Tatra Mt. Math. Publ. **39** (2005), 35–45

TATRA MOUNTAINS
Mathematical Publications

CONVERGENCE WITH A REGULATOR IN LATTICE ORDERED GROUPS

Štefan Černák — Judita Lihová

ABSTRACT. The paper deals with Cauchy completions of archimedean lattice ordered groups with respect to a convergence of sequences with a fixed regulator.

B. Z. Vulikh [5], M. Luxemburg and A. Zaanen [3], and J. Martinez [4] studied the notion of a convergence with regulators in archimedean vector lattices and in archimedean lattice ordered groups, respectively. The considered type of convergence applies variable regulators, depending on the choice of sequences.

In this note, there is introduced the notion of a (u)-convergence in archimedean lattice ordered groups. It is a sort of convergence of sequences with a fixed regulator u, that is independent of the choice of a sequence. We define a (u)-Cauchy completion of an archimedean lattice ordered group and we deal with the existence (Theorem 3.15) and uniqueness of a such completion (Theorem 3.16).

1. Preliminaries

The standard terminology and notations for lattice ordered groups will be used. The group operation will be written additively. The properties of lattice ordered groups cited in this section can be found, e.g., in [1], [2].

Let $\mathbb{N}$ be the set of all positive integers and let G be a lattice ordered group. The absolute value of $x \in G$ is defined by $|x| = x \vee (-x)$. The inequalities

$$|x \vee a - y \vee a| \leq |x - y|, \quad |x \wedge a - y \wedge a| \leq |x - y| \tag{1}$$

2000 Mathematics Subject Classification: Primary 06F15; Secondary 06F20.
Keywords: lattice ordered group, vector lattice, convergence regulator, convergent sequence, fundamental sequence, Cauchy completion.

Supported by grant VEGA 1/0423/03.

are valid in G. Lattice ordered groups are torsion free, i.e., $x \neq 0$ implies $nx \neq 0$ for each $n \in \mathbb{N}$.

A lattice ordered group G is called archimedean if for each $0 < x \in G$ and each $0 < y \in G$ there exists $n \in \mathbb{N}$ such that $nx \not\leq y$. It is well-known that each archimedean lattice ordered group is abelian. An element $e \in G$ is said to be a strong unit of G if for each $a \in G$ there exists $n \in \mathbb{N}$ with $ne > a$.

Let G be an abelian lattice ordered group, $x, y \in G$. Then $n(x \vee y) = nx \vee ny$ for each $n \in \mathbb{N}$ and dually; $|x + y| \leq |x| + |y|$; if $n \in \mathbb{N}$ and $nx < ny$, then $x < y$.

2. (u)-convergence in lattice ordered groups

Let V be a vector lattice and let (x_n) be a sequence in V. B. Z. Vulikh (cf. [4]) defines (x_n) to r-converge to $x \in V$ if there exists an element $v \in V$, $v \geq 0$ such that the following condition is satisfied:

(c_1) For each real number $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ with

$$|x_n - x| \leq \varepsilon v \qquad \text{for each } n \in \mathbb{N},\ n \geq n_0 .$$

The element v is said to be a convergence regulator of (x_n). If V is archimedean, then limits are uniquely determined.

It is easy to verify that the condition (c_1) can be replaced by the following equivalent condition:

(c_2) For each $p \in \mathbb{N}$ there exists $n_0 \in \mathbb{N}$ with

$$p|x_n - x| \leq v \qquad \text{for each } n \in \mathbb{N},\ n \geq n_0 .$$

If in Vulikh's definition of r-convergence in V the condition (c_2) is used instead of (c_1), then the definition is suitable for using also in lattice ordered groups. However, we will apply a rather different definition of convergence in lattice ordered groups.

The difference consists in the fact that the regulator will be universal, i.e., it will be independent of the choice of a sequence.

Definition 2.1. Let G be a lattice ordered group, $u \in G$, $u > 0$. We will say that a sequence (x_n) in G (u)-*converges* to $x \in G$ or that x is a (u)-limit of (x_n), written $x_n \xrightarrow{(u)} x$, if for each $p \in \mathbb{N}$ there exists $n_0 \in \mathbb{N}$ such that

$$p|x_n - x| \leq u \qquad \text{for each } n \in \mathbb{N},\ n \geq n_0 .$$

The element u will be referred to as a *convergence regulator*.

