may have a discontinuous right-hand side.

THEOREM 5. *Let $F(\cdot, x_0)$ be the unique Filippov solution of autonomous differential equation* (2.1)*, where the function f is piecewise continuous. Let the interval I be an Abelian group* (*the interval I be a semi-group*)*. Then the triplet $(\mathbb{R}^n, I, F)$, where $F(\cdot, x_0)$ means a unique Filippov solution of initial-value problem* (2.1)*, is a dynamical system* (*is a semi-dynamical system*)*.*

P r o o f . Suppose that there are unique Filippov solutions for $I = \mathbb{R}$ (when $I = [t_0, \infty)$). Below, conditions (2.6), (2.7) of Definition 5 will be checked.

At first, assume to the contrary that condition (2.6) is not fulfilled, that is $\exists x_0 \in X\colon \pi(0, x_0) \neq x_0$, i.e., $F(0, x) \neq x_0$. Then, the Filippov solution has not a property of uniqueness on this point, which contradicts the assumption. This proves that condition (2.6) is true.

Now, condition (2.7) will be investigated. By the assumptions of Theorem 5, condition (2.7) takes the form

$$F\big(t, F(s, x)\big) = F(t + s, x)\,. \tag{3.1}$$

It has been already assumed that there is the unique Filippov solution of the initial-value problem (2), where $I = \mathbb{R}$ (where $I = [t_0, \infty)$). Therefore, for any point $(t, x) \in G$ also $(t+s) \in I$. Let $F_1(t)$, $F_2(t)$ denote a left- and a right-hand side of equality (3.1), respectively, i.e.,

$$F_1(t) = F\big(t, F(s, x)\big)\,, \tag{3.2}$$

$$F_2(t) = F(t + s, x)\,. \tag{3.3}$$

Condition (2.6) has been already proved; therefore the following equalities are also true

$$F_1(0) = F\big(0, F(s, x)\big) = F(s, x)\,, \tag{3.4}$$

$$F_2(0) = F(0+s,x) = F(s,x)\,. \tag{3.5}$$

Thus, for $t = 0$ the functions F_1, F_2, being in fact Filippov solutions, have a common point. Since the assumption guarantees uniqueness of the Filippov solution, it may be concluded that $\forall(t,x) \in G\colon F_1(t) = F_2(t)$; hence, condition (2.7) is true. This completes the proof of Theorem 5. □

Note that the interval of the form $I = [t_0, \infty)$ is not a group, because the element $-a \in I$ does not exist for every $a \in I$. However, it is a semi-group.

Of course, also in periodic case by Theorem 5, if the function f is T-periodic in equation (2.1), then the system (2.1) for any x_0 generates a semidynamical system on $\mathbb{Z}+ = \{0, \pm1, \pm2, \dots\}$ in the sense, that the unique solution $F(t, x_0)$ of (2.1) is this dynamical system.

Example 1. The Filippov solutions of the closed loop control system, and the induced semi-dynamical system. Consider now the closed loop system defined by the following equations $\dot{x}(t) = 1 - u\bigl(x(t)\bigr)$, $x(t=0) = x_0$, $u\bigl(x(t)\bigr) = 2\operatorname{sgn}\bigl(x(t)\bigr)$, with $t \in [0,\infty]$. Thus, $\dot{x}(t) = \begin{cases} 3 & \text{for } x(t) < 0 \\ -1 & \text{for } x(t) \geq 0 \end{cases}$. The set of two elements $f(x_0 + \varepsilon B) = \{-1, 3\}$ is obtained by varying the quantity ε (see Def. 2.). Consequently, $\dot{x}(t) \in \mathcal{F}\bigl(f(x_0) \equiv [-1,3]$, because the closure of the convex hull of the set $\{-1,3\}$ is the closed interval. Let $F(t)$ denote the Filippov solutions of the Cauchy problem for the above mentioned equations of closed loop system. In this way, for some constants C_1, C_2:

$$x_0 \neq 0 \to x(t) \equiv F(t) = \begin{cases} 3t + C_1\,, \\ -t + C_2\,, \end{cases}$$

here, the Filippov solutions are the classical solutions,

$$x_0 = 0 \to F(t) = 0\,,$$

here, there exist only the Filippov solutions on the x-axis.

Let now $\pi(t,x)$ denote these Filippov solutions, i.e., $\pi(t,x) = F(t,x)$, thus the triplet $\bigl(X, [0,\infty), F(t,x)\bigr)$ is a semi-dynamical system.

4. Final remarks

For the discussion of connections between the different types of generalized solutions, see, e.g., [4], [6]. There are some theorems in monograph [5], and paper [3] which establish sufficient conditions for the existence and uniqueness of the Filippov solutions.

Modern control engineering takes into consideration the influence of discontinuities in real controlled objects (especially discontinuities with respect to the state variable) because of their practical importance. The presence of the dynamical systems, defined by the Filippov solutions of discontinuous mathematical models, is the basis of qualitative conclusions about the evolution of the state of the system under control. Below, an exemplary application of the problem investigated in this paper will be presented; for full presentation see [6]. A similar approach to the dynamical systems is described in [3], [9]. A geometrical meaning of the controlled dynamical systems can be found in [2].

Consider the following task, which is one of the basic problems in the practice of optimal control, especially for industrial robots and manipulators. Let the following differential equation, which expresses the second law of dynamics

$$\ddot{y}(t) = u(t) - w\ \operatorname{sgn}\big(\dot{y}(t)\big)\,, \qquad y(0) = x_{01}\,, \quad \dot{y}(0) = x_{02} \tag{4.1}$$

in case when the term $-w\operatorname{sgn}\big(\dot{y}(t)\big)$ represents a simple but discontinuous model of motion resistances, and u is a scalar control, be given. The time-optimal control problem consists here of reaching the fixed state point in a minimal and finite time. Like in the classical time-optimal problem, a switching curve can be defined as the proper solutions of the equation (4.1) for the target state, but these solutions depend on the value of the parameter w. In practice, however, it is impossible to obtain a precise estimation of the value, because the parameter w describes the whole array of physical phenomena. Therefore, the classical solution of the differential equation does not exist, and the generalized solutions in Filippov sense have to be used. A sliding mode phenomena, well known in engineering practice, appear in this case. It is impossible to leave out Filippov solutions, because a sliding mode is analysed through this type of solutions. In this paper, it was shown that such approach correctly describes optimal control problem in terms of systems theory; therefore, simultaneously, the basic concept of the dynamical system was defined properly.

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