Tatra Mt. Math. Publ. **32** (2005), 103–117

Tatra Mountains
Mathematical Publications

BILINEAR FORMS OVER SEMILOCAL DEDEKIND RINGS IN NUMBER FIELDS

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ABSTRACT. The fundamental problems in the theory of bilinear forms over commutative rings are the following: the representation problem, the classification problem and the problem of the structure of the Witt ring. We discuss them in the case of Dedekind rings, the fields of fractions of which are algebraic number fields.

Introduction

The theory of bilinear forms over commutative rings was introduced by Knebusch in [3]. It is a natural generalization of the theory of quadratic forms over fields. This theory is based on the notion of a *bilinear module* over a commutative ring R, which is a pair (M,φ), where M is a module over R and $\varphi\colon M\times M\to R$ is a symmetric bilinear form. The bilinear module (M,φ) is said to be a *bilinear space* when the module M is finitely generated projective and the bilinear form φ is *nonsingular*, i.e., the linear map

$$\hat{\varphi}\colon M\to \operatorname{Hom}(M,R)\,,\quad \hat{\varphi}(x)(y)=\varphi(x,y)\,,$$

is an isomorphism. The bilinear spaces (M,φ), (M',φ') are said to be *isometric* (write $(M,\varphi)\cong(M',\varphi')$ or simply $\varphi\underset{R}{\cong}\varphi'$, and say φ is equivalent to φ' over R) if there exists a module isomorphism $f\colon M\to M'$ such that $\varphi'\big(f(x),f(y)\big)=\varphi(x,y)$. The invertible element $a\in R$ is said to be *represented by* φ (write $\varphi\underset{R}{\sim}a$) if there exists $x\in M$ with $\varphi(x,x)=a$.

If every finitely generated projective module over R is free, there is a close interrelation between bilinear spaces over the ring R and classical integral quadratic forms over R. We recall that an integral quadratic form over the ring R of n indeterminates is a homogeneous polynomial of the form $F(X)=\mathbf{X}C\mathbf{X}^t$, where C is a symmetric $n\times n$ matrix with entries in R, $\mathbf{X}=[X_1,\dots,X_n]$ is

2000 Mathematics Subject Classification: 11E12.
Keywords: bilinear form, Witt ring.

a one-row matrix of indeterminates and $\mathbf{X}^t$ is the transpose of $\mathbf{X}$. The form F is nonsingular when the matrix C is invertible. If (M,φ) is a bilinear space over R, $(\alpha_1,\dots,\alpha_n)$ is a fixed basis of M and A_φ is a $n\times n$ matrix whose i,j entry is $a_{ij}=\varphi(\alpha_i,\alpha_j)$, then the polynomial

$$F_\varphi(X_1,\dots,X_n)=\sum_{i,j}a_{ij}X_iX_j$$

is a nonsingular integral quadratic form over R, which is referred as the *associated quadratic form* (we denote $F_\varphi(X_1,\dots,X_n)=\varphi(\mathbf{X})$ for short). It is easy to see that the bilinear form φ (and therefore the bilinear space (M,φ)) is uniquely determined by the matrix A_φ (and also the quadratic form F_φ).

The bilinear spaces (M,φ), (M',φ') are isometric if and only if there exists an invertible matrix P with entries in R such that $PA_{\varphi'}P^t=A_\varphi$, i.e., $\varphi'(\mathbf{X}P)=\varphi(\mathbf{X})$. An invertible element $a\in R$ is represented by φ over R if and only if there exists $\mathbf{x}=(x_1,\dots,x_n)\in R^n$ such that $\varphi(\mathbf{x})=a$.

If the basis $(\alpha_1,\dots,\alpha_n)$ is orthogonal and $a_i=\varphi(\alpha_i)$, we say φ has a diagonalization and write $\varphi=\langle a_1,\dots,a_n\rangle$.

Similarly as for fields, the *Witt ring* $W(R)$ of nonsingular bilinear spaces over R is the fundamental synthesis notion in the theory of bilinear forms over rings (for the construction see [6] or [5]). The commutative rings R and R' are called to be *Witt equivalent* when their Witt rings $W(R)$ and $W(R')$ are isomorphic.

The fundamental problems in the theory of bilinear forms over rings are as follows:

- First, there is the *representation problem*, which amounts to finding effective criteria for an invertible element a of a given ring R to be represented by a bilinear form over R.
- The second problem is the *classification problem*, which involves finding criteria for an isometry of bilinear spaces over a given ring R.
- Finally, there is the *problem of the structure of the Witt ring*, amounting to the description of the structure of the Witt ring of R and (or) to the classification of rings with respect to a Witt equivalence.

In this paper these three problems are discussed in the case when R is a semilocal Dedekind ring, the field of fractions K of which is an algebraic number field. It is not difficult to prove that if R is such a ring, then there exists a set $\mathcal{T}=\{\mathfrak{p}_1,\dots,\mathfrak{p}_k\}$ of nonzero prime ideals of the ring $\mathcal{O}_K$ of integers of K such that R is equal to the fraction ring $S^{-1}\mathcal{O}_K$, where $S=\mathcal{O}_K\setminus\bigcup_{i=1}^{k}\mathfrak{p}_i$. We denote the ring $S^{-1}\mathcal{O}_K$ by $\mathcal{O}_\mathcal{T}$.

In Section 1 we apply classical results of the arithmetic theory of quadratic forms over local and global fields to obtain a local-global principle for a representation of invertible elements of $\mathcal{O}_{\mathcal{T}}$ by forms over $\mathcal{O}_{\mathcal{T}}$ and for an isometry of bilinear spaces over $\mathcal{O}_{\mathcal{T}}$.

In Section 2 we study the problem of the structure of the Witt ring. Using the methods from the paper [2] we describe $W(\mathcal{O}_{\mathcal{T}})$. Moreover, we cite the main results on a Witt equivalence which are obtained in [8]. We present a local-global principle for a Witt equivalence of rings $\mathcal{O}_{\mathcal{T}}$ and $\mathcal{O}_{\mathcal{T}'}$ under the assumption that 2 is invertible in these rings.

Throughout the paper the following notation will be used. For a number field K we write $\Omega(K)$ for all primes of K. For each $\mathfrak{p} \in \Omega(K)$ we write $K_{\mathfrak{p}}$ for the $\mathfrak{p}$-adic completion of K. If $\mathfrak{p}$ is a finite prime of K, we write $\mathcal{O}_{\mathfrak{p}}$, $U_{\mathfrak{p}}$ and $\bar{K}_{\mathfrak{p}}$ for the ring of $\mathfrak{p}$-adic integers, the group of $\mathfrak{p}$-adic units and the residue class field of $K_{\mathfrak{p}}$, respectively. Let $\mathcal{T}$ be a finite set of nonzero prime ideals of $\mathcal{O}_K$. By $\mathcal{T}_{\mathrm{d}}$ and $\mathcal{T}_{\mathrm{nd}}$ we denote the set of dyadic and the set of nondyadic ideals of $\mathcal{T}$, respectively. Thus $\mathcal{T} = \mathcal{T}_{\mathrm{d}} \cup \mathcal{T}_{\mathrm{nd}}$. We write $\mathcal{U}_{\mathcal{T}}$ for the group of invertible elements of $\mathcal{O}_{\mathcal{T}}$ and we put

$$E_{\mathcal{T}}(K) = \{x \in \dot{K} \colon \operatorname{ord}_{\mathfrak{p}} x \equiv 0 \pmod 2 \text{ for every } \mathfrak{p} \in \mathcal{T}\}.$$

The set $E_{\mathcal{T}}(K)$ is a subgroup of $\dot{K}$ containing $\dot{K}^2$.