Let $\mathbb{R}$ be the additive group of all reals with the natural linear order. If $G = \mathbb{R}$, then the notion of (u)-convergence for any $0 < u \in G$ coincides with the usual convergence.

Definition 2.1 of the (u)-convergence and the above definition of r-convergence are really different in lattice ordered groups. Evidently $x_n \xrightarrow{(u)} x$ implies that (x_n) r-converges to x, but the converse fails to hold in general, as the following example shows. We can also see that $x_n \xrightarrow{(u)} x$ need not imply the existence of (u')-limit of (x_n) for another $u' \in G$, $u' > 0$.

EXAMPLE 2.2. Let G be the additive group of all real functions defined on the set $\mathbb{R}$ of all real numbers, with the pointwise ordering. If $u(x) = 1$ for all $x \in \mathbb{R}$, then evidently $f_n \xrightarrow{(u)} f$ if and only if the sequence (f_n) uniformly converges to f, while for $u'(x) = x^2 + 1$ $(x \in \mathbb{R})$ $f_n \xrightarrow{(u')} f$ means that $\left(\frac{f_n}{u'}\right)$ uniformly converges to $\frac{f}{u'}$. Let $f_n(x) = \frac{1}{n}(x^2 + 1)$ for each $n \in \mathbb{N}$ and $x \in \mathbb{R}$. Then $f_n \xrightarrow{(u')} 0$, hence (f_n) r-converges to 0, but f_n does not (u)-converge at all.

THEOREM 2.3. *Let G be an archimedean lattice ordered group and let $0 < u \in G$. Then (u)-limits in G are uniquely determined.*

Proof. Assume that $x_n \xrightarrow{(u)} x$ and $x_n \xrightarrow{(u)} y$. Then for each $p \in \mathbb{N}$ there exists n_0 with

$$2p|x_n - x| \le u\,, \; 2p|x_n - y| \le u \qquad \text{for each } n \in \mathbb{N},\; n \ge n_0\,.$$

Hence

$$2p\big(|x_{n_0} - x| + |x_{n_0} - y|\big) \le 2u\,.$$

Then for each $p \in \mathbb{N}$ we get

$$p|x - y| \le p\big(|x_{n_0} - x| + |x_{n_0} - y|\big) \le u\,.$$

Since G is archimedean, we have $|x - y| = 0$ and thus $x = y$. □

THEOREM 2.4. *Let G be a non-archimedean lattice ordered group. Then there exists a convergence regulator u in G such that (u)-limits are not uniquely determined.*

Proof. The assumption implies that there exist elements $a, b \in G$, $a > 0$, $b > 0$ with $na \le b$ for each $n \in \mathbb{N}$. Set $u = b$. Consider the sequence $(x_n) = (a, 0, a, 0, \dots)$. For each $p \in \mathbb{N}$ and each $n \in \mathbb{N}$ we have

$$p|x_n - a| = \begin{cases} 0 < b & \text{if } n \text{ is odd}\,,\\ p\,|0 - a| = pa \le b & \text{if } n \text{ is even}\,. \end{cases}$$

Analogously, we get $p|x_n - 0| \le b$ for each $p \in \mathbb{N}$ and each $n \in \mathbb{N}$. We conclude that $x_n \xrightarrow{(u)} a$ and also $x_n \xrightarrow{(u)} 0$. □

3. Completion of an archimedean lattice ordered group

In this section, G will be an archimedean lattice ordered group and $0 < u \in G$ will be a fixed convergence regulator in G. We will define a Cauchy completion of G with respect to (u)-convergence. The existence and uniqueness of the Cauchy completion of G will be dealt with.

If there is no danger of confusion, the notation $x_n \to x$ (or $x_n \to x$ in G) will be applied instead of $x_n \xrightarrow{(u)} x$, and by a convergent sequence, a (u)-convergent sequence will be meant.

