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ON A GENERALIZED KOLMOGOROFF INTEGRAL IN COMPLETE BORNOLOGICAL LOCALLY CONVEX SPACES

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ABSTRACT. We introduce a generalized Kolmogoroff integral of the first type with respect to the operator valued measure in complete bornological locally convex topological vector spaces and show that, in the equal setting, the class of integrable functions coincides with the class of integrable functions in the generalized Dobrakov integral sense, [J. Haluška: *On integration in complete bornological locally convex spaces*, Czechoslovak Math. J. **47**, (1997), 205–219].

1. Introduction

The Kolmogoroff integrals of the first and the second type belong to the basics of the integration theory, cf. [11], [4], also the well-known Dobrakov integral. This integral generalized the Lebesgue integration to Banach spaces with respect to the operator valued measure. Recall the definition of this integral, cf. [1, Definition 2]. If X, Y are two Banach spaces, Δ a δ-ring of subsets of a set $T \neq \emptyset$, $L(X,Y)$ the space of all continuous operators $L\colon X \to Y$, $m\colon \Delta \to L(X,Y)$ a measure σ-additive in the strong operator topology, then we say that a measurable function $f\colon T \to X$ is *integrable in the sense of Dobrakov* if there exists a sequence of simple functions $f_n\colon T \to X$, $n \in \mathbb{N}$, converging m-a.e. to f and the integrals $\int_E f_n dm$ are uniformly σ-additive measures on $\sigma(\Delta)$ (the σ-algebra generated by Δ). The integral of the function f on $E \in \sigma(\Delta)$ is defined by the equality $\int_E f dm = \lim\limits_{n\to\infty} \int_E f_n dm$. A very excellent review paper about Dobrakov integral is [12].

The description of the theory of complete bornological locally convex topological vector spaces (for short, CBLCS), we can find in [9], [10] and [13]. The gener-

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alization of Dobrakov integration theory to CBLCS was done by the second author in [8]. The sense of this seemingly complicated theory is that, at the present, this is the only integration theory which completely generalizes the Dobrakov integration to a class of non-metrizable locally convex topological vector spaces.

In this paper, we will deal with the Kolmogoroff integral of the first type. We will consider the Kolmogoroff integral scheme on sigma algebras of sets and, at the same time, generalize it to CBLCS for vector integrable functions and operator valued measures. The main result of this paper consists in proving that a function is integrable in this generalized Kolmogoroff integral sense if and only if it is integrable in the Dobrakov integration sense generalized to CBLCS, [8].

2. Preliminaries, lattice structures

In this section we collect the needed definitions and results from [5], [6] and [7].

CBLCS. Let X and Y be two CBLCS over the field $\mathbb{K}$ of real numbers $\mathbb{R}$ or complex numbers $\mathbb{C}$, equipped with the bornologies $\mathfrak{B}_X, \mathfrak{B}_Y$, respectively.

One of the equivalent definitions of CBLCS is to define these spaces as the inductive limits of Banach spaces. Remind that *a Banach disk* U in X is a set which is closed, absolutely convex and the linear span of which is a Banach space. Let us denote by $\mathcal{U}$ the set of all Banach disks in X such that $U \in \mathfrak{B}_X$. So, the space X is the inductive limit of Banach spaces X_U, $U \in \mathcal{U}$,

$$X = \operatorname{inj}\lim_{U \in \mathcal{U}} X_U\,, \tag{1}$$

cf. [10], where X_U is the $\mathbb{K}$-linear span of $U \in \mathcal{U}$ and the family $\mathcal{U}$ is directed by inclusion and forms the basis of bornology $\mathfrak{B}_X$ (analogously for Y; $\mathcal{W}$ forms a basis of the bornology $\mathfrak{B}_Y$). The basis $\mathcal{U}$ in the inductive limit (1) need not be unambiguous and, in particular, it can be chosen such that X_U, $U \in \mathcal{U}$, are separable.

We say that a sequence of elements $x_n \in X$, $n \in \mathbb{N}$ (the set of all natural numbers), *converges bornologically* (in the sense of Mackey; more precisely, bornologically with respect to the bornology $\mathfrak{B}_X$) to $x \in X$, if there exists $B \in \mathfrak{B}_X$ such that for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $x_n - x \in B$ for every $n \geq n_0$. Equivalently, we can deal with an arbitrary basis $\mathcal{U}$ of $\mathfrak{B}_X$ instead of the whole bornology $\mathfrak{B}_X$ in this definition.

Let $\mathcal{U}$ be a basis of bornology $\mathfrak{B}_X$. We say that a sequence of elements $x_n \in X$, $n \in \mathbb{N}$, converges bornologically with respect to the bornology $\mathfrak{B}_X$ to $x \in X$, we write $x = \mathcal{U}\text{-}\lim\limits_{n\to\infty} x_n$, if there exists $U \in \mathcal{U}$ such that for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $x_n - x \in U$ for every $n \geq n_0$.