1. Representation and classification criteria

Let K be an algebraic number field and $\mathcal{T}$ be a finite set of nonzero prime ideals of $\mathcal{O}_K$. By [5] every finitely generated projective module over the ring $\mathcal{O}_{\mathcal{T}}$ is free. This gives us the possibility to use the close interrelation between bilinear spaces over $\mathcal{O}_{\mathcal{T}}$ and classical integral quadratic forms over $\mathcal{O}_{\mathcal{T}}$.

If φ is a bilinear form, the symbols $\dim\varphi$ and $\det\varphi$ denote the rank and the determinant of φ, respectively.

1.1 Representation criterion

For $\mathbf{x} \in K_{\mathfrak{p}}^n$ denote

$$\|\mathbf{x}\|_{\mathfrak{p}} = \max\{|x_i|_{\mathfrak{p}} \colon i = 1, \dots, n\}$$

and notice that $\|\cdot\|_{\mathfrak{p}}$ has all properties of $\mathfrak{p}$-adic norm (see [7]).

Lemma 1.1. *Let φ be a nonsingular bilinear form over K with $\dim\varphi \geqslant 2$ and assume that φ represents a over K for some $a \in \dot{K}$. Let moreover $\mathbf{x}_{\mathfrak{p}} \in K_{\mathfrak{p}}^n$ and $\varphi(\mathbf{x}_{\mathfrak{p}}) = a$ for each $\mathfrak{p} \in \mathcal{T}$. Then for each $\varepsilon > 0$ there is an element $\mathbf{x} \in K^n$ such that $\varphi(\mathbf{x}) = a$ and*

$$\|\mathbf{x} - \mathbf{x}_{\mathfrak{p}}\|_{\mathfrak{p}} < \varepsilon \qquad \textit{for all } \mathfrak{p} \in \mathcal{T}.$$

P r o o f . In the case when $K = \mathbb{Q}$ the proof can be found in [1, Chapter 6, Corollary after Lemma 9.1]. The same arguments can be used in the case when K is a number field. □

Lemma 1.2. *Let $\mathfrak{p}$ be a nondyadic prime of K and φ be a nonsingular bilinear form over $\mathcal{O}_\mathfrak{p}$ with $\dim\varphi \geqslant 2$ and $v \in U_\mathfrak{p}$. Then φ represents v over $\mathcal{O}_\mathfrak{p}$.*

P r o o f . The ring $\mathcal{O}_\mathfrak{p}$ is local, so φ has a diagonalization $\varphi = \langle a_1, \dots, a_n\rangle$, where $a_i \in U_\mathfrak{p}$ (cf. [5, 2.3 Prop.]). The group $U_\mathfrak{p}/U_\mathfrak{p}^2 = \{1, u\}$, where $u \notin U_\mathfrak{p}^2$, so it suffices to observe that $\langle u, u\rangle \underset{\mathcal{O}_\mathfrak{p}}{\sim} 1$. □

From Lemmas 1.1 and 1.2 we obtain the local-global principle for a representation.

Theorem 1.3. *Let φ be a nonsingular bilinear form over $\mathcal{O}_\mathcal{T}$ and $a \in \mathcal{U}_\mathcal{T}$. Then φ represents a over $\mathcal{O}_\mathcal{T}$ if and only if*

(1) *φ represents a over $\mathcal{O}_\mathfrak{p}$ for all $\mathfrak{p} \in \mathcal{T}_\mathrm{d}$,*

(2) *φ represents a over $K_\mathfrak{p}$ for all $\mathfrak{p} \in \Omega(K) \setminus \mathcal{T}$.*

P r o o f . It suffices to prove the implication $(\Leftarrow)$.

Assume $\dim\varphi \geqslant 2$. By Lemma 1.2 $\varphi \underset{\mathcal{O}_\mathfrak{p}}{\sim} a$ for each $\mathfrak{p} \in \mathcal{T}_\mathrm{nd}$. From the Hasse principle $\varphi \underset{K}{\sim} a$. Moreover, for each $\mathfrak{p} \in \mathcal{T}$ there is $\mathbf{x}_\mathfrak{p} \in \mathcal{O}_\mathfrak{p}^n$ such that $\varphi(\mathbf{x}_\mathfrak{p}) = a$. Let $\mathbf{x} \in K^n$ be an element such that $\varphi(\mathbf{x}) = a$ and

$$\|\mathbf{x} - \mathbf{x}_\mathfrak{p}\|_\mathfrak{p} < 1 \qquad \text{for all} \quad \mathfrak{p} \in \mathcal{T}.$$

Then

$$\|\mathbf{x}\|_\mathfrak{p} = \|\mathbf{x}_\mathfrak{p} + (\mathbf{x} - \mathbf{x}_\mathfrak{p})\|_\mathfrak{p} \leqslant \max\{\|\mathbf{x}_\mathfrak{p}\|_\mathfrak{p}, \|\mathbf{x} - \mathbf{x}_\mathfrak{p})\|_\mathfrak{p}\} \leqslant 1.$$

Hence $\mathbf{x} \in \mathcal{O}_\mathcal{T}^n$.

If $\dim\varphi = 1$, then $\varphi = \langle c\rangle$ (where $c \in \mathcal{U}_\mathcal{T}$) and $ac \in \dot{K}_\mathfrak{p}^2$ for all $\mathfrak{p} \notin \mathcal{T}_\mathrm{nd}$. Therefore $ac \in \dot{K}^2$ (cf. [7, 65:15]) and $\varphi \underset{\mathcal{O}_\mathcal{T}}{\sim} a$. □

1.2 Isometry criterion

Let (M, φ) be a bilinear space over K. For a fixed $\mathfrak{p} \in \Omega(K)$ we write $(M_\mathfrak{p}, \varphi_\mathfrak{p})$ for the localization of the space (M, φ) at the prime $\mathfrak{p}$. By $O_K(\varphi)$ and $O_{K_\mathfrak{p}}(\varphi_\mathfrak{p})$ we denote the orthogonal groups of (M, φ) and $(M_\mathfrak{p}, \varphi_\mathfrak{p})$, by $O_K^+(\varphi)$ and $O_{K_\mathfrak{p}}^+(\varphi_\mathfrak{p})$ their subgroups of rotations, respectively. If $\sigma \in O_{K_\mathfrak{p}}(\varphi_\mathfrak{p})$ and $A = [a_{ij}]$ $(i, j \in \{1, \dots, n\})$ is a matrix of the automorphism σ in some basis, then we define

$$\|\sigma\|_\mathfrak{p} = \|A\|_\mathfrak{p} = \max\{|a_{ij}|_\mathfrak{p} : i, j \in \{1, \dots, n\}\}.$$

Using analogous arguments as C a s s e l s in [1, Chapter 8, Lemma 3.2] we obtain the following lemma

Lemma 1.4. *Let $\mathfrak{p} \in \Omega(K)$ be a finite prime and φ be a nonsingular bilinear form over the ring $\mathcal{O}_{\mathfrak{p}}$. Then there exists a matrix B with entries in $\mathcal{O}_{\mathfrak{p}}$ and $\det B = -1$ such that $\varphi(\mathbf{X}B) = \varphi(\mathbf{X})$.*

Lemma 1.5. *If $\mathfrak{p}$ is a nondyadic prime of K and φ, ψ are nonsingular bilinear forms over $\mathcal{O}_{\mathfrak{p}}$ such that $\dim\varphi = \dim\psi$ and $\det\varphi = \det\psi$, then the form φ is equivalent to ψ over $\mathcal{O}_{\mathfrak{p}}$.*