Lemma 3.1. *Let $x_n \to x$, $y_n \to y$. Then*

(i) $x_n + y_n \to x + y$,

(ii) $x_n \vee y_n \to x \vee y$,

(iii) $x_n \wedge y_n \to x \wedge y$.

Proof. Let $p \in \mathbb{N}$.

(i) There exists $n_0 \in \mathbb{N}$ with

$$2p|x_n - x| \le u\,,\ 2p|y_n - y| \le u \qquad \text{for each } n \in \mathbb{N},\ n \ge n_0\,.$$

Then

$$\begin{aligned} 2p\big|(x_n + y_n) - (x + y)\big| &= 2p\big|(x_n - x) + (y_n - y)\big| \\ &\le 2p\big(|x_n - x| + |y_n - y|\big) \le 2u\,. \end{aligned}$$

From this it follows

$$p\big|(x_n + y_n) - (x + y)\big| \le u \qquad \text{for each } n \in \mathbb{N},\ n \ge n_0\,.$$

Hence $x_n + y_n \to x + y$.

(ii) There exists $n_0 \in \mathbb{N}$ such that for each $n \in \mathbb{N}$, $n \ge n_0$ we have

$$-u \le p(x_n - x) \le u\,,\ -u \le p(y_n - y) \le u\,,$$
$$px - u \le px_n \le px + u\,,\ py - u \le py_n \le py + u\,.$$

So it is

$$\begin{aligned} (px \vee py) - u = (px - u) \vee (py - u) &\le px_n \vee py_n \\ &\le (px + u) \vee (py + u) = (px \vee py) + u\,, \end{aligned}$$

which implies

$$-u \le (px_n \vee py_n) - (px \vee py) \le u\,,$$

i.e.,

$$p\big|(x_n \vee y_n) - (x \vee y)\big| \le u\,.$$

Hence $x_n \vee y_n \to x \vee y$.

(iii) The proof is dual to that of (ii). □

Let $\mathbb{Z}$ be the set of all integers.

Lemma 3.2. *If $x_n \to x$, then $kx_n \to kx$ for each $k \in \mathbb{Z}$.*

Proof. Let $p \in \mathbb{N}$, $k \in \mathbb{Z}$. There exists $n_0 \in \mathbb{N}$ with

$$|k|p|x_n - x| \le u \qquad \text{for each } n \in \mathbb{N},\ n \ge n_0 .$$

We have

$$p|kx_n - kx| = |k|p|x_n - x| \le u \qquad \text{for each } n \in \mathbb{N},\ n \ge n_0 .$$

Therefore $kx_n \to kx$. □

Definition 3.3. A sequence (x_n) in G is said to be (u)-*fundamental* (or *fundamental*, for short) if for each $p \in \mathbb{N}$ there exists $n_0 \in \mathbb{N}$ such that

$$p|x_n - x_m| \le u \qquad \text{for each } m, n \in \mathbb{N},\ m \ge n \ge n_0 .$$

Lemma 3.4. *Let (x_n) be a convergent sequence in G. Then (x_n) is a fundamental sequence in G.*

Proof. Assume that $x_n \to x$. Let $p \in \mathbb{N}$. There exists $n_0 \in \mathbb{N}$ with

$$2p|x_n - x| \le u \qquad \text{for each } n \in \mathbb{N},\ n \ge n_0 .$$

Now let $m, n \in \mathbb{N}$, $m \ge n \ge n_0$. Then

$$2p|x_n - x_m| = 2p\big|(x_n - x) + (x - x_m)\big| \le 2p|x_n - x| + 2p|x_m - x| \le 2u .$$

Hence $p|x_n - x_m| \le u$ and thus (x_n) is fundamental. □

It can happen that a fundamental sequence in G is not convergent in G. In fact, it suffices to put $G = \mathbb{Q}$, where $\mathbb{Q}$ is the additive group of all rationals with the natural linear order. Let u be any rational, $u > 0$. Any sequence of rationals which converges to an irrational number in the usual sense, is such as we need.