The following examples show that each vector space X (or Y) over the field $\mathbb{K}$ can be equipped with various bornological bases of Banach disks (defining this way various CBLCS), moreover, with the property such that $U_1 \cap U_2 \neq \{\emptyset\}$ for every $U_1 \in \mathcal{U}$, $U_2 \in \mathcal{U}$.

Example 1. For the discrete bornology on the locally convex vector space X, the sets $B \in \mathfrak{B}_X$ are finite subsets of X.

Another classical bornology consists of all sets which are bounded in the von Neumann sense, i.e., for a locally convex topological vector space X equipped with a family of seminorms Q, the set B is *bounded* (in other words, belongs to the von Neumann bornology) if and only if for every $q \in Q$ there exists a constant C_q such that $q(x) \leq C_q$ for every $x \in B$.

Example 2. For a given bornology $\mathfrak{B}_X$, we may introduce a bornology $\mathfrak{B}_{X,x_0}$ of the following type.

Let the base $\mathcal{U}$ of the bornology $\mathfrak{B}_{X,x_0}$ in X consist of all Banach disks such that there exists $x_0 \neq 0$, $x_0 \in X$ such that $x_0 \in X_{U_1} \cap X_{U_2}$ for every linear spans (i.e., vector spaces) $X_{U_1} \cap X_{U_2}$ of $U_1 \in \mathcal{U}$, $U_2 \in \mathcal{U}$ in X.

In other words, each subspace X_U in the inductive limit (1) contains a point $x_0 \neq 0$, $x_0 \in X$, which is called the *marked element*. Clearly, each subspace X_U contains also a line connecting points 0 and $x_0 \in X$.

Operator spaces. On $\mathcal{U}$ *the lattice operations* are defined as follows. For $U_1, U_2 \in \mathcal{U}$ we have: $U_1 \wedge U_2 = U_1 \cap U_2$, $U_1 \vee U_2 = \operatorname{acs}(U_1 \cup U_2)$, where acs denotes the topological closure of the absolutely convex span of the set. Analogously for $\mathcal{W}$. For $(U_1, W_1), (U_2, W_2) \in \mathcal{U} \times \mathcal{W}$, we write $(U_1, W_1) \ll (U_2, W_2)$ if and only if $U_1 \subset U_2$ and $W_1 \supset W_2$.

We use Φ to denote *a class of all functions* $\mathcal{U} \to \mathcal{W}$ *with an order* $<$ defined as follows: for $\varphi, \psi \in \Phi$ we write $\varphi < \psi$ whenever $\varphi(U) \subset \psi(U)$ for every $U \in \mathcal{U}$. Denote by $L(X,Y)$ the space of all continuous linear operators $L\colon X \to Y$. We suppose $L(X,Y) \subset \Phi$. For a more detail description of the lattice structure of $L(X,Y)$ when both X, Y are CBLCS, cf. [6].

Set functions. Let $T \neq \emptyset$ be a set. Denote by 2^T the potential set of the set T and by $\Delta \subset 2^T$ a δ-ring of sets. If $\mathcal{A}$ is a system of subsets of the set T, then $\sigma(\mathcal{A})$ denotes the σ-algebra generated by the system $\mathcal{A}$. Denote by $\Sigma = \sigma(\Delta)$. We use χ_E to denote the characteristic function of the set E. By $p_U\colon X \to [0,\infty]$ we denote the Minkowski functional of the set $U \in \mathcal{U}$, i.e., $p_U = \inf\limits_{x \in \lambda U} |\lambda|$ (if U does not absorb $x \in X$, we put $p_U(x) = \infty$.). Similarly, p_W denotes the Minkowski functional of the set $W \in \mathcal{W}$.

For $(U,W) \in \mathcal{U} \times \mathcal{W}$, denote by $\hat{m}_{U,W}$ a (U,W)-*semivariation* of a charge (= finitely additive measure) $m\colon \Delta \to L(X,Y)$, where

$$\hat{m}_{U,W}(E) = \sup p_W \left(\sum_{i=1}^{I} m(E \cap E_i) x_i \right), \qquad E \in \Sigma,$$

where the supremum is taken over all finite sets $\{x_i \in X\colon x_i \in U,\ i = 1,2,\dots,I\}$ and all disjoint sets $\{E_i \in \Delta\colon i = 1,2,\dots,I\}$. It is well-known that $\hat{m}_{U,W}$ is a submeasure, i.e., a monotone, subadditive set function, and $\hat{m}_{U,W}(\emptyset) = 0$. Denote by $\Delta_{U,W} \subset \Delta$ the largest δ-ring of sets $E \in \Delta$, such that $\hat{m}_{U,W}(E) < \infty$. Denote by $\hat{m}_{\mathcal{U},\mathcal{W}} = \{\hat{m}_{U,W}\colon (U,W) \in \mathcal{U} \times \mathcal{W}\}$.

