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ON TOPOLOGIES ASSOCIATED WITH THE LEBESGUE MEASURE

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ABSTRACT. The topologies of the density type are considered on the real line. Some properties of such topologies are investigated.

Let $\mathbb{N}$ be the set of positive integers and l stand for the Lebesgue measure in $\mathbb{R}$. Let $\mathcal{L}$ be the family of Lebesgue measurable sets and $\mathbf{L}$ be the family of l-null sets. For every $A \in \mathcal{L}$ and $x, m \in \mathbb{R}$ the set $A + x$ is a translation of the set A of x and mA is an extension of the set A with the coefficient m. The ordinary density topology in $\mathbb{R}$ is denoted by $\mathcal{T}_d$. The fact that for any sets $A, B \in \mathcal{L}$, $l(A \triangle B) = 0$ will be denoted by $A \sim B$.

Let us recall one deep result of the measure theory due to von Neumann and Maharam (cf. [Kh]).

Theorem 1. *Let $(X, \mathcal{S}, \mu)$ be a measure space and μ be a nonzero complete σ-finite measure. Then there exists a topology $\mathcal{T}_\mu$ on X such that*

1) *$(X, \mathcal{T}_\mu)$ is a Baire space,*
2) *$(X, \mathcal{T}_\mu)$ satisfies the countable chain condition,*
3) *the σ-algebra of all subsets of X having the Baire property coincides with $\mathcal{S}$,*
4) *the σ-ideal of all first category subsets of X coincides with the σ-ideal of μ-null sets.*

It is well-known that the density topology $\mathcal{T}_d$ is a von Neumann topology associated with the Lebesgue measure (see [O]). In this paper we show that there are many density type topologies on the real line $\mathbb{R}$ associated with the Lebesgue measure l. In this purpose let us pay attention to the following property

Remark 1. (cf. [CLO]) Let $A \in \mathcal{L}$ and $x \in \mathbb{R}$. Then

$$\lim_{h \to 0} \frac{l(A \cap [x-h, x+h])}{2h} = 1 \quad \text{if and only if} \quad \lim_{n \to \infty} \frac{l(A \cap [x - \frac{1}{n}, x + \frac{1}{n}])}{\frac{2}{n}} = 1 .$$

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It means that the ordinary density point can be described by using a fix sequence $\{\frac{1}{n}\}_{n\in\mathbb{N}}$.

Let $\mathcal{S}$ be the family of all unbounded and nondecreasing sequences of positive reals. Every sequence $\{s_n\}\in\mathcal{S}$ is denoted by $\langle s\rangle$. In the context of the previous Remark, we have the following

Definition 1. Let $\langle s\rangle\in\mathcal{S}$. We say that $x\in\mathbb{R}$ is a *density point* of a set $A\in\mathcal{L}$ with respect to a sequence $\langle s\rangle\in\mathcal{S}$ (an $\langle s\rangle$*-density point of* A) if

$$\lim_{n\to\infty}\frac{l(A\cap[x-\frac{1}{s_n},x+\frac{1}{s_n}])}{\frac{2}{s_n}}=1;$$

x is an $\langle s\rangle$*-dispersion* of A if x is an $\langle s\rangle$-density point of A';

x is a right $\langle s\rangle$*-dispersion* of A (*a left* $\langle s\rangle$*-dispersion* of A) if

$$\lim_{n\to\infty}\frac{l(A\cap[x,x+\frac{1}{s_n}])}{\frac{1}{s_n}}=1\quad\left(\lim_{n\to\infty}\frac{l(A\cap[x-\frac{1}{s_n},x])}{\frac{1}{s_n}}=1\right).$$

It is easy to check that x is an $\langle s\rangle$-density point of $A\in\mathcal{L}$ (an $\langle s\rangle$-dispersion of A, a right $\langle s\rangle$-dispersion of A etc.) if and only if 0 is an $\langle s\rangle$-density point an $\langle s\rangle$-dispersion, a right $\langle s\rangle$-dispersion etc.) of the set $A-x=\{a-x\colon a\in A\}$. Since the proof of the fact that 0 is a right $\langle s\rangle$-dispersion of some set A is usually very similar to the proof that 0 is a left $\langle s\rangle$-dispersion of A, we will only prove that 0 is a right $\langle s\rangle$-dispersion of A.