Lemma 3.5. *Each fundamental sequence is bounded.*

Proof. Let (x_n) be a fundamental sequence. Then there exists $n_0 \in \mathbb{N}$ with

$$|x_n - x_m| \le u \qquad \text{for all } m, n \in \mathbb{N},\ m \ge n \ge n_0 .$$

So we have

$$x_{n_0} - u \le x_m \le x_{n_0} + u \qquad \text{for each } m \in \mathbb{N},\ m \ge n_0 .$$

Hence $x_1 \wedge \dots \wedge x_{n_0 - 1} \wedge (x_{n_0} - u)$ and $x_1 \vee \dots \vee x_{n_0 - 1} \vee (x_{n_0} + u)$ is a lower and an upper bound of (x_n), respectively. □

Definition 3.6. If every (u)-fundamental sequence in G is (u)-convergent, then G will be called (u)-*Cauchy complete* (shortly C-*complete*).

Definition 3.7. Let G be an archimedean lattice ordered group, $0 < u \in G$. An archimedean lattice ordered group H will be said to be a (u)-*Cauchy completion* (shortly C-*completion*) of G, if the following conditions are satisfied:

(a) G is an l-subgroup of H;

(b) H is (u)-Cauchy complete;

(c) Every element of H is a (u)-limit of some sequence in G.

Let F be the set of all fundamental sequences in G.

Lemma 3.8. *Let* $(x_n), (y_u) \in F$. *Then*

(i) $(x_n \vee y_n) \in F$,

(ii) $(x_n \wedge y_n) \in F$,

(iii) $(x_n + y_n) \in F$.

P r o o f. Let $p \in \mathbb{N}$. There exists $n_0 \in \mathbb{N}$ with

$$2p|x_n - x_m| \le u\,,\ 2p|y_n - y_m| \le u \qquad \text{for each } m, n \in \mathbb{N},\ m \ge n \ge n_0\,.$$

By using (1) we get

$$\begin{aligned} 2p|x_n \vee y_n - x_m \vee y_m| &= 2p|x_n \vee y_n - x_n \vee y_m + x_n \vee y_m - x_m \vee y_m| \\ &\le 2p|x_n \vee y_n - x_n \vee y_m| + 2p|x_n \vee y_m - x_m \vee y_m| \\ &\le 2p|y_n - y_m| + 2p|x_n - x_m| \le 2u\,. \end{aligned}$$

Hence

$$p\,|x_n \vee y_n - x_m \vee y_m| \le u \qquad \text{for each } m, n \in \mathbb{N},\ m \ge n \ge n_0\,.$$

Therefore $(x_n \vee y_n) \in F$.

The proof of (ii) is analogous and an easy verification establishes that (iii) holds, too. □

Let $(x_n), (y_n) \in F$. If we put $(x_n) + (y_n) = (x_n + y_n)$, then $(F, +)$ is a group. Further, if we set $(x_n) \le (y_n)$ if and only if $x_n \le y_n$ for each $n \in \mathbb{N}$, then $(F, +, \le)$ is an archimedean lattice ordered group; $(x_n) \vee (y_n) = (x_n \vee y_n)$ and dually.

Let E be the set of all sequences (x_n) in G such that $x_n \to 0$.

Lemma 3.9. E *is an* l-*ideal of* F.

P r o o f. According to 3.4 we get $E \subseteq F$. From 3.1 and 3.2 it follows that E is an l-subgroup of F. Now we show that E is convex in F. Let $(x_n) \in E$,

$(y_n) \in F$ with $0 \le (y_n) \le (x_n)$. Take any $p \in \mathbb{N}$. Then there exists $n_0 \in \mathbb{N}$ such that $px_n \le u$ for each $n \in \mathbb{N}$, $n \ge n_0$. Evidently $0 \le py_n \le px_n$ for each $n \in \mathbb{N}$, so that $py_n \le u$ whenever $n \in \mathbb{N}$, $n \ge n_0$. Therefore $(y_n) \in E$. □

In view of the preceding lemma we can form the factor group $G^* = F/E$, which is again a lattice ordered group. Let us denote by $(x_n)^*$ the class containing the sequence (x_n), i.e., $(x_n)^* = (x_n) + E$. Recall that $(x_n)^* \le (y_n)^*$ if and only if $(x'_n) \le (y'_n)$ for some $(x'_n) \in (x_n)^*$, $(y'_n) \in (y_n)^*$.

