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SEPARABILITY OF SPACES OF FUNCTIONS

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ABSTRACT. We consider separability of spaces of continuous function on a topological space.

Introduction

Separability of topological vector space is an important property, because polars of neighbourhoods of zero are not only compact but also metrizable in weak* topology [4, pp. 68–70]. It allows to use only sequences for describing weak* topology on weak* compact subsets of the dual X^*. For a topological space X the space of all continuous functions $C(X)$ is a vector space (under natural operations). In this paper we consider separability of $C(X)$ (or its subspaces) with respect to topologies which are described bellow.

1. Topology of uniform convergence

On the space $C(X)$ we define a metric

$$d(f,g) = \sup_{x\in X} \frac{|f(x)-g(x)|}{1+|f(x)-g(x)|}\,.$$

This metric defines so called topology of uniform convergence. Unfortunately, this topology need not be consistent with the multiplications by scalars. Namely, if the function $f\colon X \to \mathbb{R}$ is unbounded then for any n the function f/n is also unbounded and $d\big(f/n, 0\big) = 1$. It shows that multiplication by scalars is not continuous. So, this topology is not interesting from the point of view of functional analysis, if there is an unbounded continuous function on the space X. However, separability of $C(X)$ with respect to topology of uniform convergence may be solved, see [1, p. 369].

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Theorem 1.1. *Let X be a Tychonoff space. The space $C(X)$ is separable with respect to topology of uniform convergence if and only if X is a metrizable compact.*

If the space X is compact (more generally pseudocompact, i.e., any continuous function on X is bounded, see [2, p. 263]) then the topology of uniform convergence may be defined by the metric

$$d(f,g) = \sup_{x\in X} \left| f(x) - g(x) \right| .$$

2. Separability of $C_0(X)$ for a locally compact space X

A topological space X is said to be *locally compact* if it is Hausdorff and any point has a neighborhood with compact closure. Locally compact space is called *σ-compact* if it is a union of a sequence of compact sets.

Proposition 2.1. *For a locally compact topological space X the following properties are equivalent:*

(i) *X is metrizable and σ-compact,*

(ii) *X is separable metrizable,*

(iii) *X is second countable.*

Proof. Let X be a locally compact metrizable space and $X = \bigcup\limits_{n=1}^{\infty} K_n$ with compact K_n. Since K_n are metrizable compact, they are separable with a dense countable set $A_n \subset K_n$. The set $A = \bigcup\limits_{n=1}^{\infty} A_n$ is dense in X. Implication (ii)$\Rightarrow$(iii) is well known. Let X be second countable locally compact. Any locally compact space is regular, [2, p. 196]. If it is moreover second countable, it is metrizable, [2, p. 325]. Let $(U_n)_{n=1}^{\infty}$ be a countable base. From this base choose only those sets which have a compact closure $\overline{U}_n$. Since X is locally compact we obtain a new base. Clearly, X is σ-compact. □

For a locally compact (noncompact) space X, a real number L and a continuous function $f\colon X \to \mathbb{R}$ we write $\lim\limits_{n\to\infty} f(x) = L$ if for any $\varepsilon > 0$ there is a compact $K \subset X$ such that $\left| f(x) - L \right| < \varepsilon$ for all $x \notin K$. $\widetilde{C}(X)$ will denote the set of all continuous function which have $\lim\limits_{x\to\infty} f(x)$. $C_0(X)$ will denote the set of all $f \in \widetilde{C}(X)$ for which $\lim\limits_{x\to\infty} f(x) = 0$. $\widetilde{C}(X)$ and $C_0(X)$ are Banach spaces under natural operations and the norm $||f|| = \sup\limits_{x\in X} \left| f(x) \right|$. $C_0(X)$ is a closed subspace of $\widetilde{C}(X)$ and $\widetilde{C}(X) = \{\text{constants}\} + C_0(X)$.

Proposition 2.2. *For a locally compact noncompact space the space $\widetilde{C}(X)$ is separable if and only if so is $C_0(X)$.*

Let αX be the Alexandroff (one point) compactification of a space X, i.e., $\alpha X = \{\Omega\} \cup X$, where $\Omega \notin X$, see [2, p. 222].

Proposition 2.3. *Let X be a noncompact locally compact space. The space αX is metrizable if and only if X is second countable.*

Proof. If the compact αX is metrizable then it is second countable and so is X. Let X be second countable. Then there is a countable base $(U_n)_{n=1}^{\infty}$, members of which have compact closures $\overline{U}_n$. Open neighbourhoods of the point $\Omega \in \alpha X$ are of the form $\{\Omega\} \cup (X \setminus K)$, where K is a compact subset of X, [2, p. 222]. For such a set K we may choose finitely many sets U_{n_i} such that

$$K \subset \bigcup_{i=1}^{k} U_{n_i} \subset \bigcup_{i=1}^{k} \overline{U}_{n_i} .$$

The point Ω has a countable local base of the sets of the form $\{\Omega\} \cup \left(X \setminus \bigcup_{i=1}^{k} \overline{U}_{n_i}\right)$. The space αX is second countable and therefore metrizable. □

For any $f \in \widetilde{C}(X)$ put $f(\Omega) = \lim_{x \to \infty} f(x)$. Then we obtain an isometry between $\widetilde{C}(X)$ and $C(\alpha X)$.