For any $\langle s\rangle\in\mathcal{S}$ and $A\in\mathcal{L}$ let

$$\Phi_{\langle s\rangle}(A)=\{x\in\mathbb{R}\colon x \text{ is } \langle s\rangle\text{-density point of } A\}$$

and

$$\Phi(A)=\{x\in\mathbb{R}\colon x \text{ is an ordinary density point of } A\}.$$

Proposition 1. *For any $A\in\mathcal{L}$ and $\langle s\rangle\in\mathcal{S}$ the set $\Phi_{\langle s\rangle}(A)$ is a $F_{\sigma\delta}$ set*

Proof. Let

$$F(x,n)=\frac{l(A\cap[x-\frac{1}{s_n},x+\frac{1}{s_n}])}{\frac{2}{s_n}}.$$

The function $F(\cdot,n)$ is continuous with respect to x. Moreover,

$$\Phi_{\langle s\rangle}(A)=\bigcap_{k=1}^{\infty}\bigcup_{m\in\mathbb{N}}\bigcap_{n\geqslant m}\{x\colon F(x,n)\geqslant 1-\tfrac{1}{k}\}.$$

Hence $\Phi_{\langle s\rangle}(A)\in F_{\sigma\delta}$. □

Proposition 2. *For every* $A, B \in \mathcal{L}, \langle s \rangle \in \mathcal{S}$ *we have*

1. $\Phi_{\langle s \rangle}(\emptyset) = \emptyset$, $\Phi_{\langle s \rangle}(\mathbb{R}) = \mathbb{R}$;
2. $\Phi_{\langle s \rangle}(A \cap B) = \Phi_{\langle s \rangle}(A) \cap \Phi_{\langle s \rangle}(B)$;
3. $A \sim B \implies \Phi_{\langle s \rangle}(A) = \Phi_{\langle s \rangle}(B)$;
4. $\Phi_{\langle s \rangle}(A) \sim A$;
5. $\Phi(A) \subset \Phi_{\langle s \rangle}(A)$.

Proof. The properties 1, 2, 3 and 5 are the immediate consequences of the definition of $\langle s \rangle$-density point. *Since* $\Phi(A) \subset \Phi_{\langle s \rangle}(A)$ we get, by the Lebesgue Density Theorem, that $l\big(A \setminus \Phi_{\langle s \rangle}(A)\big) = 0$. Moreover,

$$\Phi_{\langle s \rangle}\big(A \cap (\mathbb{R} \setminus A)\big) = \Phi_{\langle s \rangle}(A) \cap \Phi_{\langle s \rangle}(\mathbb{R} \setminus A) = \emptyset .$$

Hence

$$\Phi_{\langle s \rangle}(A) \subset \mathbb{R} \setminus \Phi_{\langle s \rangle}(\mathbb{R} \setminus A)$$

and

$$\Phi_{\langle s \rangle}(A) \setminus A \ \subset\ \mathbb{R} \setminus A \setminus \Phi_{\langle s \rangle}(\mathbb{R} \setminus A) \ \subset\ \mathbb{R} \setminus A \setminus \Phi(\mathbb{R} \setminus A) .$$

It implies that $l\big(\Phi_{\langle s \rangle}(A) \setminus A\big) = 0$. □

Corollary 1. *The operator* $\Phi \colon \mathcal{L} \to \mathcal{L}$ *is the lower density operator in the measure space* $(\mathbb{R}, \mathcal{L}, l)$.

By the general lifting theorem (see [LMZ]) we have

Theorem 2. *For every sequence* $\langle s \rangle \in \mathcal{S}$ *the family*

$$\mathcal{T}_{\langle s \rangle} = \big\{A \in \mathcal{L} \colon A \subset \Phi_{\langle s \rangle}(A)\big\}$$

forms a topology such that $\mathcal{T}_d \subset \mathcal{T}_{\langle s \rangle}$ *and* $\mathcal{T}_{\langle s \rangle}$ *is the von Neumann topology associated with the Lebesgue measure.*

By condition 5, c.f. Proposition 2, we obtained that $\mathcal{T}_d \subset \mathcal{T}_{\langle s \rangle}$, for $\langle s \rangle \in \mathcal{S}$. The next theorem is a characterization of equality.

