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ON CERTAIN IDENTITIES FOR THE FIBONOMIAL COEFFICIENTS

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ABSTRACT. The Fibonomial coefficients belong to the category of generalized binomial coefficients. In this paper new identities for them are derived. These identities are related to the generating function of the kth powers of the Fibonacci numbers. Proofs are based on manipulation with these generating functions.

1. Introduction

Binomial coefficients are the most used tools in many combinatorial situations. Fontené in 1915 [1] and later Ward [10] introduced generalized binomial coefficients. Other authors used similar ways for this generalization. The basic idea lies in replacing the natural numbers by the terms of any sequence $\{A_n\}$ of real or complex numbers. The sequence $\{A_n\}$ is essentially arbitrary but it is required that $A_0 = 0$ and $A_n \neq 0$ for $n \geqslant 1$.

Fontené considered the generalized binomial coefficients defined by

$$\left\{ {n \atop k} \right\} = \frac{A_n A_{n-1} \dots A_{n-k+1}}{A_1 A_2 \dots A_k}$$

and he gave the fundamental recurrence for them in the form

$$\left\{ {n \atop k} \right\} - \left\{ {n-1 \atop k} \right\} = \left\{ {n-1 \atop k-1} \right\} \frac{A_n - A_k}{A_{n-k}} .$$

Setting $A_n = n$ we recover the ordinary binomial coefficients, while for $A_n = q^n - 1$ we obtain so-called Gauss q-binomial coefficients.

Jarden in [5] considered the general second order recurrence relation

$$y_{n+2} = g y_{n+1} - h y_n , \tag{1}$$

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where $h \neq 0$ and ε, ω are roots of its auxiliary equation. He denoted $U_n = \dfrac{\varepsilon^n - \omega^n}{\varepsilon - \omega}$ the solution of (1) for $\varepsilon \neq \omega$ and generalized binomial coefficients

$$\left\{ {m \atop j} \right\} = \frac{U_m U_{m-1} \dots U_{m-j+1}}{U_1 U_2 \dots U_j}\,, \quad \text{with} \quad \left\{ {m \atop 0} \right\} = 1\,.$$

He showed that the kth order recurrence relation

$$\sum_{j=0}^{k} (-1)^j \left\{ {k \atop j} \right\} h^{\frac{j}{2}(j+1)} z_{n+k-j} = 0$$

holds for the product z_n of the nth terms of $k-1$ sequences satisfying (1).

Torretto and Fuchs in [9] established the following identity for generalized binomial coefficients

$$\begin{aligned}\sum_{j=0}^{k} (-1)^j \left\{ {k \atop j} \right\} h^{\frac{j}{2}(j+1)}\, &U_{a_1+k-j} U_{a_2+k-j} \cdots U_{a_k+k-j}\, y_{n+k-j} \\ &= U_1 U_2 \cdots U_k\, y_{n+a_1+a_2+\dots+a_k+\frac{k}{2}(k+1)}\,,\end{aligned}$$

where n, a_1, $\dots$, a_k are any integers. Gould in [2] took a sequence $\{u_n\}$ of real numbers generated by

$$u_0 = 0\,, \quad u_1 = 1\,, \quad u_{n+2} = a u_{n+1} + b u_n \tag{2}$$

for integers $n \geqslant 0$, where a, b are real numbers. One may define the generalized factorial $[n]! = u_1 u_2 \cdots u_n$, with $[0]! = 1$, and for $n \geqslant k \geqslant 0$ the generalized binomial coefficients

$$\left\{ {n \atop k} \right\} = \frac{[n]!}{[k]!\,[n-k]!}\,.$$

Holte in [3] proved a basic recurrence for $n \geqslant k \geqslant 1$

$$\left\{ {n \atop k} \right\} = \left\{ {n-1 \atop k} \right\} u_{k+1} + b \left\{ {n-1 \atop k-1} \right\} u_{n-k-1}\,. \tag{3}$$

The validity of this recurrence shows that if a and b are integers, then the binomial coefficients are all integers. The Fibonomial coefficients correspond to the choice $u_n = F_n$, where F_n are the Fibonacci numbers defined by (2) for $a = b = 1$. In this case we use the symbol [.] instead of {.}. The Fibonacci numbers can be also expressed by Binet formula $F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}$, where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. Similarly, the Lucas numbers L_n satisfy the basic Fibonacci recurrence but $L_0 = 2$, $L_1 = 1$ and therefore $L_n = \alpha^n + \beta^n$.