For $W \in \mathcal{W}$, denote by $|\mu|_W$ a W-*semivariation* of a charge $\mu\colon \Sigma \to Y$, where

$$|\mu|_W(E) = \sup p_W \left(\sum_{i=1}^{I} \lambda_i \mu(E \cap E_i) \right), \qquad E \in \Sigma,$$

where the supremum is taken over all finite sets of scalars $\{\lambda_i \in \mathbb{K}\colon |\lambda_i| \le 1,\ i = 1,2,\dots,I\}$ and all disjoint sets $\{E_i \in \Delta\colon i = 1,2,\dots,I\}$. The W-semivariation $|\mu|_W$ is a submeasure. Denote by $\mu_{\mathcal{W}} = \{\mu_W\colon W \in \mathcal{W}\}$.

For a more detail description the basic $L(X,Y)$-measure set structures when both X and Y are CBLCS, cf. [5].

In the theory of integration in Banach spaces we suppose the generalizations of the classical notions, such as almost uniform convergence, almost everywhere convergence, and convergence in measure of measurable functions and relations among them as commonly well-known. All this theory can be generalized to CBLCS as follows.

Null sets. Let $\beta_{\mathcal{U},\mathcal{W}}$ be a lattice of submeasures $\beta_{U,W}\colon \Sigma \to [0,\infty]$, $(U,W) \in \mathcal{U} \times \mathcal{W}$, such that $\beta_{U_2,W_2} \wedge \beta_{U_3,W_3} = \beta_{U_2 \wedge U_3, W_2 \vee W_3}$, $\beta_{U_2,W_2} \vee \beta_{U_3,W_3} = \beta_{U_2 \vee U_3, W_2 \wedge W_3}$, $(U_2,W_2),(U_3,W_3) \in \mathcal{U} \times \mathcal{W}$. For instance, $\beta_{\mathcal{U},\mathcal{W}} = \hat{m}_{\mathcal{U},\mathcal{W}}$, where $X_{\mathcal{U}}$ and $Y_{\mathcal{W}}$ have marked elements.

Denote by $\mathcal{O}(\beta_{U,W}) = \{N \in \Sigma\colon \beta_{U,W}(N) = 0\}$, $(U,W) \in \mathcal{U} \times \mathcal{W}$. The set $N \in \Sigma$ is called $\beta_{\mathcal{U},\mathcal{W}}$-*null* if there exists a couple $(U,W) \in \mathcal{U} \times \mathcal{W}$, such that $\beta_{U,W}(N) = 0$. We say that an assertion holds $\beta_{\mathcal{U},\mathcal{W}}$-*almost everywhere*, shortly $\beta_{\mathcal{U},\mathcal{W}}$-a.e., if it holds everywhere except in a $\beta_{\mathcal{U},\mathcal{W}}$-null set. A set $E \in \Sigma$ is said to be of *finite submeasure* $\beta_{\mathcal{U},\mathcal{W}}$ if there exists a couple $(U,W) \in \mathcal{U} \times \mathcal{W}$, such that $\beta_{U,W}(E) < \infty$.

Convergences of functions, measurable functions, simple functions. For $E \in \Sigma$, $R \in \mathcal{U}$, $(U,W) \in \mathcal{U} \times \mathcal{W}$, we say that a sequence $f_n\colon T \to X$, $n \in \mathbb{N}$, of functions (R,E)-*converges* $\beta_{U,W}$-a.e. to a function $f\colon T \to X$ if $\lim_{n\to\infty} p_R\big(f_n(t) - f(t)\big) = 0$ for every $t \in E \setminus N$, where $N \in \mathcal{O}(\beta_{U,W})$. We say that a sequence $f_n\colon T \to X$, $n \in \mathbb{N}$, of functions $(\mathcal{U}, E)$-*converges* $\beta_{\mathcal{U},\mathcal{W}}$-a.e. to a function $f\colon T \to X$ if there exist $R \in \mathcal{U}$, $(U,W) \in \mathcal{U} \times \mathcal{W}$, such that the sequence f_n, $n \in \mathbb{N}$, of functions (R,E)-converges $\beta_{U,W}$-a.e. to f. We write $f = \mathcal{U}\text{-}\lim_{n\to\infty} f_n$ $\beta_{\mathcal{U},\mathcal{W}}$-a.e. If $E = T$, then we will simply say that the sequence R-converges $\beta_{U,W}$-a.e., resp. $\mathcal{U}$-converges $\beta_{\mathcal{U},\mathcal{W}}$-a.e.