Theorem 3. (*cf.* [W]) *Let* $\langle s \rangle \in \mathcal{S}$ *be a sequence.*

$$\mathcal{T}_d = \mathcal{T}_{\langle s \rangle} \quad \text{if and only if} \quad \liminf_{n \to \infty} \frac{s_n}{s_{n+1}} > 0 .$$

Proof. Necessity. Let $\langle s \rangle \in \mathcal{S}$ and $\liminf_{n\to\infty} \frac{s_n}{s_{n+1}} = \lambda > 0$. It is sufficient to show that $\Phi(A) = \Phi_{\langle s \rangle}(A)$ for any $A \in \mathcal{L}$. We will show that if 0 is a right $\langle s \rangle$-dispersion point of the set A then 0 is a right $\langle u \rangle$-dispersion point of the set A for arbitrary sequence $\langle u \rangle \in \mathcal{S}$.

Suppose that $\lim\limits_{n\to\infty} s_n \cdot l\big(A\cap[0,\frac{1}{s_n}]\big) = 0$. Let $\langle u\rangle \in \mathcal{S}$ and $\varepsilon > 0$. There exists $n_0 \in \mathbb{N}$ such that

$$s_n \cdot l\left(A\cap\left[0,\frac{1}{s_n}\right]\right) < \frac{\varepsilon\lambda}{2} \quad\text{and}\quad \frac{s_n}{s_{n+1}} > \frac{\lambda}{2} \qquad \text{for } n > n_0 .$$

There exists $k_0 \in \mathbb{N}$ such that $s_{n_0} \leqslant u_{k_0}$. Fix $k > k_0$. There exists $n \geqslant n_0$ such that $s_n \leqslant u_k \leqslant s_{n+1}$. Hence

$$\begin{aligned} u_k \cdot l\left(A\cap\left[0,\frac{1}{u_k}\right]\right) &\leqslant s_{n+1}\cdot l\left(A\cap\left[0,\frac{1}{s_n}\right]\right) \\ &= s_n \cdot l\left(A\cap\left[0,\frac{1}{s_n}\right]\right)\cdot\frac{s_{n+1}}{s_n} < \varepsilon\cdot\frac{\lambda}{2}\cdot\frac{2}{\lambda} = \varepsilon . \end{aligned}$$

Sufficiency. Let $\mathcal{T}_d = \mathcal{T}_{\langle s\rangle}$. We prove that $\liminf\limits_{n\to\infty} \frac{s_n}{s_{n+1}} > 0$. Let us suppose that $\liminf\limits_{n\to\infty}\frac{s_n}{s_{n+1}} = 0$. There exists a subsequence $\{s_{k_n}\}$ of $\langle s\rangle$ such that $\lim\limits_{n\to\infty}\frac{s_{k_n}}{s_{k_n+1}} = 0$. Define

$$A = \bigcup_{j=1}^{\infty}\left[\frac{1}{s_{k_j+1}}, \frac{1}{\sqrt{s_{k_j}\cdot s_{k_j+1}}}\right].$$

We will check that 0 is a right $\langle s\rangle$-dispersion point of A. Let $j_n = \min\{j\colon k_j+1>n\}$. Then $k_{j_n} \geqslant n$ and

$$\begin{aligned} s_n\cdot l&\left(\bigcup_{j=1}^{\infty}\left[\frac{1}{s_{k_j+1}}, \frac{1}{\sqrt{s_{k_j}\cdot s_{k_j+1}}}\right]\cap\left[0,\frac{1}{s_n}\right]\right) \\ &= s_n\cdot l\left(\bigcup_{j=j_n}^{\infty}\left[\frac{1}{s_{k_j+1}}, \frac{1}{\sqrt{s_{k_j}\cdot s_{k_j+1}}}\right]\cap\left[0,\frac{1}{s_n}\right]\right) \\ &\leqslant \frac{s_n}{\sqrt{s_{k_{j_n}}\cdot s_{k_{j_n}+1}}} \leqslant \frac{s_{k_{j_n}}}{\sqrt{s_{k_{j_n}}\cdot s_{k_{j_n}+1}}} \underset{n\to\infty}{\longrightarrow} 0 . \end{aligned}$$

It implies that 0 is a $\langle s\rangle$-dispersion point of the set $X = -A\cup A$. Let

$$B = \bigcup_{j=1}^{\infty}\left(\left(\frac{1}{\sqrt{s_{k_j}\cdot s_{k_j+1}}}, \frac{1}{s_{k_j}}\right)\cup\left(\frac{1}{s_{k_{j+1}}}, \frac{1}{s_{k_j+1}}\right)\right).$$

Then $A\cap B=\emptyset$, $A\cup B = \left(0,\frac{1}{s_{k_1}}\right)$ and putting $Y = -B\cup B\cup\{0\}$, we get that $Y\in\mathcal{T}_{\langle s\rangle}$.