Thus, the Fibonomial coefficients can be expressed for integers $n \geqslant k \geqslant 1$ as

$$\begin{bmatrix} n \\ k \end{bmatrix} = \prod_{i=0}^{k-1} \frac{F_{n-i}}{F_{k-i}} = \frac{F_n F_{n-1} \cdots F_{n-k+1}}{F_k F_{k-1} \cdots F_1},$$

with $\begin{bmatrix} n \\ 0 \end{bmatrix} = 1$ and $\begin{bmatrix} n \\ k \end{bmatrix} = 0$ for $n < k$. It is easy to write the key recurrence formula (3) in the form

$$\begin{bmatrix} n \\ k \end{bmatrix} = F_{k+1} \begin{bmatrix} n-1 \\ k \end{bmatrix} + F_{n-k-1} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}. \tag{4}$$

By addition of formula (4) and the symmetric formula for $\begin{bmatrix} n \\ n-k \end{bmatrix}$ we have the recurrence relation

$$2 \begin{bmatrix} n \\ k \end{bmatrix} = L_k \begin{bmatrix} n-1 \\ k \end{bmatrix} + L_{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix},$$

as $F_{k-1} + F_{k+1} = L_k$.

It is possible to get the following relation for the generating function of the integer powers of Fibonacci numbers

$$f_k(x) = \frac{\sum\limits_{i=0}^{k} \sum\limits_{j=0}^{i} (-1)^{\frac{j(j+1)}{2}} \begin{bmatrix} k+1 \\ j \end{bmatrix} F_{i-j}^k \, x^i}{\sum\limits_{i=0}^{k+1} (-1)^{\frac{i(i+1)}{2}} \begin{bmatrix} k+1 \\ i \end{bmatrix} x^i}, \tag{5}$$

as a special case of Horadam's result in [4].

Shannon obtained in [8] some special forms for the numerator and the denominator in the expression of the generating function $f_k(x)$. For any odd integer k it is easy to obtain

$$f_k(x) = 5^{-\frac{k-1}{2}} \sum_{j=0}^{\frac{k-1}{2}} \binom{k}{j} \frac{F_{k-2j}\, x}{1 - (-1)^j L_{k-2j}\, x - x^2} \tag{6}$$

and for any positive even integer k

$$f_k(x) = 5^{-\frac{k}{2}} \sum_{j=0}^{\frac{k-2}{2}} (-1)^j \binom{k}{j} \frac{2 - (-1)^j L_{k-2j}\, x}{1 - (-1)^j L_{k-2j}\, x + x^2} + \binom{k}{\frac{k}{2}} \frac{(-5)^{-\frac{k}{2}}}{1 - (-1)^{\frac{k}{2}} x} \tag{7}$$

after simplification of one of Shannon's results.

We will denote the denominator in (5) $D_{k+1}(x) = \sum\limits_{i=0}^{k+1} d_i x^i$, where integers $d_i = (-1)^{\frac{i}{2}(i+1)} \begin{bmatrix} k+1 \\ i \end{bmatrix}$ are terms of the sequence which was named as "signed

Fibonomial triangle" in Sloane's encyclopedia of sequences with ID Number A055870 (see [11]).

From (5), (6) and (7) we get the following generating functions

$$D_{k+1}(x) = \prod_{j=0}^{\frac{k-1}{2}} \left(1 - (-1)^j L_{k-2j}\, x - x^2\right)$$

for any odd positive integer k and

$$D_{k+1}(x) = \left(1 - (-1)^{\frac{k}{2}} x\right) \prod_{j=0}^{\frac{k}{2}-1} \left(1 - (-1)^j L_{k-2j}\, x + x^2\right)$$

for any even positive integer k.

These expressions are effective sources to finding of various relations for the Fibonomial coefficients (see, e.g., [6] and [7]).

Throughout the paper we adopt the conventions that the sum and the product over an empty set is 0 and 1, respectively, and that $\lfloor x \rfloor$ represents the greatest integer less than or equal to x.

2. The main results

The main aim of the paper is to prove the following theorems including some new identities for the Fibonomial coefficients.