P r o o f. The forms φ and ψ have diagonalizations $\varphi = \langle a_1, \dots, a_n\rangle$ and $\psi = \langle b_1, \dots, b_n\rangle$, where $a_i, b_i \in \{1, u\} \bmod \dot{K}_{\mathfrak{p}}^2$ and u is a $\mathfrak{p}$-adic unit such that $u \notin \dot{K}_{\mathfrak{p}}^2$. Let

$$k = \#\{a_i \colon a_i = u \bmod \dot{K}_{\mathfrak{p}}^2\} \quad \text{and} \quad l = \#\{b_i \colon b_i = u \bmod \dot{K}_{\mathfrak{p}}^2\}.$$

Since $\det\varphi = \det\psi$, so $k \equiv l \pmod 2$. Moreover $\langle u, u\rangle \underset{\mathcal{O}_{\mathfrak{p}}}{\sim} 1$, therefore

$$\varphi \underset{\mathcal{O}_{\mathfrak{p}}}{\cong} \langle 1, \dots, 1\rangle \underset{\mathcal{O}_{\mathfrak{p}}}{\cong} \psi \quad \text{or} \quad \varphi \underset{\mathcal{O}_{\mathfrak{p}}}{\cong} \langle 1, \dots, 1, u\rangle \underset{\mathcal{O}_{\mathfrak{p}}}{\cong} \psi.$$

□

Lemmas 1.4 and 1.5 allow us to prove the local-global principle for an isometry.

Theorem 1.6. *Let φ and ψ be nonsingular bilinear forms over the ring $\mathcal{O}_{\mathcal{T}}$. Then φ is equivalent to the form ψ over $\mathcal{O}_{\mathcal{T}}$ if and only if*

(1) *φ is equivalent to ψ over $\mathcal{O}_{\mathfrak{p}}$ for all $\mathfrak{p} \in \mathcal{T}_{\mathrm{d}}$,*

(2) *φ is equivalent to ψ over $K_{\mathfrak{p}}$ for all $\mathfrak{p} \in \Omega(K) \setminus \mathcal{T}$.*

P r o o f. The implication $(\Rightarrow)$ is obvious.

Notice that by the assumption and Lemma 1.5 the form φ is equivalent to ψ over the ring $\mathcal{O}_{\mathfrak{p}}$ for all $\mathfrak{p} \in \mathcal{T}$. Therefore, for each $\mathfrak{p} \in \mathcal{T}$ there exists a matrix $P_{\mathfrak{p}}$ with entries in the ring $\mathcal{O}_{\mathfrak{p}}$ such that $\psi(\mathbf{X}P_{\mathfrak{p}}) = \varphi(\mathbf{X})$ and $|\det P_{\mathfrak{p}}|_{\mathfrak{p}} = 1$.

More, by the local-global principle the form φ is equivalent to ψ over K. Thus there exists an invertible matrix P_0 with entries in the field K such that

$$\psi(\mathbf{X}P_0) = \varphi(\mathbf{X}).$$

For every $\mathfrak{p} \in \mathcal{T}$ we put $Q_{\mathfrak{p}} = P_0 P_{\mathfrak{p}}^{-1}$ and observe that

$$\varphi(\mathbf{X}Q_{\mathfrak{p}}) = \psi(\mathbf{X}P_0) = \varphi(\mathbf{X}).$$

Therefore $Q_{\mathfrak{p}} \in O_{K_{\mathfrak{p}}}(\varphi_{\mathfrak{p}})$.

From Lemma 1.4 it follows that for each $\mathfrak{p} \in \mathcal{T}$ there exists an invertible matrix $S_{\mathfrak{p}}$ with entries in $\mathcal{O}_{\mathfrak{p}}$ such that $\varphi(\mathbf{X}S_{\mathfrak{p}}) = \varphi(\mathbf{X})$ and $\det S_{\mathfrak{p}} = -1$. Considering possibly $S_{\mathfrak{p}}P_{\mathfrak{p}}$ instead of $P_{\mathfrak{p}}$ we can assume that $Q_{\mathfrak{p}} \in O^{+}_{K_{\mathfrak{p}}}(\varphi_{\mathfrak{p}})$.

From the Weak Approximation Theorem for Rotations (cf. [7, 101:7]) it follows that there exists an invertible matrix D with entries in K such that

$$\|D - Q_{\mathfrak{p}}^{-1}\|_{\mathfrak{p}} < \frac{\varepsilon}{\|P_0\|_{\mathfrak{p}}} \qquad \text{for every } \mathfrak{p} \in \mathcal{T}$$

and $\varphi(\mathbf{X}D) = \varphi(\mathbf{X})$.

Taking $B = DP_0$ we get

$$\|B - P_{\mathfrak{p}}\|_{\mathfrak{p}} = \|D - Q_{\mathfrak{p}}^{-1}\|_{\mathfrak{p}} \cdot \|P_0\|_{\mathfrak{p}} < \varepsilon .$$

Hence $\|B\|_{\mathfrak{p}} \leqslant 1$ for each $\mathfrak{p} \in \mathcal{T}$. Moreover, the following equality holds:

$$\psi(\mathbf{X}B) = \psi(\mathbf{X}DP_0) = \varphi(\mathbf{X}D) = \varphi(\mathbf{X}),$$

which proves the implication $(\Leftarrow)$. $\square$

2. Witt ring

Let K be an algebraic number field and $\mathcal{T}$ be a finite set of nonzero prime ideals of $\mathcal{O}_K$. Every element of the Witt ring $W(\mathcal{O}_{\mathcal{T}})$ is represented by the isometry class of a form $\langle a_1, \dots, a_n \rangle$, where $a_i \in U_{\mathcal{T}}$ (cf. [5, 2.7 Cor.]).

For the Witt class of a nonsingular bilinear form as well as for its isometry class we use the same symbol and will refer to them for convenience as to forms. If φ is a nonsingular bilinear form over the field K, then by $\operatorname{disc}(\varphi)$ and $e(\varphi)$ we denote its discriminant and dimension index, respectively. We write $I(K)$ and $I^n(K)$ for the fundamental ideal of K and its nth power and we write $C(K) = C_K$, $C_{\mathcal{T}}(K) = C_{\mathcal{T}}$ for the ideal class group and $\mathcal{T}$-class group of K, respectively. By $\Omega_{\mathrm{r}}(K)$, $\Omega_{\mathrm{nc}}(K)$, $\Omega_{\mathrm{fin}}(K)$ and $\Omega_{\mathrm{d}}(K)$ we denote the sets of all real, all noncomplex, all finite and all dyadic primes of K. If $\mathfrak{p} \in \Omega_{\mathrm{fin}}(K)$ the symbol $\operatorname{ord}_{\mathfrak{p}} a$ means the $\mathfrak{p}$-adic order of $a \in \dot{K}$.

2.1 Milnor-Knebusch sequence

By [6] the following sequence is exact

$$0 \longrightarrow W(\mathcal{O}_{\mathcal{T}}) \longrightarrow W(K) \xrightarrow{\ \partial\ } \bigoplus_{\mathfrak{p} \in \mathcal{T}} W(\bar{K}_{\mathfrak{p}}),$$

where $\partial = \bigoplus_{\mathfrak{p} \in \mathcal{T}} \partial_{\mathfrak{p}}$ is the direct sum of the second residue class homomorphisms of Witt groups $\partial_{\mathfrak{p}} : W(K) \to W(\bar{K}_{\mathfrak{p}})$. This sequence enables us to treat the ring $W(\mathcal{O}_{\mathcal{T}})$ in the natural way as a subring of $W(K)$ and identify it with the kernel of the homomorphism ∂.

