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FAMILIES OF ψ-APPROXIMATELY CONTINUOUS FUNCTIONS

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ABSTRACT. Families of ψ-approximately continuous functions for different ψ-density topologies are discussed. It is also constructed an example of pair of topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ such that there exists an interval set belonging to $\mathcal{T}_1 \setminus \mathcal{T}_2$ but families of $\mathcal{T}_1$-continuous and $\mathcal{T}_2$-continuous functions are equal.

Let $\mathbb{N}$ be the set of positive integers, $m(A)$ denotes the Lebesgue measure of a set $A \subset \mathbb{R}$, A'—a complement of A. Let $\mathcal{T}_0$ stand for the natural topology on $\mathbb{R}$. We will denote by C_0 the family of continuous functions $(\mathbb{R}, \mathcal{T}_0) \to (\mathbb{R}, \mathcal{T}_0)$ and by $\mathcal{A}$ the family of approximately continuous functions.

Between topologies $\mathcal{T}_0$ and $\mathcal{T}_d$ there are a lot of ψ-density topologies $\mathcal{T}_\psi$ introduced by M. Terepeta and E. Wagner-Bojakowska in [2]. Let us recall the basic definitions.

We denote by C the family of continuous, nondecreasing functions $\psi\colon (0,\infty) \to (0,\infty)$ such that $\lim\limits_{h\to 0+} \psi(h) = 0$. For any $\psi \in C$ we say that $x \in \mathbb{R}$ is a *ψ-density* (*ψ-dispersion*) *point* of a measurable set A if $\lim\limits_{h\to 0+} \frac{m(A'\cap[x-h,x+h])}{2h\cdot\psi(2h)} = 0$ $\left(\lim\limits_{h\to 0+} \frac{m(A\cap[x-h,x+h])}{2h\cdot\psi(2h)} = 0\right)$.

We denote

$$\Phi_\psi(A) = \{x \in \mathbb{R} : x \text{ is a } \psi\text{-density point of } A\}$$

and

$$\mathcal{T}_\psi = \{A \in \mathcal{S} \colon A \subset \Phi_\psi(A)\}.$$

For any $\psi \in C$ a topology $\mathcal{T}_\psi$ is bigger than $\mathcal{T}_0$ and smaller than $\mathcal{T}_d$ (cf. Theorem 1.4 in [2]). We say that a function $f\colon \mathbb{R} \to \mathbb{R}$ is *ψ-approximately continuous* at x_0 if x_0 is the inner ψ-density point of $f^{-1}\big(f(x_0)-\varepsilon, f(x_0)+\varepsilon\big)$ for each $\varepsilon > 0$, i.e., for any $\varepsilon > 0$ there exists a measurable set B such that $B \subset f^{-1}\big(f(x_0)-\varepsilon, f(x_0)+\varepsilon\big)$ and $x_0 \in \Phi_\psi(B)$. As usual, we will say that a function f is ψ-approximately continuous if it is ψ-approximately continuous at each point.

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THEOREM. (Theorem 3.6 in [2]) *A function* $f\colon \mathbb{R} \to \mathbb{R}$ *is* ψ*-approximately continuous if and only if a set* $f^{-1}\big((a,b)\big)$ *belongs to* $\mathcal{T}_\psi$ *for every open interval* (a,b).

It is known that there exist different functions $\psi_1, \psi_2 \in C$ such that $\mathcal{T}_{\psi_1} = \mathcal{T}_{\psi_2}$. We will show that if $\mathcal{T}_{\psi_1} \setminus \mathcal{T}_{\psi_2} \neq \emptyset$, then there exist an interval set $A \in \mathcal{T}_{\psi_1} \setminus \mathcal{T}_{\psi_2}$ and a function $f\colon \mathbb{R} \to \mathbb{R}$ which is ψ_1-approximately continuous and is not ψ_2-approximately continuous function.