Further, we will show that $Y \notin \mathcal{T}_d$ proving that 0 is not the right density point of Y. Namely,

$$\sqrt{s_{k_n} \cdot s_{k_n+1}} \cdot l\left(\bigcup_{j=1}^{\infty}\left(\frac{1}{\sqrt{s_{k_j} \cdot s_{k_j+1}}}, \frac{1}{s_{k_j}}\right) \cap \left[0, \frac{1}{\sqrt{s_{k_n} \cdot s_{k_n+1}}}\right]\right)$$
$$= \sqrt{s_{k_n} \cdot s_{k_n+1}} \cdot l\left(\bigcup_{j=n+1}^{\infty}\left(\frac{1}{\sqrt{s_{k_j} \cdot s_{k_j+1}}}, \frac{1}{s_{k_j}}\right) \cap \left[0, \frac{1}{\sqrt{s_{k_n} \cdot s_{k_n+1}}}\right]\right)$$
$$\leqslant \frac{\sqrt{s_{k_n} \cdot s_{k_n+1}}}{s_{k_{n+1}}} \leqslant \sqrt{\frac{s_{k_n}}{s_{k_{n+1}}}} \underset{n\to\infty}{\longrightarrow} 0\,.$$

Also,

$$\sqrt{s_{k_{n+1}} \cdot s_{k_{n+1}+1}} \cdot l\left(\bigcup_{j=1}^{\infty}\left(\frac{1}{s_{k_{j+1}}}, \frac{1}{s_{k_j+1}}\right) \cap \left[0, \frac{1}{\sqrt{s_{k_{n+1}} \cdot s_{k_{n+1}+1}}}\right]\right)$$
$$= \sqrt{s_{k_{n+1}} \cdot s_{k_{n+1}+1}} \cdot l\left(\bigcup_{j=n+1}^{\infty}\left(\frac{1}{s_{k_{j+1}}}, \frac{1}{s_{k_j+1}}\right) \cap \left[0, \frac{1}{\sqrt{s_{k_{n+1}} \cdot s_{k_{n+1}+1}}}\right]\right)$$
$$\leqslant \frac{\sqrt{s_{k_{n+1}} \cdot s_{k_{n+1}+1}}}{s_{k_{n+1}+1}} = \sqrt{\frac{s_{k_{n+1}}}{s_{k_{n+1}+1}}} \underset{n\to\infty}{\longrightarrow} 0\,. \qquad \square$$

Let $\mathcal{S}_0 = \left\{\langle s\rangle \in \mathcal{S} \colon \liminf_{n\to\infty} \frac{s_n}{s_{n+1}} = 0\right\}$. By the previous theorem we have

Corollary 2. *For every* $\langle s\rangle \in \mathcal{S} \setminus S_0$ *and every* $\langle u\rangle \in \mathcal{S}_0$

$$\mathcal{T}_{\langle s\rangle} \subsetneqq \mathcal{T}_{\langle u\rangle}\,.$$

In the following theorem the natural properties of $\mathcal{T}_{\langle s\rangle}$-topologies are listed.