We will study when any element $\varphi \in W(K)$ belongs to $\ker \partial$ (i.e., to $W(\mathcal{O}_{\mathcal{T}})$). If $\varphi \in \ker \partial_{\mathfrak{p}}$, then

$$\operatorname{ord}_{\mathfrak{p}}\bigl(\operatorname{disc}(\varphi)\bigr) \equiv 0 \,(\mathrm{mod}\ 2)\,.$$

If $\mathfrak{p}$ is a dyadic prime of K, then $W(\bar{K}_{\mathfrak{p}}) \cong \mathbb{Z}_2$, so

$$\varphi \in \ker \partial_{\mathfrak{p}} \Longleftrightarrow \operatorname{ord}_{\mathfrak{p}}\bigl(\operatorname{disc}(\varphi)\bigr) \equiv 0 \,(\mathrm{mod}\ 2)\,.$$

Assume $\mathfrak{p}$ is nondyadic and denote

$$W_e(K_{\mathfrak{p}}) = \bigl\{\varphi \in W(K_{\mathfrak{p}}) \colon \operatorname{ord}_{\mathfrak{p}}\bigl(\operatorname{disc}(\varphi)\bigr) \equiv 0 \,(\mathrm{mod}\ 2)\bigr\}\,.$$

The set $W_e(K_{\mathfrak{p}})$ is a subgroup of the group $W(K_{\mathfrak{p}})$. Immediately from the definition it follows that the restriction of the $\mathfrak{p}$th Hasse invariant $h_{\mathfrak{p}}$ to $W_e(K_{\mathfrak{p}})$ is a group homomorphism and is equal to the Hasse–Witt invariant. Let $\varphi \in W_e(K_{\mathfrak{p}})$. Notice that

$$e(\varphi)\langle -1\rangle + \varphi = \langle 1, -\operatorname{disc}(\varphi)\rangle + \psi\,,$$

where $\psi \in I^2(K_{\mathfrak{p}})$. Moreover,

$$\varphi \in \ker \partial_{\mathfrak{p}} \Longleftrightarrow \psi \in \ker \partial_{\mathfrak{p}}$$

and $h_{\mathfrak{p}}(\varphi) = h_{\mathfrak{p}}(\psi)$. Since the restriction of $h_{\mathfrak{p}}$ to $I^2(K_{\mathfrak{p}})$ can be identified with $\partial_{\mathfrak{p}}$ (cf. [6, p. 96]),

$$\varphi \in \ker \partial_{\mathfrak{p}} \Longleftrightarrow h_{\mathfrak{p}}(\varphi) = 1\,.$$

Hence we obtain

Proposition 2.1. *Let $\varphi \in W(K)$. Then φ belongs to $W(\mathcal{O}_{\mathcal{T}})$ if and only if*

(1) $\operatorname{ord}_{\mathfrak{p}}\bigl(\operatorname{disc}(\varphi)\bigr) \equiv 0 (\mathrm{mod}\ 2)$ *for all* $\mathfrak{p} \in \mathcal{T}$,
(2) $h_{\mathfrak{p}}(\varphi) = 1$ *for all* $\mathfrak{p} \in \mathcal{T}_{\mathrm{nd}}$.

We have the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & W(\mathcal{O}_K) \arrow[r] \arrow[d] & W(\mathcal{O}_{\mathcal{T}}) \arrow[r, "\hat{\partial}"] \arrow[d] & \bigoplus\limits_{\mathfrak{p}\notin\mathcal{T}} W(\bar{K}_{\mathfrak{p}}) \arrow[r, "\hat{\lambda}"] \arrow[d, "\mu"] & C_{\mathcal{T}}/C_{\mathcal{T}}^2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & W(\mathcal{O}_K) \arrow[r] & W(K) \arrow[r, "\bar{\partial}"] & \bigoplus\limits_{\mathfrak{p}\in\Omega(K)} W(\bar{K}_{\mathfrak{p}}) \arrow[r, "\lambda"] & C_K/C_K^2 \arrow[r] & 0
\end{tikzcd}$$

where

$$\hat{\partial} = \bigoplus_{\mathfrak{p}\notin\mathcal{T}} \partial_{\mathfrak{p}}\,, \qquad \bar{\partial} = \bigoplus_{\mathfrak{p}\in\Omega(K)} \partial_{\mathfrak{p}}\,,$$

$$\hat{\lambda}\left((\eta_{\mathfrak{p}})_{\mathfrak{p}\notin\mathcal{T}}\right) = \mathrm{cl}\Bigg(\prod_{\mathfrak{p}\notin\mathcal{T}} \mathfrak{p}^{e(\eta_{\mathfrak{p}})}\Bigg) C_{\mathcal{T}}^2, \qquad \lambda\left((\eta_{\mathfrak{p}})_{\mathfrak{p}\in\Omega(K)}\right) = \mathrm{cl}\Bigg(\prod_{\mathfrak{p}\in\Omega(K)} \mathfrak{p}^{e(\eta_{\mathfrak{p}})}\Bigg) C_K^2$$

and

$$\mu((\eta_{\mathfrak{p}})_{\mathfrak{p}\notin\mathcal{T}}) = \big((\eta_{\mathfrak{p}})_{\mathfrak{p}\notin\mathcal{T}}, \underbrace{0,\ldots,0}_{\mathfrak{p}\in\mathcal{T}}\big)\,.$$

Observe that this diagram is commutative and the second line is exact (cf. [6, (3.4) Ex.]). Hence we obtain

Proposition 2.2. *The sequence*

$$0 \longrightarrow W(\mathcal{O}_K) \longrightarrow W(\mathcal{O}_{\mathcal{T}}) \xrightarrow{\ \hat{\partial}\ } \bigoplus_{\mathfrak{p}\notin\mathcal{T}} W(\bar{K}_{\mathfrak{p}}) \xrightarrow{\ \hat{\lambda}\ } C_{\mathcal{T}}/C_{\mathcal{T}}^2 \longrightarrow 0$$

is exact.

2.2 Structure of the ring $W(\mathcal{O}_{\mathcal{T}})$

For the field K, denote by $\Gamma(K)$ the set of all finite subsets of $\Omega_{\mathrm{nc}}(K)$ which contain even number of elements. The set $\Gamma(K)$ with the symmetric difference, $A+B=(A\cup B)\setminus(A\cap B)$, is a group. By the symbol $\Gamma^+(K)$ denote the subgroup of the group $\Gamma(K)$ containing subsets of the set $\Omega_{\mathrm{fin}}(K)$. Moreover, for $a,b\in\dot{K}$ let

$$(a\mid b)=\big\{\mathfrak{p}\in\Omega_{\mathrm{nc}}(K)\colon (a,b)_{\mathfrak{p}}=-1\big\}\,.$$

In the paper [2] it was proved that there is an isomorphism of groups $w\colon I^2(K)/I^3(K)\to\Gamma(K)$ such that

$$w\big(\langle 1,-a,-b,ab\rangle + I^3(K)\big) = (a\mid b)\,.$$

Consider $\Omega_{\mathcal{T}}(K)=\Omega_{\mathrm{nc}}(K)\setminus\mathcal{T}_{\mathrm{nd}}$. Denote

$$\Gamma_{\mathcal{T}}(K)=\big\{A\subseteq\Omega_{\mathcal{T}}(K)\colon \#A\equiv 0\,(\mathrm{mod}\ 2)\big\}$$

and

$$\Gamma_{\mathcal{T}}^+(K)=\big\{A\subseteq\Omega_{\mathcal{T}}(K)\cap\Omega_{\mathrm{fin}}(K)\colon \#A\equiv 0\,(\mathrm{mod}\ 2)\big\}\,.$$

It is not difficult to notice that $\Gamma_{\mathcal{T}}(K)$ is a subgroup of the group $\Gamma(K)$ and $\Gamma_{\mathcal{T}}^+(K)$ is a subgroup of $\Gamma^+(K)$.