Theorem 1. *Let k, m be any positive integers and A, B, C be any real numbers. Let $\{x_n\}$ be a sequence satisfying the recurrence*

$$x_n = A x_{n-1} + B x_{n-2} + C x_{n-3}\,,$$

with the initial conditions $x_0 = 0$, $x_1 = 0$, $x_2 = 1$. Then

$$\sum_{i=0}^{k+m} x_{i+2}\, (-1)^{\frac{n-i}{2}(n-i-1)} \Bigg((-1)^{n-i} \begin{bmatrix} k+1 \\ n-i \end{bmatrix} - A \begin{bmatrix} k+1 \\ n-i-1 \end{bmatrix} + (-1)^{n-i} B \begin{bmatrix} k+1 \\ n-i-2 \end{bmatrix} + C \begin{bmatrix} k+1 \\ n-i-3 \end{bmatrix} \Bigg) = 0\,.$$

Theorem 2. *Let k, m and $n > k+m+1$ be any positive integers, λ be any real number and $\vartheta(i,m,n) = (-1)^{i - \frac{im}{2}\left(2n+1-m(i+2)\right)}$. Then*

$$\sum_{i=0}^{\lfloor \frac{n}{m} \rfloor} \lambda^i \vartheta(i,m,n) \Bigg((-1)^{\frac{m}{2}(2n+1-m-2im)} \begin{bmatrix} k+1 \\ n-mi \end{bmatrix} + \lambda \begin{bmatrix} k+1 \\ n-(i+1)m \end{bmatrix} \Bigg) = 0\,.$$

3. The preliminary results

Lemma 1. *Let A, B, C be any real numbers. Let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be any sequences satisfying the recurrence*

$$u_n = Au_{n-1} + Bu_{n-2} + Cu_{n-3}\,,$$

with the initial conditions $x_0 = 0$, $x_1 = 0$, $x_2 = 1$, $y_0 = 0$, $y_1 = 1$, $y_2 = 0$ and $z_0 = 1$, $z_1 = 0$, $z_2 = 0$. Then

$$y_n = x_{n+1} - Ax_n \qquad \text{and} \qquad z_n = x_{n+2} - Ax_{n+1} - Bx_n\,.$$

P r o o f . We show the proof of the first relation only as the second one can be proved analogously. We use induction on n. It is easy to see that the equality holds for $n = 0$, $n = 1$ and $n = 2$. Suppose that the identity holds for $n-1$, $n-2$ and n. Then we have with respect to the definition of sequence $\{y_n\}$

$$\begin{aligned} y_{n+1} &= Ay_n + By_{n-1} + Cy_{n-2} \\ &= A(x_{n+1} - Ax_n) + B(x_n - Ax_{n-1}) + C(x_{n-1} - Ax_{n-2}) \\ &= (Ax_{n+1} + Bx_n + Cx_{n-1}) - A(Ax_n + Bx_{n-1} + Cx_{n-2}) \\ &= x_{n+2} - Ax_{n+1}\,. \end{aligned}$$

□

Lemma 2. *Let $\{a_n\}$, $\{b_n\}$ be any sequences of real numbers and $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ be any sequences satisfying the recurrence from Lemma 1. Then for $n > 2$*

$$b_n = a_n - Aa_{n-1} - Ba_{n-2} - Ca_{n-3} \tag{8}$$

if and only if

$$a_n = \sum_{i=0}^{n-3} x_{i+2}b_{n-i} + x_n a_2 + y_n a_1 + z_n a_0\,. \tag{9}$$

P r o o f . Let us show that identity (8) implies identity (9). Moving indexes in sums and using Lemma 1 we have

$$\begin{aligned} &\sum_{i=0}^{n-3} x_{i+2}b_{n-i} + x_n a_2 + y_n a_1 + z_n a_0 = \\ &= \sum_{i=0}^{n-3} x_{i+2}(a_{n-i} - Aa_{n-i-1} - Ba_{n-i-2} - Ca_{n-i-3}) + x_n a_2 + y_n a_1 + z_n a_0 \\ &= \sum_{i=0}^{n-3} x_{i+2}a_{n-i} - A\sum_{i=0}^{n-3} x_{i+2}a_{n-i-1} - B\sum_{i=0}^{n-3} x_{i+2}a_{n-i-2} - C\sum_{i=0}^{n-3} x_{i+2}a_{n-i-3} \end{aligned}$$