Theorem 4. (i) $\underset{\langle s\rangle\in\mathcal{S}}{\forall}\ \underset{x\in\mathbb{R}}{\forall} \left(A \in \mathcal{T}_{\langle s\rangle} \Rightarrow A + x \in \mathcal{T}_{\langle s\rangle}\right)$,

(ii) $\underset{\langle s\rangle\in\mathcal{S}}{\forall} \left(A \in \mathcal{T}_{\langle s\rangle} \Rightarrow -A \in \mathcal{T}_{\langle s\rangle}\right)$,

(iii) $\underset{\langle s\rangle\in\mathcal{S}\setminus S_0}{\forall}\ \underset{m\in\mathbb{R}\setminus\{0\}}{\forall} \left(A \in \mathcal{T}_{\langle s\rangle} \Rightarrow mA \in \mathcal{T}_{\langle s\rangle}\right)$,

(iv) $\underset{\langle s\rangle\in\mathcal{S}}{\forall}\ \underset{|m|\geqslant 1}{\forall} \left(A \in \mathcal{T}_{\langle s\rangle} \Rightarrow mA \in \mathcal{T}_{\langle s\rangle}\right)$,

(v) $\underset{\langle s\rangle\in\mathcal{S}_0}{\forall}\ \underset{|m|<1}{\forall}\ \underset{A\in\mathcal{L}}{\exists} \left(A \in \mathcal{T}_{\langle s\rangle} \wedge mA \notin \mathcal{T}_{\langle s\rangle}\right)$.

P r o o f. Only the condition (v) needs to be proved. The others are the consequence of the definition of topology $\mathcal{T}_{\langle s\rangle}$ and Theorem 3. Let us assume that $\langle s\rangle \in \mathcal{S}_0$.

If $m = 0$ then we can put $A = \mathbb{R}$. Let $0 < m < 1$. We shall show that there exists a set $A \in \mathcal{L}$ such that 0 is an $\langle s \rangle$-dispersion point of A and is not an $\langle s \rangle$-dispersion point of mA. Let $\{s_{k_n}\}_{n\in\mathbb{N}}$ be a subsequence of $\langle s \rangle$ such that

$$\lim_{n\to\infty} \frac{s_{k_n}}{s_{k_n+1}} = 0 .$$

Let n_0 be such a positive integer that

$$\sqrt{\frac{s_{k_n}}{s_{k_n+1}}} < m$$

for any $n > n_0$. Putting

$$X = \bigcup_{j=n_0}^{\infty} \left[\frac{1}{s_{k_j+1}}, \frac{1}{\sqrt{s_{k_j} \cdot s_{k_j+1}}} \right]$$

we claim, similarly as in Theorem 3, that 0 is an $\langle s \rangle$-dispersion point of a set X. Hence, defining $Y = -X \cup X$, we get that $A = \{0\} \cup (\mathbb{R} \setminus Y) \in \mathcal{T}_{\langle s \rangle}$. Since

$$\frac{1}{s_{k_j+1}} < \frac{m}{\sqrt{s_{k_j} \cdot s_{k_j+1}}} \quad \text{and} \quad \frac{m}{s_{k_j+1}} < \frac{1}{s_{k_j+1}} \qquad \text{for any } j \geqslant n_0 ,$$

$$mA \supset \bigcup_{j=n_0}^{\infty} \left(\frac{m}{s_{k_j+1}}, \frac{1}{s_{k_j+1}} \right) \qquad \text{for any } j \geqslant n_0 .$$

Therefore

$$s_{k_n+1} \cdot l \left(mA \cap \left[0, \frac{1}{s_{k_n+1}} \right] \right) \geqslant s_{k_n+1} \cdot l \left(\bigcup_{j=n_0}^{\infty} \left(\frac{m}{s_{k_j+1}}, \frac{1}{s_{k_j+1}} \right) \cap \left[0, \frac{1}{s_{k_n+1}} \right] \right)$$

$$= l \left(\bigcup_{j=n_0}^{\infty} \left(\frac{m \cdot s_{k_n+1}}{s_{k_j+1}}, \frac{s_{k_n+1}}{s_{k_j+1}} \right) \cap [0,1] \right) \geqslant 1 - m > 0$$

for $n \geqslant n_0$. It means that 0 is not a right $\langle s \rangle$-dispersion point of the set mX and

$$mA = \{0\} \cup m(\mathbb{R} \setminus Y) = \{0\} \cup (\mathbb{R} \setminus mY) \notin \mathcal{T}_{\langle s \rangle} .$$

□

Corollary 3. *For every $\langle s \rangle \in \mathcal{S}_0$ and every $m \in (0,1)$ there exists a set $Y \in \mathcal{L}$ such that $\{0\} \cup (\mathbb{R} \setminus Y) \in \mathcal{T}_{\langle s \rangle}$ and $0 \notin \Phi_{\langle s \rangle}(\mathbb{R} \setminus mY)$.*

Theorem 5. *If* $\langle s\rangle, \langle t\rangle \in \mathcal{S}_0$ *and*

$$\lim_{n\to\infty} \frac{s_n}{t_n} = \alpha \in (0,\infty)\,,$$

then $\mathcal{T}_{\langle s\rangle} = \mathcal{T}_{\langle t\rangle}$ *if and only if* $\alpha = 1$.