Theorem 2.4. *For a locally compact noncompact topological space X the following properties are equivalent:*

(i) *X is metrizable and σ-compact,*

(ii) *X is separable metrizable,*

(iii) *X is second countable,*

(iv) *$\widetilde{C}(X)$ is separable,*

(v) *$C_0(X)$ is separable.*

3. Separability of $C(X)$ with compact-open topology

Let X be a Tychonoff space. Let $C(X)$ be a set of all continuous functions on X. Consider on $C(X)$ the topology of uniform convergence on compact subsets $K \subset X$ (compact-open topology). This topology is generated by a family of seminorms

$$p_K(f) = \sup_{x \in K} \left| f(x) \right| ,$$

where K is compact. Obviously, if X is compact then this topology coincide with topology of uniform convergence.

If Y is a subspace of X then $C(Y)$ may contain a function which can not be extended onto X. To see this, it is sufficient to take $X = \mathbb{R}$, $Y = \mathbb{Q}$ and $f\colon Y \to \mathbb{R}$, $f(y) = \frac{1}{y-\sqrt{2}}$. Nevertheless, the following theorem holds.

THEOREM 3.1. *Let X be a Tychonoff space and $Y \subset X$. If $C(X)$ is separable then $C(Y)$ is separable.*

Proof. Let $(f_n)_{n=1}^{\infty}$ be a dense sequence in $C(X)$. Let φ_n be the restriction of f_n onto Y. We denote by Φ the set of all functions

$$\sum_{i_1,i_2,\cdots,i_n=0}^{m} a_{i_1,i_2,\cdots,i_n} \varphi_1^{i_1} \varphi_2^{i_2} \cdots \varphi_n^{i_n}\,, \qquad \text{where } a_{i_1,i_2,\cdots,i_n} \in \mathbb{Q}\,.$$

Obviously, Φ is countable. We prove that Φ is dense in $C(Y)$. Let $K \subset Y$ be compact. Take $y_1, y_2 \in K$. There is a continuous function $f \in C(X)$ such that $f(y_1) = 0$ and $f(y_2) = 1$. Since the set $\{y_1, y_2\}$ is compact, there is a continuous f_n such that $\left|f(y_i) - f_n(y_i)\right| < \frac{1}{2}$ for $i = 1, 2$. It means that

$$f_n(y_1) < \frac{1}{2} \quad \text{and} \quad f_n(y_2) > \frac{1}{2} \qquad \text{i.e.,} \qquad \varphi_n(y_1) < \frac{1}{2} \quad \text{and} \quad \varphi_n(y_2) > \frac{1}{2}\,.$$

The sequence $(\varphi_n)_{n=1}^{\infty}$ separates the points of K. The Stone–Weierstrass theorem implies that the set Φ is dense in $C(K)$. Particularly, for any $f \in C(Y)$ and any $\varepsilon > 0$ there is a $\varphi \in \Phi$ such that

$$p_K(f - \varphi) < \varepsilon\,.$$

$\square$

COROLLARY 3.2. *Let X be a Tychonoff space and let $C(X)$ be separable. Then any compact $K \subset X$ is metrizable.*

Any separable metrizable space X is embeddable into the Hilbert cube $I^{\aleph_0}$, [2, p. 325].

COROLLARY 3.3. *If X is a separable metrizable space then $C(X)$ is separable.*

Remark. Take the set $\mathbb{Q}$ of all rationals with usual metric. The space $C(\mathbb{Q})$ is separable but not metrizable. Using the notion of Cantor–Bendixon rank, [3, p. 34], it may be proved that for any sequence $(K_n)_{n=1}^{\infty}$ of compact subsets of $\mathbb{Q}$ there is a compact $K \subset \mathbb{Q}$ such that $K \not\subset K_n$ for any n. But the following theorem shows that this contradicts the metrizability of $C(\mathbb{Q})$.