PROPOSITION 1. *If* $\mathcal{T}_{\psi_1} \setminus \mathcal{T}_{\psi_2} \neq \emptyset$, *then there exists an interval set* $A = \bigcup_{n=1}^{\infty} [a_n, b_n]$ *with* $0 < b_{n+1} < a_n < b_n$ *for* $n \in \mathbb{N}$ *and* $\lim_{n\to\infty} b_n = 0$ *such that zero is a* ψ_1*-dispersion point of* A *but*

$$\liminf_{n\to\infty} \frac{m\big(A \cap [0, b_n]\big)}{2b_n \cdot \psi_1(2b_n)} > 0 .$$

P r o o f. Since $\mathcal{T}_{\psi_1} \setminus \mathcal{T}_{\psi_2} \neq \emptyset$, there exists a measurable set E such that $E \subset \Phi_{\psi_1}(E)$ and a point $x \in E$ such that $x \notin \Phi_{\psi_2}(E)$. We can assume that $x = 0$. Hence, there exists a sequence $(t_n) \searrow 0$ and a number $a > 0$ such that

$$\frac{m\big(E' \cap [0, t_n]\big)}{2t_n \cdot \psi_2(2t_n)} > a$$

or a sequence $(t_n) \searrow 0$ and a number $a > 0$ such that

$$\frac{m\big(E' \cap [-t_n, 0]\big)}{2t_n \cdot \psi_2(2t_n)} > a .$$

Suppose that the first possibility holds. Let for any $n \in \mathbb{N}$, $b_n = t_n$, $a_n = t_n - m\big(E' \cap [t_{n+1}, t_n]\big)$ and

$$A = \bigcup_{n=1}^{\infty} [a_n, b_n] .$$

Naturally, for any $n \in \mathbb{N}$, $m\big(A \cap [0, b_n]\big) = m\big(E' \cap [0, t_n]\big)$. Moreover, for any $t \in (0, t_1]$, $m\big(A \cap [0, t]\big) \leq m\big(E' \cap [0, t]\big)$, what completes the proof.

If the second possibility holds, we can use the set

$$-E = \{x \in \mathbb{R} \colon -x \in E\} . \qquad \square$$

THEOREM 2. *If $\mathcal{T}_{\psi_1} \setminus \mathcal{T}_{\psi_2} \neq \emptyset$, then there exists a function which is ψ_1-approximately continuous and is not ψ_2-approximately continuous.*

Proof. From Proposition 1 we know that there exists an interval set $A = \bigcup_{n=1}^{\infty} [a_n, b_n]$ such that zero is a ψ_1-dispersion point of A but

$$\liminf_{n\to\infty} \frac{m\big(A \cap [0, b_n]\big)}{2b_n \cdot \psi_2(2b_n)} > 0 .$$

Therefore there exist $a > 0$ and $n_0 \in \mathbb{N}$ such that

$$\frac{m\big(A \cap [0, b_n]\big)}{2b_n \cdot \psi_2(2b_n)} > a$$

for every $n \geq n_0$.

Set, for any $n \in \mathbb{N}$, $c_n = a_n + \frac{b_n - a_n}{4}$, $d_n = a_n + \frac{3}{4}(b_n - a_n)$. Of course, $d_n - c_n = \frac{1}{2}(b_n - a_n)$. Let

$$C = \bigcup_{n=1}^{\infty} [c_n, d_n] .$$

For any B such that $C \subset B \subset A$ and any $h > 0$

$$m\big(B \cap [0,h]\big) \leq m\big(A \cap [0,h]\big)$$

and zero is a ψ_1-dispersion point of B. Moreover, for every $n \geq n_0$,

$$\frac{m\big(B \cap [0, b_n]\big)}{2b_n \cdot \psi_2(2b_n)} \geq \frac{m\big(C \cap [0, b_n]\big)}{2b_n \cdot \psi_2(2b_n)} > \frac{a}{2} ,$$

so zero is not a ψ_2-dispersion point of B.

We will define the function f by the formula

$$f(t) = \begin{cases} 0 \ \text{ for } t \leq 0 \text{ or } t \geq b_1 , & \\ 1 \ \text{ for } t \in [c_n, d_n] , & n = 1, 2, \dots , \\ 0 \ \text{ for } t \in [b_{n+1}, b_n] , & n = 1, 2, \dots , \\ \text{linear on intervals } [a_n, c_n] \text{ and } [d_n, b_n] \text{ for} & n = 1, 2, \dots \end{cases}$$

The function f is continuous at every point x different from zero. So, it is ψ_1- and ψ_2-approximately continuous.