$$
\begin{aligned}
&+ x_n a_2 + y_n a_1 + z_n a_0 \\
&= \sum_{i=0}^{n-3} x_{i+2} a_{n-i} - A \sum_{i=1}^{n-2} x_{i+1} a_{n-i} - B \sum_{i=2}^{n-1} x_i a_{n-i} - C \sum_{i=3}^{n} x_{i-1} a_{n-i} \\
&+ x_n a_2 + y_n a_1 + z_n a_0 \\
&= \sum_{i=3}^{n-3} a_{n-i}(x_{i+2} - A x_{i+1} - B x_i - C x_{i-1}) + (x_2 a_n + x_3 a_{n-1} + x_4 a_{n-2}) \\
&- A(x_2 a_{n-1} + x_3 a_{n-2} + x_{n-1} a_2) - B(x_2 a_{n-2} + x_{n-1} a_1 + x_{n-2} a_2) \\
&- C(x_{n-1} a_0 + x_{n-2} a_1 + x_{n-3} a_2) + x_n a_2 + y_n a_1 + z_n a_0 \\
&= a_n x_2 + a_{n-1}(x_3 - A x_2) + a_{n-2}(x_4 - A x_3 - B x_2) \\
&+ a_2(x_n - A x_{n-1} - B x_{n-2} - C x_{n-3}) + a_1(-B x_{n-1} - C x_{n-2} + y_n) \\
&+ a_0(-C x_{n-1} + z_n) = a_n + a_1(A x_n - x_{n+1} + y_n) \\
&+ a_0(A x_{n+1} + B x_n - x_{n+2} + z_n) = a_n \,.
\end{aligned}
$$

Thus, this part of the statement is true and similarly we can prove the reversed implication. □

Corollary 1. *Let $\{a_n\}$ be any sequence of real numbers and $\{x_n\}$ be an arbitrary sequence from Lemma 2. Then for any nonnegative integer n*

$$
a_n = \sum_{i=0}^{n-3} x_{i+2}(a_{n-i} - A a_{n-i-1} - B a_{n-i-2} - C a_{n-i-3}) + x_n a_2 + (x_{n+1} - A x_n) a_1 + (x_{n+2} - A x_{n+1} - B x_n) a_0 \,.
$$

Proof. It is easy to see that the assertion holds for $n = 0,\ 1$ and 2 and that for $n > 2$ it follows from Lemma 1 and Lemma 2. □

Lemma 3. *Let $\{a_n\}$, $\{b_n\}$ be any sequences of real numbers. Let m be any positive integer and $\lambda \neq 0$ be any real number. Then for any positive integer $n \geqslant m$*

$$
a_n = b_n + \lambda b_{n-m} \tag{10}
$$

if and only if

$$
b_n = \sum_{i=0}^{\lfloor \frac{n-m}{m} \rfloor} (-\lambda)^i a_{n-mi} + (-\lambda)^{\lfloor \frac{n}{m} \rfloor} b_{n-m\lfloor \frac{n}{m} \rfloor} \,. \tag{11}
$$

P r o o f . We show only that identity (10) implies identity (11). It is obvious that the reversed implication can be proved similarly. We write successively

$$\begin{aligned}
&\sum_{i=0}^{\lfloor\frac{n-m}{m}\rfloor}(-\lambda)^i\big(b_{n-mi}+\lambda b_{n-(i+1)m}\big)+(-\lambda)^{\lfloor\frac{n}{m}\rfloor}b_{n-m\lfloor\frac{n}{m}\rfloor}\\
&\quad=\sum_{i=0}^{\lfloor\frac{n-m}{m}\rfloor}(-\lambda)^i b_{n-mi}-\sum_{i=0}^{\lfloor\frac{n}{m}\rfloor-1}(-\lambda)^{i+1}b_{n-m(i+1)}+(-\lambda)^{\lfloor\frac{n}{m}\rfloor}b_{n-m\lfloor\frac{n}{m}\rfloor}\\
&\quad=\sum_{i=0}^{\lfloor\frac{n}{m}\rfloor-1}(-\lambda)^i b_{n-mi}-\sum_{i=1}^{\lfloor\frac{n}{m}\rfloor}(-\lambda)^i b_{n-mi}+(-\lambda)^{\lfloor\frac{n}{m}\rfloor}b_{n-m\lfloor\frac{n}{m}\rfloor}\\
&\quad=b_n\,.
\end{aligned}$$

and this part of the proof is over. □

Corollary 2. *Let $\{b_n\}$ be any sequence of real numbers. Let m be any positive integer and $\lambda \neq 0$ be any real number. Then for any positive integer n*

$$b_n=\sum_{i=0}^{\lfloor\frac{n-m}{m}\rfloor}(-\lambda)^i\big(b_{n-mi}+\lambda b_{n-(i+1)m}\big)+(-\lambda)^{\lfloor\frac{n}{m}\rfloor}b_{n-m\lfloor\frac{n}{m}\rfloor}\,.$$