The following assertion is evident.

Lemma 3.10. *Let $(x_n), (y_n) \in F$. The following conditions are equivalent:*

(i) $(x_n)^* \le (y_n)^*$;

(ii) *for each $(x'_n) \in (x_n)^*$ there exists $(y'_n) \in (y_n)^*$ with $(x'_n) \le (y'_n)$;*

(iii) $(x_n) \le (y_n) + (t_n)$ *for some $(t_n) \in E$, $(t_n) \ge 0$.*

Further, it is $(x_n)^* + (y_n)^* = (x_n + y_n)^*$, $(x_n)^* \vee (y_n)^* = (x_n \vee y_n)^*$ and dually, $|(x_n)^*| = (|x_n|)^*$ for all $(x_n), (y_n) \in F$. Finally, if $(x_n) \in F$, then also $(x_k, x_{k+1}, x_{k+2}, \dots) \in F$, $(y_1, y_2, \dots, y_{k-1}, x_k, x_{k+1}, \dots) \in F$ and $(x_n)^* = (x_k, x_{k+1}, x_{k+2}, \dots)^* = (y_1, y_2, \dots, y_{k-1}, x_k, x_{k+1}, \dots)^*$ holds for any $k \in \mathbb{N}$ and $y_1, \dots, y_{k-1} \in G$.

It is well-known that the class of all archimedean lattice ordered groups is not closed with respect to homomorphic images. So it is not guaranteed that G^* is archimedean. Nevertheless, couples (G, u) with archimedean G^* form a proper class, as the following theorem shows. Namely, if G is any non-zero lattice ordered group and $0 < u \in G$, then u is a strong unit in the lattice ordered group $G_1 = \{a \in G\colon |a| \le nu \text{ for some } n \in \mathbb{N}\}$.

Theorem 3.11. *Let G be an archimedean lattice ordered group. If the regulator u is a strong unit of G, then the lattice ordered group G^* is archimedean.*

Proof. Let $0 \le (x_n)^* \in G^*$, $0 \le (y_n)^* \in G^*$. Suppose that $m(x_n)^* \le (y_n)^*$ holds for each $m \in \mathbb{N}$. We have to show that $(x_n)^* = 0$. We can suppose that $(x_n) \ge 0$, $(y_n) \ge 0$. By 3.10,

$$m(x_n) \le (y_n) + (t_n^m),$$

where $(t_n^m) \in E$, $(t_n^m) \ge 0$. As $(y_n) \in F$, by 3.5 there exists $a \in G$ with $y_n \le a$ for all $n \in \mathbb{N}$. The assumption that u is a strong unit implies that $ku > a$ for some $k \in \mathbb{N}$. Now let $p \in \mathbb{N}$. We have

$$p(k+1)(x_n) \le (y_n) + \left(t_n^{p(k+1)}\right).$$

Since $t_n^{p(k+1)} \to 0$, there exists $n_0 \in \mathbb{N}$ with $t_n^{p(k+1)} \le u$ for all $n \in \mathbb{N}$, $n \ge n_0$. Thus

$$p(k+1)x_n < ku + u = (k+1)u$$

for all $n \in \mathbb{N}$, $n \geq n_0$. Consequently $px_n < u$ for all $n \in \mathbb{N}$, $n \geq n_0$.
We have proved that $x_n \to 0$, i.e., $(x_n)^* = 0$. □

The question whether the assertion of 3.11 is true also without the assumption that u is a strong unit, remains open.

In what follows we will suppose that (G, u) has archimedean G^*.

Every constant sequence in G is evidently fundamental in G. For x and x_n in G let us denote $X = (x, x, \dots)^*$, $X_n = (x_n, x_n, \dots)^*$. The element $\mathcal{U} = (u, u, \dots)^*$ will be taken as a regulator in G^*. Let (Z^n) be a sequence in G^* and let $(z_n)^* \in G^*$. In the case that $Z^n \xrightarrow{(\mathcal{U})} (z_n)^*$, we shall write briefly $Z^n \to (z_n)^*$.