To prove that f is ψ_1-approximately continuous at zero, it is sufficient to observe that for any $\varepsilon > 0$ the set

$$B = \mathbb{R} \setminus f^{-1}\big((-\varepsilon, \varepsilon)\big)$$

is measurable and contained in A, so zero is a ψ_1-density point of $f^{-1}\big((-\varepsilon,\varepsilon)\big)$. On the other hand,

$$\mathbb{R}\setminus f^{-1}\left(\left(\frac{1}{2},\frac{1}{2}\right)\right)\supset C\,,$$

and zero is not a ψ_2-density point of $f^{-1}\left(\left(\frac{1}{2},\frac{1}{2}\right)\right)$. Hence f is not ψ_2-approximately continuous. □

The fact that there exists an interval set A such that $\mathbb{R}\setminus A\in\mathcal{T}_1\setminus\mathcal{T}_2$ plays the fundamental role in the previous proof. However, the existence of such a set is not a sufficient condition for the existence a function continuous from $\mathbb{R}$ equipped with topology $\mathcal{T}_1$ to $\mathbb{R}$ with natural topology, which is not continuous as a function from $(\mathbb{R},\mathcal{T}_2)$ to $(\mathbb{R},\mathcal{T}_0)$.

Theorem 3. *For any interval set* $A=\bigcup\limits_{n=1}^{\infty}[a_n,b_n]$ *with* $b_{n+1}<a_n<b_n$, $\lim\limits_{n\to\infty}b_n=0$, *there exist topologies* $\mathcal{T}_1$ *and* $\mathcal{T}_2$ *such that*

(a) $\mathcal{T}_0\subset\mathcal{T}_1\subset\mathcal{T}_2\subset 2^{\mathbb{R}}$,

(b) $\mathbb{R}\setminus A\in\mathcal{T}_2\setminus\mathcal{T}_1$,

(c) *families of continuous functions* $(\mathbb{R},\mathcal{T}_1)\to(\mathbb{R},\mathcal{T}_0)$ *and* $(\mathbb{R},\mathcal{T}_2)\to(\mathbb{R},\mathcal{T}_0)$ *are equal.*

P r o o f . Define topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ using bases of neighbourhoods at points.

Let for every $x\neq 0$

$$B_1(x)=\big\{(x-\varepsilon,x+\varepsilon)\colon\varepsilon>0\big\},\quad B_1(x)=B_2(x)\,,$$

and

$$B_1(0)=\left\{(-\varepsilon,\varepsilon)\setminus C\colon C=\bigcup_{n=1}^{\infty}[c_n,d_n]\,,\ \lim_{n\to\infty}d_n=0\,,\ C\subset\bigcup_{n=1}^{\infty}(a_n,b_n)\right\},$$

$$B_2(0)=\left\{(-\varepsilon,\varepsilon)\setminus C\colon C=\bigcup_{n=1}^{\infty}[c_n,d_n]\,,\ \lim_{n\to\infty}d_n=0\,,\ C\subset A\right\}.$$

It is not difficult to check that topology $\mathcal{T}_1$ generated by $\big\{B_1(x)\colon x\in\mathbb{R}\big\}$ is smaller than topology $\mathcal{T}_2$ generated by $\big\{B_2(x)\colon x\in\mathbb{R}\big\}$ and

$$\mathbb{R}\setminus A\in\mathcal{T}_2\setminus\mathcal{T}_1\,.$$

Suppose, to the contrary to our claim, that $f\colon\mathbb{R}\to\mathbb{R}$ is a continuous function considered as $f\colon(\mathbb{R},\mathcal{T}_2)\to(\mathbb{R},\mathcal{T}_0)$ but is not continuous as $f\colon(\mathbb{R},\mathcal{T}_1)\to$

$(\mathbb{R}, \mathcal{T}_0)$. It means that there exists $\alpha \in \mathbb{R}$ such that $f^{-1}\big((\alpha,\infty)\big) \in \mathcal{T}_2 \setminus \mathcal{T}_1$ or $f^{-1}\big((-\infty,\alpha)\big) \in \mathcal{T}_2 \setminus \mathcal{T}_1$. Assume that the first case occurs (the second is similar). From definitions of $\mathcal{T}_1$ and $\mathcal{T}_2$ it follows that

$$f^{-1}\big((\alpha,\infty)\big) \cap (-\varepsilon,\varepsilon) = (\mathbb{R} \setminus A) \cap (-\varepsilon,\varepsilon)$$

for some $\varepsilon > 0$. Since $0 \in (\mathbb{R} \setminus A) \cap (-\varepsilon,\varepsilon)$, $f(0) > \alpha$. Let n_0 be such a natural number that $[a_n, b_n] \subset (-\varepsilon,\varepsilon)$, for every $n \geq n_0$. $f|(0,\varepsilon)$ is continuous in usual sense, so it has the Darboux property. Hence $f(a_n) = f(b_n) = \alpha$ for any $n \geq n_0$.