Let $I(\mathcal{O}_{\mathcal{T}})=W(\mathcal{O}_{\mathcal{T}})\cap I(K)$, what means that

$$I(\mathcal{O}_{\mathcal{T}})=\big\{\varphi\in W(\mathcal{O}_{\mathcal{T}}):\ \dim\varphi\equiv 0\,(\mathrm{mod}\ 2)\big\}$$

is the set of all forms of the ring $W(\mathcal{O}_{\mathcal{T}})$ with even rank. Observe that $I(\mathcal{O}_{\mathcal{T}})$ is an ideal of the ring $W(\mathcal{O}_{\mathcal{T}})$ of index 2. Denote by $I^n(\mathcal{O}_{\mathcal{T}})$ the nth power of the ideal $I(\mathcal{O}_{\mathcal{T}})$. Since the ring $W(\mathcal{O}_{\mathcal{T}})$ is additively generated by one-dimensional forms $\langle a\rangle$, where $a\in E_{\mathcal{T}}(K)$, in an analogous way as in [10, Prop. 15.1.5] observe that the ideal $I(\mathcal{O}_{\mathcal{T}})$ is generated by forms $\langle 1,a\rangle$, $a\in E_{\mathcal{T}}(K)$. Hence it is easy to draw the conclusion that $I^2(\mathcal{O}_{\mathcal{T}})$ is generated by $\langle 1,a\rangle\cdot\langle 1,b\rangle$, where $a,b\in E_{\mathcal{T}}(K)$.

Theorem 2.3. *The discriminant* disc *determines an epimorphism of groups* disc: $I(\mathcal{O}_{\mathcal{T}}) \to E_{\mathcal{T}}(K)/\dot{K}^2$. *The kernel of* disc *is the ideal* $I^2(\mathcal{O}_{\mathcal{T}})$.

Proof. If $\varphi \in W(\mathcal{O}_{\mathcal{T}})$, it is easy to verify that $\operatorname{disc}(\varphi) \in E_{\mathcal{T}}(K)/\dot{K}^2$.

The second part of the theorem we prove in an analogous way as in [10, Theorem 15.2.1]. □

From the above we get the following corollary

Corollary 2.4. $I^2(\mathcal{O}_{\mathcal{T}}) = I^2(K) \cap W(\mathcal{O}_{\mathcal{T}})$.

Assume that K *is a nonreal* field. In this case $I^3(K) = 0$, so there is an isomorphism $w\colon I^2(K) \to \Gamma(K)$ such that

$$w\big(\langle 1, -a, -b, ab\rangle\big) = (a \mid b).$$

If $a, b \in E_{\mathcal{T}}(K)$, then $(a,b)_{\mathfrak{p}} = 1$ for each $\mathfrak{p} \in \mathcal{T}_{\mathrm{nd}}$. Therefore $(a \mid b) \in \Gamma_{\mathcal{T}}(K)$. Taking the restriction of the isomorphism w to the ideal $I^2(\mathcal{O}_{\mathcal{T}})$, we get a monomorphism

$$w\colon I^2(\mathcal{O}_{\mathcal{T}}) \to \Gamma_{\mathcal{T}}(K)$$

such that $w\big(\langle 1, -a, -b, ab\rangle\big) = (a \mid b)$.

Theorem 2.5. *The monomorphism* $w\colon I^2(\mathcal{O}_{\mathcal{T}}) \to \Gamma_{\mathcal{T}}(K)$ *is an isomorphism.*

Proof. It suffices to show that w is a surjection. For this purpose fix some $A \in \Gamma_{\mathcal{T}}(K)$. For each $\mathfrak{p} \in A \cap \mathcal{T}_{\mathrm{d}}$ let $u_{\mathfrak{p}} \in U_{\mathfrak{p}}$ be a unique $(\operatorname{mod} \dot{K}_{\mathfrak{p}}^2)$ element such that the extension $K_{\mathfrak{p}}\big(\sqrt{u_{\mathfrak{p}}}\big)/K_{\mathfrak{p}}$ is quadratic unramified. Using the Strong Approximation Theorem (cf. [7, 21:2]) we find an element $a \in E_{\mathcal{T}}(K)$ such that

(1) $a \notin \dot{K}_{\mathfrak{p}}^2$, if $\mathfrak{p} \in A$,

(2) $a \neq u_{\mathfrak{p}} \bmod \dot{K}_{\mathfrak{p}}^2$, if $\mathfrak{p} \in A \cap \mathcal{T}_{\mathrm{d}}$.

Let

$$M = \big\{\mathfrak{p} \in \Omega_{\mathrm{fin}}(K)\colon \operatorname{ord}_{\mathfrak{p}} a \equiv 1 \pmod 2\big\}.$$

Choose elements $\alpha_{\mathfrak{p}} \in \dot{K}_{\mathfrak{p}}$ for $\mathfrak{p} \in A \cup \mathcal{T} \cup M \cup \Omega_{\mathrm{d}}(K) \cup \Omega_{\mathrm{r}}(K)$ in the following way:

(1) for $\mathfrak{p} \in A \setminus \mathcal{T}_{\mathrm{d}}$ choose $\alpha_{\mathfrak{p}} \in \dot{K}_{\mathfrak{p}}$ such that $(a, \alpha_{\mathfrak{p}})_{\mathfrak{p}} = -1$,

(2) for $\mathfrak{p} \in A \cap \mathcal{T}_{\mathrm{d}}$ choose $\alpha_{\mathfrak{p}} \in U_{\mathfrak{p}}$ such that $(a, \alpha_{\mathfrak{p}})_{\mathfrak{p}} = -1$,

(3) for $\mathfrak{p} \in \big(\mathcal{T} \cup M \cup \Omega_{\mathrm{d}}(K) \cup \Omega_{\mathrm{r}}(K)\big) \setminus A$ choose $\alpha_{\mathfrak{p}} = 1$.

By [4, Lemma 2.1] there is a nondyadic finite prime $\mathfrak{q} \in \Omega(K) \setminus (A \cup \mathcal{T} \cup M)$ and an element $b \in \dot{K}$ such that

(1) $b\dot{K}_{\mathfrak{p}}^2 = \alpha_{\mathfrak{p}}\dot{K}_{\mathfrak{p}}^2$ for $\mathfrak{p} \in A \cup \mathcal{T} \cup M \cup \Omega_{\mathrm{d}}(K) \cup \Omega_{\mathrm{r}}(K)$,

(2) $\operatorname{ord}_{\mathfrak{q}} b = 1$,

(3) $\operatorname{ord}_{\mathfrak{p}} b \equiv 0 \pmod 2$ for each $\mathfrak{p} \in \Omega_{\mathrm{fin}}(K) \setminus \big(A \cup \mathcal{T} \cup M \cup \Omega_{\mathrm{d}}(K) \cup \{\mathfrak{q}\}\big)$.