Lemma 3.12. *Let (t_n) be a sequence in G. If $t_n \to 0$ in G, then $T_n \to E$ in G^*.*

P r o o f. Let $p \in \mathbb{N}$. There exists $n_0 \in \mathbb{N}$ with $p|t_n| \leq u$ for each $n \in \mathbb{N}$, $n \geq n_0$. We have $p|T_n| = p\bigl|(t_n, t_n, \dots)^*\bigr| = p\bigl(|t_n|, |t_n|, \dots\bigr)^* \leq (u, u, \dots)^* = \mathcal{U}$ for each $n \in \mathbb{N}$, $n \geq n_0$. Hence $T_n \to E$ in G^*. □

Denote by F^* the set of all fundamental sequences in G^*.

Lemma 3.13. *Let (x_n) be a sequence in G. Then $(x_n) \in F$ if and only if $(X_n) \in F^*$.*

P r o o f. Let $p \in \mathbb{N}$. First assume that $(x_n) \in F$. There exists $n_0 \in \mathbb{N}$ with

$$p|x_n - x_m| \leq u \qquad \text{for each } m, n \in \mathbb{N},\ m \geq n \geq n_0 .$$

Hence

$$p\bigl(|x_n - x_m|, |x_n - x_m|, \dots\bigr) \leq (u, u, \dots),$$
$$p\bigl(|x_n - x_m|, |x_n - x_m|, \dots\bigr)^* \leq \mathcal{U}$$

and then we have

$$\begin{aligned} p|X_n - X_m| &= p\bigl|(x_n, x_n, \dots)^* - (x_m, x_m, \dots)^*\bigr| \\ &= p\bigl|(x_n - x_m, x_n - x_m, \dots)^*\bigr| = p\bigl(|x_n - x_m|, |x_n - x_m|, \dots\bigr)^* \\ &\leq \mathcal{U} \qquad \text{for each } m, n \in \mathbb{N},\ m \geq n \geq n_0 . \end{aligned}$$

Thus $(X_n) \in F^*$.

Now suppose that $(X_n) \in F^*$. Then there exists $n_0 \in \mathbb{N}$ with

$$2p|X_n - X_m| \leq \mathcal{U} \qquad \text{for each } m, n \in \mathbb{N},\ m \geq n \geq n_0 .$$

Consequently

$$2p\bigl(|x_n - x_m|, |x_n - x_m|, \dots\bigr)^* = 2p|X_n - X_m| \leq (u, u, \dots)^*$$

for each $m,n \in \mathbb{N}$, $m \ge n \ge n_0$.

Now let m,n be fixed, $m \ge n \ge n_0$. With respect to 3.10 there exists a sequence $0 \le (t_s) \in E$ (depending on p,m,n) such that

$$2p|x_n - x_m| \le u + t_s \qquad \text{for each } s \in \mathbb{N}.$$

As for $(t_s) \in E$, there exists $s_0 \in \mathbb{N}$ with $t_s \le u$ for each $s \in \mathbb{N}$, $s \ge s_0$. Set $\bar{t}_s = t_s$ for each $s \in \mathbb{N}$, $s \ge s_0$ and $\bar{t}_s = t_{s_0}$ for each $s \in \mathbb{N}$, $s < s_0$. Then $0 \le (\bar{t}_s) \in E$, $\bar{t}_s \le u$ for each $s \in \mathbb{N}$ and so

$$2p|x_n - x_m| \le u + \bar{t}_s \le 2u\,.$$

Hence

$$p|x_n - x_m| \le u \qquad \text{for each } m,n \in \mathbb{N},\ m \ge n \ge n_0$$

and so $(x_n) \in F$. □

Define the mapping $\varphi\colon G \to G^*$ as follows: $\varphi(x) = X$ for each $x \in G$. Then evidently φ is an isomorphism of the lattice ordered group G into G^*.