Let $\beta = \frac{f(0)+\alpha}{2}$. Clearly

$$0 \in f^{-1}\big((\beta,\infty)\big) \quad \text{and} \quad b_n \notin f^{-1}\big((\beta,\infty)\big) \qquad \text{for any } n \geq n_0 .$$

It gives a contradiction, because

$$f^{-1}\big((\beta,\infty)\big) \notin \mathcal{T}_2 . \qquad \square$$

Let go back to our main topic. For any $\psi \in \mathcal{C}$ we will denote by $\mathcal{A}_\psi$ the class of all ψ-approximately continuous functions. By Theorem 2, if $\mathcal{T}_{\psi_1} \subsetneq \mathcal{T}_{\psi_2}$ then $\mathcal{A}_{\psi_1} \subsetneq \mathcal{A}_{\psi_2}$. According to Theorem 2.6 from [2] we have, that if $\lim\limits_{h\to 0+} \frac{\psi_1(h)}{\psi_2(h)} = \infty$ then $\mathcal{T}_{\psi_2} \subsetneq \mathcal{T}_{\psi_1}$. Denote, for any $a \in (0,1)$, by $\psi_a(h) = h^a$ for $h \in (0,\infty)$. By the above, if $a < b$ then $\mathcal{T}_{\psi_b} \subsetneq \mathcal{T}_{\psi_a}$. Therefore

PROPERTY 4. *There exists continuum different classes of functions* $\{\mathcal{A}_{\psi_a} : a \in (0,1)\}$ *such that* $C_0 \subsetneq \mathcal{A}_{\psi_b} \subsetneq \mathcal{A}_{\psi_a} \subsetneq \mathcal{A}$ *if* $0 < a < b < 1$.

Finally we will construct the second family of uncountable many classes of ψ-approximately continuous functions (with another order).

PROPERTY 5. *For any* $\psi_0 \in \mathcal{C}$, *there exist sequences* $(\psi_\alpha)_{1\leq\alpha<\Omega}$ *and* $(\bar{\psi}_\alpha)_{1\leq\alpha<\Omega}$ *of functions from family* $\mathcal{C}$ *such that if* $1 \leq \alpha < \beta < \Omega$, *then*

$$C_0 \subsetneq \mathcal{A}_{\psi_\beta} \subsetneq \mathcal{A}_{\psi_\alpha} \subsetneq \mathcal{A}_{\psi_0} \subsetneq \mathcal{A}_{\bar{\psi}_\alpha} \subsetneq \mathcal{A}_{\bar{\psi}_\beta} \subsetneq \mathcal{A} .$$

P r o o f. Observe first, that for any $\psi \in \mathcal{C}$ there exist functions $\breve{\psi} \in \mathcal{C}$ and $\hat{\psi} \in \mathcal{C}$ such that

$$\lim_{h\to 0+} \frac{\psi(h)}{\breve{\psi}(h)} = \infty \qquad \text{and} \qquad \lim_{h\to 0+} \frac{\hat{\psi}(h)}{\psi(h)} = \infty .$$

We can assume, that $1 \in \psi\big((0,\infty)\big)$. Let

$$\breve{\psi}(h) = \begin{cases} \frac{1}{n^2} \ \text{ if } \psi(h) = \frac{1}{n}, & n = 1,2,\dots\,, \\ \text{linear on intervals } \psi^{-1}\Big(\big[\frac{1}{n+1}, \frac{1}{n}\big]\Big), & n = 1,2,\dots\,, \\ 1 \ \text{ for } h \in \psi^{-1}\big([1,\infty)\big); & \end{cases} \tag{1}$$

$$\hat{\psi}(h) = \begin{cases} \frac{1}{n} \ \text{ if } \psi(h) = \frac{1}{n^2}, & n = 1, 2, \dots, \\ \text{linear on intervals } \psi^{-1}\Big(\big[\frac{1}{(n+1)^2}, \frac{1}{n^2}\big]\Big), & n = 1, 2, \dots, \\ 1 \ \text{ for } h \in \psi^{-1}\big([1, \infty)\big). \end{cases} \tag{2}$$

It is not difficult to check that $\breve{\psi}$ and $\hat{\psi}$ fulfil required conditions.