Of course $\operatorname{ord}_{\mathfrak{p}} b \equiv 0 \pmod 2$ for each $\mathfrak{p} \in \mathcal{T}$, so $b \in E_{\mathcal{T}}(K)$.

Notice that $(a,b)_{\mathfrak{p}} = -1$ for each $\mathfrak{p} \in A$ and $(a,b)_{\mathfrak{p}} = 1$ for $\mathfrak{p} \in \big(\mathcal{T} \cup M \cup \Omega_{\mathrm{d}}(K) \cup \Omega_{\mathrm{r}}(K)\big) \setminus A$. If $\mathfrak{p} \in \Omega_{\mathrm{fin}}(K) \setminus \big(A \cup \mathcal{T} \cup M \cup \Omega_{\mathrm{d}}(K) \cup \{\mathfrak{q}\}\big)$, then a and b are $\mathfrak{p}$-adic units $(\mathrm{mod} \dot{K}_{\mathfrak{p}}^2)$, so $(a,b)_{\mathfrak{p}} = 1$. The set A has even number of elements, so by the Hilbert reciprocity $(a,b)_{\mathfrak{q}} = 1$. Hence $(a \mid b) = A$. $\square$

According to [2] consider the ring

$$V(K) = \mathbb{Z}_2 \times \dot{K}/\dot{K}^2 \times \Gamma(K)$$

with the addition

$$\begin{aligned}(e_1, a_1, A) &+ (e_2, a_2, B) \\ &= \big(e_1 + e_2,\, (-1)^{e_1 e_2} a_1 a_2,\, A + B + \big((-1)^{e_1 e_2} \mid a_1 a_2\big) + (a_1 \mid a_2)\big)\end{aligned}$$

and the multiplication

$$(e_1, a_1, A) \cdot (e_2, a_2, B) = \big(e_1 e_2,\, a_1^{e_2} a_2^{e_1},\, e_2 A + e_1 B + (e_1 e_2 + 1)(a_1 \mid a_2)\big)$$

for $(e_1, a_1, A), (e_2, a_2, B) \in V(K)$. An arbitrary element $\varphi \in W(K)$ has the canonical decomposition:

$$\varphi = e(\varphi)\langle 1 \rangle + \langle 1, -\operatorname{disc}(\varphi)\rangle + \varphi',$$

where $\varphi' \in I^2(K)$. From [2] it follows that the map $\phi \colon W(K) \to V(K)$,

$$\phi(\varphi) = \big(e(\varphi), \operatorname{disc}(\varphi), w(\varphi')\big)\,,$$

is an isomorphism of rings. By this and our previous observations we obtain

THEOREM 2.6. *The image of the ring $W(\mathcal{O}_{\mathcal{T}})$ by the isomorphism ϕ is the ring $\mathbb{Z}_2 \times E_{\mathcal{T}}(K)/\dot{K}^2 \times \Gamma_{\mathcal{T}}(K)$.*

P r o o f. If $\varphi \in W(\mathcal{O}_{\mathcal{T}})$, then φ has the canonical decomposition

$$\varphi = e(\varphi)\langle 1 \rangle + \langle 1, -\operatorname{disc}(\varphi)\rangle + \varphi',$$

where $\operatorname{disc}(\varphi) \in E_{\mathcal{T}}(K)/\dot{K}^2$ and $\varphi' \in I^2(K) \cap W(\mathcal{O}_{\mathcal{T}}) = I^2(\mathcal{O}_{\mathcal{T}})$. Hence $w(\varphi') \in \Gamma_{\mathcal{T}}(K)$.

The inclusion $\supseteq$ follows from Theorem 2.5. $\square$

Denoting

$$V_{\mathcal{T}}(K) = \mathbb{Z}_2 \times E_{\mathcal{T}}(K)/\dot{K}^2 \times \Gamma_{\mathcal{T}}(K)$$

we finally get

Corollary 2.7. *The ring* $V_{\mathcal{T}}(K)$ *is a subring of* $V(K)$ *isomorphic to* $W(\mathcal{O}_{\mathcal{T}})$.

Assume K *is real.* By K_+ denote the set of totally positive elements of K. Let $W_{\mathrm{tor}}(K)$ denote the nilradical of the ring $W(K)$, i.e.,

$$W_{\mathrm{tor}}(K) = \{\varphi \in W(K) \colon \varphi \text{ is torsion}\}.$$

Since $I^3(K) \cap W_{\mathrm{tor}}(K) = 0$, there is a monomorphism $w \colon I^2(K) \cap W_{\mathrm{tor}}(K) \to \Gamma(K)$ with the image $\Gamma^+(K)$ such that

$$w\big(\langle 1, -a, -b, ab\rangle\big) = (a \mid b).$$

Taking the restriction of w to $I^2(\mathcal{O}_{\mathcal{T}}) \cap W_{\mathrm{tor}}(K)$, we get a monomorphism

$$w \colon I^2(\mathcal{O}_{\mathcal{T}}) \cap W_{\mathrm{tor}}(K) \to \Gamma_{\mathcal{T}}(K)$$

such that $w\big(\langle 1, -a, -b, ab\rangle\big) = (a \mid b)$. We have to show that the image of the monomorphism w is the subgroup $\Gamma^+_{\mathcal{T}}(K)$. For this purpose observe the following fact

Lemma 2.8. *Let* $A \in \Gamma^+_{\mathcal{T}}(K)$. *Then there exist elements* $a, b \in E_{\mathcal{T}}(K) \cap K_+$ *such that*

$$(-a, b)_{\mathfrak{p}} = \begin{cases} 1 & \text{if } \mathfrak{p} \notin A, \\ -1 & \text{if } \mathfrak{p} \in A. \end{cases}$$

P r o o f. For $\mathfrak{p} \in A$ we choose elements $\alpha_{\mathfrak{p}}, \beta_{\mathfrak{p}} \in \dot{K}_{\mathfrak{p}}$ such that $(\alpha_{\mathfrak{p}}, \beta_{\mathfrak{p}})_{\mathfrak{p}} = -1$. If $\mathfrak{p} \in A \cap \mathcal{T}$, then $\mathfrak{p}$ is a dyadic prime, so we can assume additionally that $\alpha_{\mathfrak{p}}, \beta_{\mathfrak{p}} \in U_{\mathfrak{p}}$.

From [4, Lemma 2.1] it follows that there exists a nondyadic finite prime $\mathfrak{q} \in \Omega(K) \setminus (A \cup \mathcal{T})$ and an element $a \in \dot{K}$ such that

(1) $-a \in K_+$,

(2) $-a\dot{K}_{\mathfrak{p}}^2 = \alpha_{\mathfrak{p}} \dot{K}_{\mathfrak{p}}^2$ for $\mathfrak{p} \in A$,

(3) $-a \in \dot{K}_{\mathfrak{p}}^2$ for $\mathfrak{p} \in \big(\mathcal{T} \cup \Omega_{\mathrm{d}}(K)\big) \setminus A$,

(4) $\operatorname{ord}_{\mathfrak{q}} a = 1$,

(5) $\operatorname{ord}_{\mathfrak{p}} a \equiv 0 \pmod 2$ for each $\mathfrak{p} \in \Omega_{\mathrm{fin}}(K) \setminus \big(A \cup \{\mathfrak{q}\}\big)$.