P r o o f . For $m = 1$ and $n = 0$ the assertion is obvious, for $m > 1$ and $n < m$ it follows from the fact that $\lfloor\frac{n}{m}\rfloor = 0$, and for $n \geqslant m$ it follows from Lemma 3. □

4. The proof of the main theorems

P r o o f o f T h e o r e m 1. First, we prove that the relation

$$d_n=\sum_{i=0}^{n-3}x_{i+2}\,p_{n-i}(k) \tag{12}$$

holds for any positive integer n. We define polynomials $P_{k+4}(x)=\sum\limits_{i=0}^{k+4}p_i(k)x^i$ as

$$P_{k+4}(x)=(1-Ax-Bx^2-Cx^3)D_{k+1}(x)=\Big(1-Ax-Bx^2-Cx^3\Big)\sum_{i=0}^{k+1}d_i x^i\,.$$

Expanding the product on the right side and comparing the expressions on the both sides we have

$$\begin{aligned}
p_0(k) &= d_0\,,\\
p_1(k) &= d_1 - A d_0\,,\\
p_2(k) &= d_2 - A d_1 - B d_0\,,\\
p_i(k) &= d_i - A d_{i-1} - B d_{i-2} - C d_{i-3}\,, \quad i = 3,\,4,\,\dots,\,k+1\,,\\
p_{k+2}(k) &= -A d_{k+1} - B d_k - C d_{k-1}\,,\\
p_{k+3}(k) &= -B d_{k+1} - C d_k\,,\\
p_{k+4}(k) &= -C d_{k+1}\,.
\end{aligned}$$

Setting $p_n(k) = 0$ for $n < 0$ and $n > k+4$ we obtain the general recurrence relation

$$p_n(k) = d_n - A d_{n-1} - B d_{n-2} - C d_{n-3}$$

for any nonnegative integer n. Now identity (12) follows from Lemma 1 and Lemma 2 with respect to $d_n = 0$ for $n < 0$. Corollary 1 implies

$$d_n = \sum_{i=0}^{n} x_{i+2}\big(d_{n-i} - A d_{n-i-1} - B d_{n-i-2} - C d_{n-i-3}\big)$$

and as $d_n = 0$ for $n > k+m$ we get the assertion of theorem by setting $d_i = (-1)^{\frac{i}{2}(i+1)} \begin{bmatrix} k+1 \\ i \end{bmatrix}$ and after simplification. $\square$

P r o o f o f T h e o r e m 2. We start the proof by deriving the relation

$$d_n = \sum_{i=0}^{\lfloor \frac{n}{m} \rfloor} (-\lambda)^i p_{n-mi}(k,m)\,. \tag{13}$$

Define polynomials $R_{k+m+1}(x) = \sum\limits_{i=0}^{k+m+1} p_i(k,m) x^i$ as

$$R_{k+m+1}(x) = (1+\lambda x^m) D_{k+1}(x) = (1+\lambda x^m) \sum_{i=0}^{k+1} d_i x^i\,.$$

Comparing the coefficients of polynomials in the previous equality we have

$$\begin{aligned}
p_i(k,m) &= d_i\,, \quad i = 0,\,1,\,\dots,\,m-1\,,\\
p_i(k,m) &= d_i + \lambda d_{i-m}\,, \quad i = m,\, m+1,\,\dots,\,k+1\,,\\
p_i(k,m) &= \lambda d_{k+1}\,, \quad i = k+2,\ k+3,\,\dots,\,k+m+1\,.
\end{aligned}$$

Setting $p_n(k,m) = 0$ for $n < 0$ and $n > k+m+1$ we have the general recurrence relation

$$p_n(k,m) = d_n + \lambda d_{n-m}$$

for any nonnegative integer n. Identity (13) follows from Lemma 5. Corollary 2 implies

$$\sum_{i=0}^{\lfloor \frac{n}{m} \rfloor} (-\lambda)^i (d_{n-mi} + \lambda d_{n-(i+1)m}) = 0$$

for $n > k+m+1$ and we get the assertion after simplification. □

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