THEOREM 3.14. *Every element of G^* is a limit of some sequence in $\varphi(G)$.*

Proof. Assume that $(x_n)^* \in G^*$. Then (X_n) is a sequence in $\varphi(G)$ and we intend to show that $X_n \to (x_n)^*$ in G^*. Let $p \in \mathbb{N}$. Since $(x_n) \in F$, there exists $n_0 \in \mathbb{N}$ with $p|x_n - x_m| \le u$ for each $m,n \in \mathbb{N}$, $m \ge n \ge n_0$. Now let $k \in \mathbb{N}$, $k \ge n_0$. Then

$$\begin{aligned} p\big|X_k - (x_n)^*\big| &= p\big|(x_k, x_k, \dots)^* - (x_1, x_2, \dots)^*\big| \\ &= p\big|(x_k - x_1, x_k - x_2, \dots)^*\big| = p\big(|x_k - x_1|, |x_k - x_2|, \dots\big)^* \\ &= p\big(|x_k - x_k|, |x_k - x_{k+1}|, \dots\big)^* \le (u, u, \dots)^* = \mathcal{U}. \end{aligned}$$

The proof is finished. □

THEOREM 3.15. *G^* is a C-complete lattice ordered group.*

Proof. Let $X^1 = (x^1_m)^*$, $X^2 = (x^2_m)^*, \dots$ be a sequence from F^*. Take any $n \in \mathbb{N}$. As it was shown in the proof of 3.14, the sequence $(X^n_1, X^n_2, \dots)$, where $X^n_m = (x^n_m, x^n_m, \dots)^* \in \varphi(G)$ for each $m \in \mathbb{N}$, converges to X^n. Hence there exists $m_n \in \mathbb{N}$ such that $n|X^n_m - X^n| \le \mathcal{U}$ for each $m \in \mathbb{N}$, $m \ge m_n$. If we let run n over $\mathbb{N}$, we can choose $m_1 \le m_2 \le \cdots$. Put $Z_n = X^n_{m_n}$ for each $n \in \mathbb{N}$. Then $n|Z_n - X^n| \le \mathcal{U}$ for each $n \in \mathbb{N}$. Observe that if $n' \in \mathbb{N}$, $n' \le n$, then $n'|Z_n - X^n| \le \mathcal{U}$ is valid, too.

Now we show that $(Z_n) \in F^*$. Let $p \in \mathbb{N}$. Fundamentality of (X^n) implies that there exists $n_0 \in \mathbb{N}$, $n_0 \ge 3p$ with

$$3p|X^n - X^m| \le \mathcal{U} \qquad \text{for each } m,n \in \mathbb{N},\ m \ge n \ge n_0\,.$$

for each $m, n \in \mathbb{N}$, $m \geq n \geq n_0$ we get

$$3p|Z_n - Z_m| = 3p|Z_n - X^n + X^n - X^m + X^m - Z_m| \leq 3p|Z_n - X^n| + 3p|X^n - X^m| + 3p|X^m - Z_m| \leq 3\mathcal{U}.$$

Hence

$$p|Z_n - Z_m| \leq \mathcal{U} \qquad \text{for each } m, n \in \mathbb{N},\ m \geq n \geq n_0$$

and thus $(Z_n) \in F^*$.

If we denote $z_n = x^n_{m_n}$, then $Z_n = (z_n, z_n, \dots)^*$. Since $(Z_n) \in F^*$, 3.13 yields that $(z_n) \in F$ and $Z_n \to (z_n)^*$.

Let $p \in \mathbb{N}$. There exists $n_0 \in \mathbb{N}$, $n_0 \geq 2p$ such that for each $k \in \mathbb{N}$, $k \geq n_0$ we have

$$2p|X^k - (z_n)^*| = 2p|X^k - Z_k + Z_k - (z_n)^*| \leq 2p|X^k - Z_k| + 2p|Z_k - (z_n)^*| \leq 2\mathcal{U}$$

and hence

$$p|X^k - (z_n)^*| \leq \mathcal{U}.$$

We have proved that $X^n \to (z_n)^*$. □

In what follows x and $\varphi(x)$ will be identified for each $x \in G$. Then G is an l-subgroup of G^*. Using 3.15 and 3.14 we obtain the following result.