Using [4, Lemma 2.1] we find a nondyadic finite prime $\mathfrak{r} \in \Omega(K) \setminus (A \cup \mathcal{T} \cup \{\mathfrak{q}\})$ and an element $b \in \dot{K}$ such that

(1) $b \in K_+$,

(2) $b\dot{K}_{\mathfrak{p}}^2 = \beta_{\mathfrak{p}} \dot{K}_{\mathfrak{p}}^2$ for $\mathfrak{p} \in A$,

(3) $b \in \dot{K}_{\mathfrak{p}}^2$ for $\mathfrak{p} \in (\mathcal{T} \cup \Omega_{\mathrm{d}}(K)) \setminus A$,

(4) $b \in \dot{K}_{\mathfrak{q}}^2$,

(5) $\operatorname{ord}_{\mathfrak{r}} a = 1$,

(6) $\operatorname{ord}_{\mathfrak{p}} a \equiv 0 \pmod 2$ for each $\mathfrak{p} \in \Omega_{\mathrm{fin}}(K) \setminus (A \cup \{\mathfrak{r}\})$.

Of course the elements a, b belong to $E_{\mathcal{T}}(K)$. Moreover $(-a, b)_{\mathfrak{p}} = -1$ for every $\mathfrak{p} \in A$ and $(-a, b)_{\mathfrak{p}} = 1$ for every $\mathfrak{p} \in \Omega(K) \setminus (A \cup \{\mathfrak{r}\})$. Thus by the Hilbert reciprocity we have $(-a, b)_{\mathfrak{r}} = 1$. This completes the proof. □

Lemma 2.8 allows us to prove the following theorem

Theorem 2.9. *The image of the monomorphism* $w\colon I^2(\mathcal{O}_{\mathcal{T}}) \cap W_{\mathrm{tor}}(K) \to \Gamma_{\mathcal{T}}(K)$ *is the subgroup* $\Gamma_{\mathcal{T}}^{+}(K)$.

Proof. By [9, 14.14] the ideal $I^2(\mathcal{O}_{\mathcal{T}}) \cap W_{\mathrm{tor}}(K)$ is additively generated by forms $\langle 1, a\rangle \cdot \langle 1, -b\rangle$, where $a, b \in E_{\mathcal{T}}(K)$ and $b \in K_{+}$. Hence $\operatorname{im} w \subseteq \Gamma_{\mathcal{T}}^{+}(K)$.

The inclusion $\operatorname{im} w \supseteq \Gamma_{\mathcal{T}}^{+}(K)$ follows from Lemma 2.8. □

Let $\tau_1, \dots, \tau_r$ be all embeddings of K into the field $\mathbb{R}$. For $a \in \dot{K}$ let $\sigma_i(a)$ denote the sign of the number $\tau_i(a)$. Similarly as in [2], by [4, Lemma 2.1] we find *in succession* elements $\varepsilon_1, \dots, \varepsilon_{r-1} \in \dot{K}$ and finite primes $\mathfrak{p}_1, \dots, \mathfrak{p}_{r-1} \in \Omega(K) \setminus (\mathcal{T} \cup \Omega_{\mathrm{d}}(K))$ such that for every $i \in \{1, \dots, r-1\}$ we have

(1) $\sigma_i(\varepsilon_i) = -1$,

(2) $\sigma_j(\varepsilon_i) = 1$ for $j \in \{1, \dots, r\}$, $j \neq i$,

(3) $\varepsilon_i \in \dot{K}_{\mathfrak{q}}^2$ for each $\mathfrak{q} \in (\mathcal{T} \cup \Omega_{\mathrm{d}}(K)) \cup \{\mathfrak{p}_1, \dots, \mathfrak{p}_{i-1}\}$,

(4) $\operatorname{ord}_{\mathfrak{p}_i} \varepsilon_i = 1$,

(5) $\operatorname{ord}_{\mathfrak{q}} \varepsilon_i \equiv 0 \pmod 2$ for each finite prime $\mathfrak{q} \neq \mathfrak{p}_i$.

Moreover, let $\varepsilon_r = -\varepsilon_1 \cdots \varepsilon_{r-1}$. Then every ε_i is an element of the ring $E_{\mathcal{T}}(K)$.

Consider the set

$$W_{\mathrm{red}}(K) = \left\{ e\langle 1\rangle + \sum_{i=1}^{r} n_i \langle 1, -\varepsilon_i\rangle \in W(K)\colon e = 0, 1;\ n_i \in \mathbb{Z} \right\}.$$

By [2] the set $W_{\mathrm{red}}(K)$ is a subring of $W(K)$ isomorphic to the subring

$$(\mathbb{Z}^r)_0 = \{\bar{m} = (m_1, \dots, m_r) \in \mathbb{Z}^r\colon m_1 \equiv \cdots \equiv m_r \pmod 2\}$$

of $\mathbb{Z}^r$. Let $\Psi\colon W_{\mathrm{red}}(K) \to (\mathbb{Z}^r)_0$ be this isomorphism. Denoting by $\dot{K}_{-}$ the subgroup of $\dot{K}$ generated by the set $\{\varepsilon_1, \dots, \varepsilon_r\}$ according to [2] consider the dimension index $e\colon (\mathbb{Z}^r)_0 \to \mathbb{Z}_2$ and the discriminant $\operatorname{disc}\colon (\mathbb{Z}^r)_0 \to \dot{K}_{-}\dot{K}^2/\dot{K}^2$ defined in the following way:

$$e(\bar{m}) = e(\Psi^{-1}(\bar{m})) \quad \text{and} \quad \operatorname{disc}(\bar{m}) = \operatorname{disc}(\Psi^{-1}(\bar{m})).$$

Consider the ring
$$V(K) = (\mathbb{Z}^r)_0 \times \dot{K}_+/\dot{K}^2 \times \Gamma^+(K)$$
with the addition
$$(\bar{m}_1, a_1, A) + (\bar{m}_2, a_2, B) = \big(\bar{m}_1 + \bar{m}_2, a_1 a_2, (a_1 \mid a_2) + A + B\big)$$
and the multiplication
$$(\bar{m}_1, a_1, A) \cdot (\bar{m}_2, a_2, B) = \big(\bar{m}_1 \bar{m}_2, a_1^{e(\bar{m}_2)} a_2^{e(\bar{m}_1)}, C\big)\,,$$
where $C = \big(\operatorname{disc}(\bar{m}_2) \mid a_1\big) + \big(\operatorname{disc}(\bar{m}_1) \mid a_2\big) + \big(e(\bar{m}_1)e(\bar{m}_2) + 1\big)(a_1 \mid a_2) + e(\bar{m}_2)A + e(\bar{m}_1)B$.