P r o o f . Necessity. Let us suppose that $0 < \alpha < 1$. There exists $0 < m < 1$ and $n_0 \in \mathbb{N}$ such that for $n \geqslant n_0$, $s_n \leqslant m t_n$. By the previous corollary there exists a set Y such that $\{0\} \cup (\mathbb{R} \setminus Y) \in \mathcal{T}_{\langle t\rangle}$ and $0 \notin \Phi_{\langle t\rangle}(\mathbb{R} \setminus mY)$. Hence

$$0 \notin \Phi_{\langle mt\rangle}(\mathbb{R} \setminus Y)\,,$$

because

$$\Phi_{\langle mt\rangle}(\mathbb{R} \setminus Y) = \frac{1}{m}\Phi_{\langle t\rangle}(\mathbb{R} \setminus mY)\,.$$

We shall observe that $0 \notin \Phi_{\langle s\rangle}(\mathbb{R}\setminus Y)$ which implies that $\{0\}\cup(\mathbb{R}\setminus Y) \notin \mathcal{T}_{\langle s\rangle}$. Namely,

$$m\cdot t_n \cdot l\left(Y \cap \left[-\frac{1}{mt_n}, \frac{1}{mt_n}\right]\right) \leqslant m \cdot \frac{t_n}{s_n}\cdot s_n \cdot l\left(Y \cap \left[-\frac{1}{s_n}, \frac{1}{s_n}\right]\right).$$

Since $\liminf\limits_{n\to\infty}\, m\cdot t_n \cdot l\Big(Y \cap \big[-\frac{1}{mt_n}, \frac{1}{mt_n}\big]\Big) > 0$,

$$\liminf_{n\to\infty}\, m\cdot \frac{t_n}{s_n}\cdot s_n \cdot l\left(Y \cap \left[-\frac{1}{s_n}, \frac{1}{s_n}\right]\right) > 0$$

and

$$\liminf_{n\to\infty}\, s_n \cdot l\left(Y \cap \left[-\frac{1}{s_n}, \frac{1}{s_n}\right]\right) > 0\,.$$

If $\alpha > 1$ then $\lim\limits_{n\to\infty}\frac{t_n}{s_n} < 1$ and, as we have already proved, $\mathcal{T}_{\langle t\rangle} \neq \mathcal{T}_{\langle s\rangle}$.

Sufficiency. Let $\alpha = 1$. We will prove that $\mathcal{T}_{\langle s\rangle} = \mathcal{T}_{\langle t\rangle}$. Let $A \in \mathcal{L}$. We will show that

$$\Phi_{\langle s\rangle}(A) = \Phi_{\langle t\rangle}(A)\,.$$

On this purpose it is sufficient to prove that

$$\lim_{n\to\infty} t_n \cdot l\left(A \cap \left[0, \frac{1}{t_n}\right]\right) = 0$$

if and only if

$$\lim_{n\to\infty} s_n \cdot l\left(A \cap \left[0, \frac{1}{s_n}\right]\right) = 0\,.$$

Let
$$\lim_{n\to\infty} t_n \cdot l\left(A\cap\left[0,\frac{1}{t_n}\right]\right)=0\,.$$
We have
$$l\left(A\cap\left[0,\frac{1}{s_n}\right]\right)\leqslant l\left(A\cap\left[0,\frac{1}{t_n}\right]\right)+l\left(\left[\frac{1}{t_n},\frac{1}{s_n}\right]\right)=l\left(A\cap\left[0,\frac{1}{t_n}\right]\right)+\left|\frac{1}{s_n}-\frac{1}{t_n}\right|\,.$$
Hence
$$\begin{aligned} s_n\cdot l\left(A\cap\left[0,\frac{1}{s_n}\right]\right)&\leqslant s_n\cdot l\left(A\cap\left[0,\frac{1}{t_n}\right]\right)+s_n\left|\frac{1}{s_n}-\frac{1}{t_n}\right|\\ &=t_n\cdot l\left(A\cap\left[0,\frac{1}{t_n}\right]\right)\frac{s_n}{t_n}+\left|1-\frac{s_n}{t_n}\right|\,.\end{aligned}$$
Therefore
$$\lim_{n\to\infty} s_n \cdot l\left(A\cap\left[0,\frac{1}{s_n}\right]\right)=0\,.$$
Analogously, from $\lim_{n\to\infty} s_n\cdot l\left(A\cap\left[0,\frac{1}{s_n}\right]\right)=0$ we conclude that
$$\lim_{n\to\infty} t_n \cdot l\left(A\cap\left[0,\frac{1}{t_n}\right]\right)=0\,.$$
□