THEOREM 3.16. *If G^* is archimedean, then it is a C-completion of G.*

THEOREM 3.17. *Let G_1 and G_2 be (u)-Cauchy completions of G. Then there exists an isomorphism ϕ of the lattice ordered group G_1 onto G_2 leaving all elements of G fixed.*

Proof. Consider the conditions (a), (b) and (c) of 3.7. By (a), G is an l-subgroup of G_1 and G_2. Let $x^1 \in G_1$. Then (c) yields that there exists a sequence (x_n) in G with $x_n \to x^1$ in G_1. From 3.4 it follows that (x_n) is fundamental in G_1 and then also in G. By using (b) we get that there exists $x^2 \in G_2$ with $x_n \to x_2$ in G_2. We set $\phi(x^1) = x^2$.

First we want to verify that ϕ is correctly defined. Let (y_n) be another sequence in G with $y_n \to x^1$ in G_1. Analogously as above we obtain that there exists $y^2 \in G_2$ with $y_n \to y^2$ in G_2. Then 3.1 and 3.2 imply that $x_n - y_n \to 0$ in G_1 and $x_n - y_n \to x^2 - y^2$ in G_2. Consequently $x_n - y_n \to 0$ in G and hence $x_n - y_n \to 0$ in G_2. According to 2.3 we have $x^2 - y^2 = 0$, so that $x^2 = y^2$.

Let $x^2 \in G_2$. There exists a sequence (x_n) in G with $x_n \to x^2$ in G_2 and an element $x^1 \in G_1$ with $x_n \to x^1$ in G_1. Hence $\phi(x^1) = x^2$ and so ϕ is surjective.

Assume that $x^1, y^1 \in G_1$, $\phi(x^1) = x^2$, $\phi(y^1) = y^2$. There are sequences (x_n) and (y_n) in G with $x_n \to x^1$, $y_n \to y^1$ in G_1, $x_n \to x^2$, $y_n \to y^2$ in G_2.

Suppose that $x^2 = y^2$. Consequently $x_n - y_n \to 0$ in G and also in G_1. Since $x_n - y_n \to x^1 - y^1$ in G_1, we obtain $x^1 = y^1$. Therefore ϕ is injective.

According to 3.1 we have $x_n \wedge y_n \to x^1 \wedge y^1$ in G_1 and $x_n \wedge y_n \to x^2 \wedge y^2$ in G_2. Thus $\phi(x^1 \wedge y^1) = x^2 \wedge y^2 = \phi(x^1) \wedge \phi(y^1)$. Analogously we get $\phi(x^1 \vee y^1) = \phi(x^1) \vee \phi(y^1)$ and $\phi(x^1 + y^1) = \phi(x^1) + \phi(y^1)$.

It is evident that $\phi(x) = x$ for each $x \in G$. □

REFERENCES

[1] DARNEL, M. R.: *Theory of Lattice Ordered Groups*, M. Dekker, Inc. New York, 1995.

[2] FUCHS, L.: *Partially Ordered Algebraic Systems*, Pergamon Press, Oxford, 1963.

[3] LUXEMBURG, M.—ZAANEN, A.: *Riesz Spaces*, Vol. 1, North Holland, Amsterdam 1971.

[4] MARTINEZ, J.: *Polar Functions, III: On irreducible maps vs. essential extensions of archimedean l-groups with unit*, Tatra Mt. Math. Publ. **27** (2003), 189–211.

[5] VULIKH, B. Z.: *Introduction to the Theory of Partially Ordered Spaces*, Wolters-Noordhoff Scientific Publications, Groningen, 1967.

Received September 24, 2004

Štefan Černák
Department of Mathematics
Faculty of Civil Engineering
Technical University
Vysokoškolská 4
SK–042 02 Košice
SLOVAKIA
E-mail: svfkm@tuke.sk

Judita Lihová
Institute of Mathematics
Faculty of Science
P. J. Šafárik University
Jesenná 5
SK–041 54 Košice
SLOVAKIA
E-mail: lihova@science.upjs.sk

Mathematical Institute
Slovak Academy of Sciences
Grešákova 6
SK–040 01 Košice
SLOVAKIA