We have chosen the elements $\varepsilon_1, \ldots, \varepsilon_r$ in such the way that every element $\varphi \in W(K)$ has the representation
$$\varphi = \varphi_{\mathrm{red}} + \varphi_{\mathrm{tor}}\,,$$
where $\varphi_{\mathrm{red}} \in W_{\mathrm{red}}(K)$ and $\varphi_{\mathrm{tor}} \in W_{\mathrm{tor}}(K)$. Moreover
$$\varphi_{\mathrm{red}} = e(\varphi)\langle 1\rangle + \sum_{i=1}^{r} n_i \langle 1, -\varepsilon_i\rangle\,, \quad \varphi_{\mathrm{tor}} = \langle 1, -\operatorname{disc}(\varphi_{\mathrm{tor}})\rangle + \varphi'_{\mathrm{tor}}\,,$$
where $\varphi'_{\mathrm{tor}} \in I^2(K) \cap W_{\mathrm{tor}}(K)$. By [2] it follows that the map $\phi\colon W(K) \to V(K)$,
$$\begin{aligned}\phi\Bigg(e(\varphi)\langle 1\rangle + \sum_{i=1}^{r} n_i\langle 1, -\varepsilon_i\rangle + \langle 1, -\operatorname{disc}(\varphi_{\mathrm{tor}})\rangle + \varphi'_{\mathrm{tor}}\Bigg)\\ = \Big(\big(e(\varphi) + 2n_1, \ldots, e(\varphi) + 2n_r\big), \operatorname{disc}(\varphi_{\mathrm{tor}}), w(\varphi'_{\mathrm{tor}})\Big)\end{aligned}$$
is an isomorphism of groups. By this and our previous observations we reach

THEOREM 2.10. *The image of the ring $W(\mathcal{O}_{\mathcal{T}})$ by the isomorphism ϕ is the ring*
$$(\mathbb{Z}^r)_0 \times E_{\mathcal{T}}(K) \cap \dot{K}_+/\dot{K}^2 \times \Gamma^+_{\mathcal{T}}(K)\,.$$

P r o o f. If $\varphi \in W(\mathcal{O}_{\mathcal{T}})$, then φ has the canonical decomposition
$$\varphi = e(\varphi)\langle 1\rangle + \sum_{i=1}^{r} n_i\langle 1, -\varepsilon_i\rangle + \langle 1, -\operatorname{disc}(\varphi_{\mathrm{tor}})\rangle + \varphi'_{\mathrm{tor}}\,,$$
where $\operatorname{disc}(\varphi_{\mathrm{tor}}) \in E_{\mathcal{T}}(K) \cap \dot{K}_+/\dot{K}^2$ and $\varphi'_{\mathrm{tor}} \in W(\mathcal{O}_{\mathcal{T}}) \cap I^2(K) \cap W_{\mathrm{tor}}(K) = I^2(\mathcal{O}_{\mathcal{T}}) \cap W_{\mathrm{tor}}(K)$.

The thesis follows from Theorem 2.9. □

Denoting
$$V_{\mathcal{T}}(K) = (\mathbb{Z}^r)_0 \times E_{\mathcal{T}}(K) \cap \dot{K}_+/\dot{K}^2 \times \Gamma^+_{\mathcal{T}}(K)\,,$$
we finally reach

COROLLARY 2.11. *The ring $V_{\mathcal{T}}(K)$ is a subring of $V(K)$ isomorphic to $W(\mathcal{O}_{\mathcal{T}})$.*

2.3 Witt equivalence

Let K and L be number fields. Assume that $\mathcal{T}$ and $\mathcal{T}'$ are finite sets of nonzero prime ideals of the rings $\mathcal{O}_K$ and $\mathcal{O}_L$, respectively. *Hilbert equivalence* with respect to $(\mathcal{T}, \mathcal{T}')$ is a pair (t, ϕ), where

(1) $t\colon E_{\mathcal{T}}(K)/\dot{K}^2 \to E_{\mathcal{T}'}(L)/\dot{L}^2$ is a group isomorphism such that $t(-1) = -1$,

(2) $\phi\colon \Omega_{\mathcal{T}}(K) \to \Omega_{\mathcal{T}'}(L)$ is a bijection such that
$$(a,b)_{\mathfrak{p}} = (ta, tb)_{\phi\mathfrak{p}} \quad \text{for all } a, b \in E_{\mathcal{T}}(K) \text{ and } \mathfrak{p} \in \Omega_{\mathcal{T}}(K).$$

If $\mathfrak{p}$ is a finite prime of the field K, by $u_{\mathfrak{p}}$ denote a unique $(\bmod \dot{K}_{\mathfrak{p}}^2)$ $\mathfrak{p}$-adic unit such that the extension $K_{\mathfrak{p}}(\sqrt{u_{\mathfrak{p}}})/K_{\mathfrak{p}}$ is quadratic unramified. Moreover, for a dyadic prime $\mathfrak{p}$ we define
$$H_{\mathfrak{p}} = \begin{cases} \{1, u_{\mathfrak{p}}\} \lhd \dot{K}_{\mathfrak{p}}/\dot{K}_{\mathfrak{p}}^2 & \text{if } \mathfrak{p} \in \mathcal{T}, \\ \{1\} \lhd \dot{K}_{\mathfrak{p}}/\dot{K}_{\mathfrak{p}}^2 & \text{if } \mathfrak{p} \notin \mathcal{T}. \end{cases}$$

Let
$$n_{\mathfrak{p}} = \begin{cases} [K_{\mathfrak{p}} : \mathbb{Q}_2] & \text{if } \mathfrak{p} \in \mathcal{T}_{\mathrm{d}}, \\ [K_{\mathfrak{p}} : \mathbb{Q}_2] + 2 & \text{if } \mathfrak{p} \notin \mathcal{T}_{\mathrm{d}} \text{ and } \mathfrak{p} \text{ is dyadic}, \end{cases}$$

and
$$\Omega_2(K) = \{\mathfrak{p} \in \Omega_{\mathcal{T}}(K)\colon \mathfrak{p} \text{ is dyadic and } n_{\mathfrak{p}} \neq 2\}.$$

THEOREM 2.12. *Let K and L be number fields, $\mathcal{T}$ and $\mathcal{T}'$ be finite sets of nonzero prime ideals of the rings $\mathcal{O}_K$ and $\mathcal{O}_L$, respectively. The following statements are equivalent.*

(i) *The rings $\mathcal{O}_{\mathcal{T}}$ and $\mathcal{O}_{\mathcal{T}'}$ are Witt equivalent.*

(ii) *There exists an isomorphism between $W(\mathcal{O}_{\mathcal{T}})$ and $W(\mathcal{O}_{\mathcal{T}'})$ which sends one-dimensional forms onto one-dimensional forms.*

(iii) *There is a Hilbert equivalence (t, ϕ) with respect to $(\mathcal{T}, \mathcal{T}')$ such that the restriction ϕ to the set of all real primes of K is a bijection onto the set of all real primes of L.*

(iv) *The following conditions hold:*

  (1) *$-1 \in \dot{K}^2$ if and only if $-1 \in \dot{L}^2$,*

  (2) *the number of real primes of K is equal to the number of real primes of L,*

  (3) *there exists a bijection $\phi\colon \Omega_2(K) \to \Omega_2(L)$ such that:*

    (a) *$-1 \in H_{\mathfrak{p}}$ if and only if $-1 \in H_{\phi\mathfrak{p}}$,*

    (b) *$n_{\mathfrak{p}} = n_{\phi\mathfrak{p}}$.*

Proof. The proof can be found in [8]. □

From Theorem 2.12 we obtain the local-global principle for a Witt equivalence of rings $\mathcal{O}_{\mathcal{T}}$ and $\mathcal{O}_{\mathcal{T}'}$ under the assumption that 2 is invertible in them.

THEOREM 2.13. *Let K and L be number fields, $\mathcal{T}$ and $\mathcal{T}'$ be finite sets of nonzero nondyadic prime ideals of the rings $\mathcal{O}_K$ and $\mathcal{O}_L$, respectively. The following statements are equivalent.*

(i) *The rings $W(\mathcal{O}_{\mathcal{T}})$ and $W(\mathcal{O}_{\mathcal{T}'})$ are isomorphic.*

(ii) *There exists a bijection $\phi\colon \Omega_{\mathcal{T}}(K) \to \Omega_{\mathcal{T}'}(L)$ such that $W(K_{\mathfrak{p}})$ and $W(K_{\phi\mathfrak{p}})$ are isomorphic for every $\mathfrak{p} \in \Omega_{\mathcal{T}}(K)$.*

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Received January 10, 2005

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