Moreover, for any sequence $(\psi_n)_{n \in \mathbb{N}}$ of functions from family $\mathcal{C}$ there exist functions $\breve{\psi} \in \mathcal{C}$ and $\hat{\psi} \in \mathcal{C}$ such that

$$\lim_{h \to 0+} \frac{\psi_n(h)}{\breve{\psi}(h)} = \infty \qquad \text{and} \qquad \lim_{h \to 0+} \frac{\hat{\psi}(h)}{\psi_n(h)} = \infty$$

for any $n \in \mathbb{N}$.

Suppose, that $1 \in \psi_n\big((0, \infty)\big)$ for every $n \in \mathbb{N}$. Let

$$\breve{\psi}(h) = \begin{cases} \min\big\{\psi_i^2\big(\frac{1}{k+1}\big) \colon i \le k\big\} \text{ if } h = \frac{1}{k}, & k = 1, 2, \dots, \\ \text{linear on intervals } \big[\frac{1}{k+1}, \frac{1}{k}\big], & k = 1, 2, \dots, \\ \breve{\psi}(1) \ \text{ for } h \in [1, \infty). \end{cases} \tag{3}$$

Fix $n \in \mathbb{N}$. For any $k > 0$ and $h \in \big[\frac{1}{k+1}, \frac{1}{k}\big]$,

$$\frac{\psi_n(h)}{\breve{\psi}(h)} \ge \frac{\psi_n\big(\frac{1}{k+1}\big)}{\breve{\psi}\big(\frac{1}{k}\big)} \ge \frac{\psi_n\big(\frac{1}{k+1}\big)}{\psi_n^2\big(\frac{1}{k+1}\big)} = \frac{1}{\psi_n\big(\frac{1}{k+1}\big)}.$$

If $h \to 0^+$ then $k \to \infty$ and $\lim\limits_{k \to \infty} \psi_n(\frac{1}{k+1}) = 0$. Therefore

$$\lim_{h \to 0+} \frac{\psi_n(h)}{\breve{\psi}(h)} = \infty.$$

To define $\hat{\psi}$, let us denote $t_1 = \inf\{t > 0 \colon \psi_1(t) = 1\}$, $t_k = \min\{\inf\{t > 0 \colon \psi_i(t) = \frac{1}{k^2}\} \colon i = 1, \dots, k\}$. Put

$$\hat{\psi}(h) = \begin{cases} \frac{1}{k} \ \text{ for } \ h = t_k, & k = 1, 2, \dots, \\ \text{linear on intervals } [t_{k+1}, t_k], & k = 1, 2, \dots, \\ 1 \ \text{ for } h \in [t_1, \infty). \end{cases} \tag{4}$$

Fix $n \in \mathbb{N}$. For any $k > n$ and $h \in [t_{k+1}, t_k]$

$$\frac{\hat{\psi}(h)}{\psi_n(h)} \ge \frac{\hat{\psi}(t_{k+1})}{\psi_n(t_k)} \ge \frac{\frac{1}{k+1}}{\big(\frac{1}{k}\big)^2} = \frac{k^2}{k+1}.$$

Since $\lim\limits_{k \to \infty} \frac{k^2}{k+1} = \infty$ and $k \to \infty$ if $h \to 0^+$, we have

$$\lim_{h \to 0+} \frac{\hat{\psi}(h)}{\psi_n(h)} = \infty.$$

Now we will define a sequence $(\psi_\alpha)_{1\le\alpha<\Omega}$ using an transfinite induction. To define ψ_1 we will use formula (1) for $\psi = \psi_0$. Let $\psi_1(h) = \breve{\psi}(h)$ for $h > 0$. Since $\lim\limits_{h\to 0+} \frac{\psi(h)}{\psi_1(h)} = \infty$, $\mathcal{T}_{\psi_1} \subsetneq \mathcal{T}_{\psi_0}$ and $\mathcal{A}_{\psi_1} \subsetneq \mathcal{A}_{\psi_0}$.

Assume that we have defined ψ_β for $\beta < \alpha$. If α has the predecessor we use formula (1). In other case, since the family $\{\psi_\beta \colon \beta < \alpha\}$ is countable, we can use formula (3). In both cases we obtain function ψ_α with the required condition.

In the same manner we can define, using (2) and (4), a sequence $(\bar{\psi}_\alpha)_{1\le\alpha<\Omega}$. $\square$

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