Theorem 3.4. *Let X be a Tychonoff space. Then $C(X)$ is metrizable if and only if there is a sequence of compact sets $(K_n)_{n=1}^{\infty}$ such that for any compact set $K \subset X$ there is an integer n for which $K \subset K_n$.*

Proof. Let $(K_n)_{n=1}^{\infty}$ be a sequence of compact sets such that for any compact set $K \subset X$ there is an integer n for which $K \subset K_n$. Then $p_K < p_{K_n}$ and $C(X)$ is countably normable and metrizable. Let $C(X)$ be metrizable. Then the zero has a local countable base of the form

$$B(K_n, \varepsilon_n) = \left\{ f \in C(X) \colon p_{K_n}(f) < \varepsilon_n \right\},$$

where K_n are compact and $\varepsilon_n > 0$. Assume there is a compact $K \subset X$ such that $K \not\subset K_n$ for all n. Take $x_n \in K \setminus K_n$. There is a continuous function $f_n \colon X \to \langle 0, 1 \rangle$ such that $f_n(x_n) = 1$ and f_n vanishes on K_n. We have

$$f_n \in B(K_n, \varepsilon_n),$$
$$f_n \notin B(K, 1).$$

Particularly $B(K_n, \varepsilon_n) \not\subset B(K, 1)$ for all n. □

Separability of a metrizable space X is not necessary for the separability of the space $C(X)$. Let X be a discrete space. Then any function $f \colon X \to \mathbb{R}$ is continuous and the topology of the uniform convergence on compacts coincides with the pointwise convergence, because any compact set $K \subset X$ is finite. Therefore, the space $C(X)$ may be identified with the Cartesian product $\mathbb{R}^X$ equipped with the Tychonoff topology.

Proposition 3.5. *Let X be a discrete space. The space $C(X)$ is separable if and only if $|X| \le \mathfrak{c}$.*

Proof. See [2, pp. 112 and 122]. □

Theorem 3.6. *Let X be a Tychonoff space for which there is an increasing sequence of metrizable compact sets $(K_n)_{n=1}^{\infty}$ such that for any compact set $K \subset X$ there is an integer n for which $K \subset K_n$. Then $C(X)$ is separable.*

Proof. Let n be a fixed integer. Since K_n is metrizable compact, the space $C(K_n)$ is separable. Take a dense sequence $(\varphi_m)_{m=1}^{\infty} \subset C(K_n)$. For $f \in C(X)$ let $f|_{K_n}$ be the restriction of f onto K_n. Let $R(K_n)$ be the set of all such restrictions. Then $R(K_n)$ is a ring which contains all constants, and separates the points of K_n. By the Stone–Weierstrass theorem $R(K_n)$ is dense in $C(K_n)$. For any natural i, m take a function $g \in R(K_n)$ such that $\left|\varphi_m(x) - g(x)\right| < 2^{-i}$ for all $x \in K_n$. The function g is a restriction of some $f \in C(X)$. Denote this function by $f_{n,m,i}$. Clearly, the triple sequence $(f_{n,m,i})_{n,m,i=1}^{\infty}$ is dense in $C(X)$. □

From Proposition 3.5 we see that conditions in the preceding theorem are only sufficient but not necessary for the separability of $C(X)$. Topological vector spaces allow to prove another theorems about separability of $C(X)$.

THEOREM 3.7. *Let X be a topological vector space the dual X^* of which separates the points of X and is separable with respect to the weak* topology. Then $C(X)$ is separable.*

P r o o f . Let $(\varphi_n)_{n=1}^{\infty}$ be a sequence of continuous linear functionals which is dense in the weak* topology. Take $x, y \in X$, $x \neq y$. Put $z = x - y \neq 0$. Since X^* separates the points of X, there is $f \in X^*$ for which $f(z) \neq 0$. We may assume $f(z) = 1$. The set

$$\left\{g \colon g \in X^*,\ \left|g(z)\right| < \frac{1}{2}\right\}$$

is a weak* neighbourhood of 0 and the set

$$\left\{g \colon g \in X^*,\ \left|f(z) - g(z)\right| < \frac{1}{2}\right\}$$

is a weak* neighbourhood of f. There is some φ_n from this neighbourhood, i.e.,

$$\frac{1}{2} > \left|f(z) - \varphi_n(z)\right| = \left|1 - \varphi_n(z)\right|,$$

which means $\frac{1}{2} < \varphi_n(z)$ and $0 \neq \varphi_n(z) = \varphi_n(x) - \varphi_n(y)$ So, the sequence $(\varphi_n)_{n=1}^{\infty}$ separates the points of X. For a compact set $K \subset X$ we use the Stone–Weierstrass theorem to the family of all functions

$$\sum_{i_1, i_2, \cdots, i_n = 0}^{m} a_{i_1, i_2, \cdots, i_n} \varphi_1^{i_1} \varphi_2^{i_2} \cdots \varphi_n^{i_n}, \qquad \text{where } a_{i_1, i_2, \cdots, i_n} \in \mathbb{Q},$$

and we see that this family is dense in $C(X)$. $\square$

THEOREM 3.8. *Let X be a separable topological vector space the dual X^* of which separates the points and is equipped with the weak* topology. Then $C(X^*)$ is separable.*

P r o o f . Let Y be X^* with the weak* topology. Then Y^* with the weak* topology may be identified with X equipped with weak topology. Since X is separable in the original topology, it is also separable in the weak topology. So, we may apply the preceding theorem. $\square$

If X is a separable Banach space then the last theorem may be also deduced from Theorem 3.6.

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