Remark 2. The previous property is not true if $\langle s\rangle,\langle t\rangle\in\mathcal{S}\setminus\mathcal{S}_0$. For example, if $\langle s\rangle=2^n$ and $\langle t\rangle=3^n$, then
$$\mathcal{T}_{\langle s\rangle}=\mathcal{T}_{\langle t\rangle}=\mathcal{T}_d \quad\text{and}\quad \lim_{n\to\infty}\frac{s_n}{t_n}=\frac{2}{3}\,.$$

Observing the proof of sufficiency we get

Corollary 4. *If $\langle s\rangle,\langle t\rangle\in\mathcal{S}$ and $\lim_{n\to\infty}\frac{s_n}{t_n}=1$ then $\mathcal{T}_{\langle s\rangle}=\mathcal{T}_{\langle t\rangle}$.*

Proposition 3. *For any $\langle s\rangle\in\mathcal{S}$ there exists a nondecreasing sequence $\langle t\rangle$ of natural numbers, such that $\mathcal{T}_{\langle s\rangle}=\mathcal{T}_{\langle t\rangle}$.*

Proof. Let $\langle s\rangle\in\mathcal{S}$ and let $\langle t\rangle=\{t_n\}_{n\in\mathbb{N}}$, where $t_n=[s_n]$, for $n\in\mathbb{N}$. Then $\lim_{n\to\infty}\frac{s_n}{t_n}=1$ and, by the previous corollary, $\mathcal{T}_{\langle s\rangle}=\mathcal{T}_{\langle t\rangle}$. □

Let
$$\mathcal{T}=\left\{\mathcal{T}_{\langle s\rangle}\colon \langle s\rangle\in\mathcal{S}\right\}.$$
It is clear that
$$\mathcal{T}=\{\mathcal{T}_d\}\cup\left\{\mathcal{T}_{\langle s\rangle}\colon \langle s\rangle\in\mathcal{S}_0\right\}.$$

Theorem 6. $\operatorname{card}(\mathcal{T}) = \mathfrak{c}$.

P r o o f . Naturally, $\operatorname{card}(\mathcal{T}) \leqslant \mathfrak{c}$. We will prove that $\operatorname{card}(\mathcal{T}) \geqslant \mathfrak{c}$.

If $\langle s\rangle \in \mathcal{S}_0$ then, for any $\alpha > 0$, $\langle \alpha s\rangle = \{\alpha s_n\}_{n\in\mathbb{N}} \in \mathcal{S}_0$. By Theorem 5 we obtain that, for any $0 < \alpha, \beta < 1$, such that $\alpha \neq \beta$ we have

$$\mathcal{T}_{\langle \alpha s\rangle} \neq \mathcal{T}_{\langle \beta s\rangle}\,.$$

Hence, $\operatorname{card}(\mathcal{T}) = \mathfrak{c}$. □

Theorem 7. $\bigcap\limits_{\langle s\rangle\in\mathcal{S}_0} \mathcal{T}_{\langle s\rangle} = \mathcal{T}_d$.

P r o o f . It is sufficient to show that

$$\bigcap_{\langle s\rangle\in\mathcal{S}_0} \mathcal{T}_{\langle s\rangle} \subset \mathcal{T}_d\,.$$

Let us suppose that there exists a set $X \in \mathcal{T}_{\langle s\rangle}$ for an arbitrary $\langle s\rangle \in \mathcal{S}_0$ and $X \notin \mathcal{T}_d$. Thus there exists a point $x_0 \in X$ such that

$$\liminf_{n\to\infty} \frac{l(X\cap[x_0-\frac{1}{n},\, x_0+\frac{1}{n}])}{\frac{2}{n}} < 1\,.$$

Therefore there exists a sequence $\langle s\rangle \in \mathcal{S}$ such that

$$\liminf_{n\to\infty} \frac{l(X\cap[x_0-\frac{1}{s_n},\, x_0+\frac{1}{s_n}])}{\frac{2}{s_n}} < 1\,.$$

It is easy to find a subsequence $\langle t\rangle \subset \langle s\rangle$ such that

$$\lim_{n\to\infty} \frac{l(X\cap[x_0-\frac{1}{t_n},\, x_0+\frac{1}{t_n}])}{\frac{2}{t_n}} < 1$$

and $\langle t\rangle \in \mathcal{S}_0$. It implies that $X \notin \mathcal{T}_{\langle t\rangle}$. □

Theorem 8. *Let $\mathcal{T}^*$ be a topology generated by $\bigcup\limits_{\langle s\rangle\in\mathcal{S}} \mathcal{T}_{\langle s\rangle}$. Then*

$$\bigcup_{\langle s\rangle\in\mathcal{S}} \mathcal{T}_{\langle s\rangle} \neq \mathcal{T}^* = 2^{\mathbb{R}}\,.$$

P r o o f . First notice, that

$$\left(\mathop{\forall}_{x\in A} \limsup_{h\to0^+} \frac{l(A\cap[x-h,x+h])}{2h} = 1\right) \implies A\in\mathcal{T}^* \qquad (*)$$

for an arbitrary set $A \in \mathcal{L}$. Suppose that

$$\limsup_{h\to 0^+} \frac{l\big(A\cap[x-h,x+h]\big)}{2h} = 1\,.$$

Then there exists a sequence $\{h_n\} \searrow 0$ such that

$$\lim_{n\to\infty} \frac{l(A\cap[x-h_n,x+h_n])}{2h_n} = 1\,.$$

Obviously,

$$\langle t\rangle = \left\{\frac{1}{h_n}\right\} \in \mathcal{S}\,.$$

Since $\mathcal{T}_d \subset \mathcal{T}_{\langle s\rangle}$ and $l\big(A\cap\Phi(A)\big) = l(A)$,

$$\big(A\cap\Phi(A)\big)\cup\{x\} \in \mathcal{T}_{\langle t\rangle} \subset \mathcal{T}^*.$$

Hence

$$A = \bigcup_{x\in A}\Big(\big(A\cap\Phi(A)\big)\cup\{x\}\Big) \in \mathcal{T}^*.$$

Further, we will show that the singletons are $\mathcal{T}^*$-open. There exists an interval set

$$B = \bigcup_{n=1}^{\infty}(a_n,b_n)\,,\qquad 0<b_{n+1}<a_n<b_n\,,\quad n\in\mathbb{N}\,,\quad \lim_{n\to\infty} a_n = 0\,,$$

such that

$$\lim_{n\to\infty}\frac{l(B\cap[0,b_n])}{b_n} = 1 \quad\text{and}\quad \lim_{n\to\infty}\frac{l(B\cap[0,a_n])}{a_n} = 0\,.$$

Let

$$C = -B\cup\{0\}\cup B$$

and

$$D = \bigcup_{n=1}^{\infty}\big((b_{n+1},a_n)\cup(-a_n,-b_n)\big)\cup\{0\}\,.$$

By the property $(*)$ we obtained that $C\in\mathcal{T}^*$ and $D\in\mathcal{T}^*$. Hence

$$\{0\} = C\cap D \in \mathcal{T}^*.$$

Similarly, taking into account that the topology $\mathcal{T}^*$ is invariant with respect to translation, we get that

$$\{x\} = (C+x)\cap(D+x) \in \mathcal{T}^*.$$

Then,

$$\mathcal{T}^* = 2^{\mathbb{R}}.$$

Since $\{0\}\notin\mathcal{T}^*$, for any $\langle s\rangle\in\mathcal{S}$, we have

$$\mathcal{T}^* \neq \bigcup_{\langle s\rangle\in\mathcal{S}} \mathcal{T}_{\langle s\rangle}\,.$$

$\square$

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