For $E \in \Sigma$, $R \in \mathcal{U}$, $(U,W) \in \mathcal{U} \times \mathcal{W}$, we say that a sequence $f_n\colon T \to X$, $n \in \mathbb{N}$, of functions (R,E)-*converges uniformly* to a function $f\colon T \to X$ if $\lim_{n\to\infty} \|f_n - f\|_{E,R} = 0$, where $\|f\|_{E,R} = \sup_{t\in E} p_R\big(f(t)\big)$. We say that a sequence $f_n\colon T \to X$, $n \in \mathbb{N}$, of functions (R,E)-*converges* $\beta_{U,W}$-*almost uniformly* to a function $f\colon T \to X$ if for every $\varepsilon > 0$ there exists a set $N \in \Sigma$, such that $\beta_{U,W}(N) < \varepsilon$ and the sequence f_n, $n \in \mathbb{N}$, of functions $(R, E \setminus N)$-converges uniformly to f. We say that a sequence $f_n\colon T \to X$, $n \in \mathbb{N}$, of functions $(\mathcal{U}, E)$-*converges* $\beta_{\mathcal{U},\mathcal{W}}$-*almost uniformly* to a function $f\colon T \to X$ if there exist $R \in \mathcal{U}$, $(U,W) \in \mathcal{U} \times \mathcal{W}$, such that the sequence f_n, $n \in \mathbb{N}$, of functions (R,E)-converges $\beta_{U,W}$-almost uniformly to f. If $E = T$, then we will simply say that the sequence of functions R-converges uniformly, resp. R-converges $\beta_{U,W}$-almost uniformly, resp. $\mathcal{U}$-converges $\beta_{\mathcal{U},\mathcal{W}}$-almost uniformly.

For a more detail description of described convergences of functions in CBLCS, cf. [7].

We use $\mathcal{M}_{\mathcal{U}}$ to denote *the space of all $\mathcal{U}$-measurable functions*, the largest vector space of functions $f\colon T \to X$ with the property: there exists $R \in \mathcal{U}$, such that for every $U \supset R$, $U \in \mathcal{U}$ and $\delta > 0$ the set $\big\{t \in T\colon p_U\big(f(t)\big) \geq \delta\big\} \in \Sigma$. In what follows we deal only with functions which are $\mathcal{U}$-measurable.

A function $f\colon T \to X$ is called Δ-*simple* if $f(T)$ is a finite set and $f^{-1}(x) \in \Delta$ for every $x \in X \setminus \{0\}$. The space of all Δ-simple functions is denoted by $\mathcal{S}$. For $(U,W) \in \mathcal{U} \times \mathcal{W}$, a function $f\colon T \to X$ is said to be $\Delta_{U,W}$-*simple* if $f = \sum_{i=1}^{I} x_i \chi_{E_i}$, where $x_i \in X_U$, $E_i \in \Delta_{U,W}$, $E_i \cap E_j = \emptyset$, for $i \neq j$, $i,j = 1,2,\dots,I$. The space of all $\Delta_{U,W}$-simple functions is denoted by $\mathcal{S}_{U,W}$. A function $f \in \mathcal{S}$ is said to be $\Delta_{\mathcal{U},\mathcal{W}}$-*simple* if there exists a couple $(U,W) \in \mathcal{U} \times \mathcal{W}$, such that $f \in \mathcal{S}_{U,W}$. The space of all $\Delta_{\mathcal{U},\mathcal{W}}$-simple functions is denoted by $\mathcal{S}_{\mathcal{U},\mathcal{W}}$.

Measure structures. The Dobrakov integral is defined in Banach spaces. Since X and Y are inductive limits of Banach spaces, there is a natural question whether an integral in CBLCS can be defined as a finite sum of Dobrakov's integrals in various Banach spaces, the choice of which may depend on the function which we integrate. In the paper [5] we showed that such an integral can be constructed. A suitable class of operator measures in CBLCS which allow such a generalization is a class of all σ_Φ-additive measures. The idea consists of the fact that the σ-finiteness of measure in the classical Lebesgue integration brings no new quality. In the case of CBLCS, we put and use an additional condition about σ-finiteness of measure and this enables us the generalization of the whole Dobrakov integration to CBLCS.

For $(U,W) \in \mathcal{U} \times \mathcal{W}$, we say that a charge m is *of σ-finite (U,W)-semivariation* if there exist sets $E_i \in \Delta_{U,W}$, $i \in \mathbb{N}$, such that $T = \bigcup_{i=1}^{\infty} E_i$.

For $\varphi \in \Phi$, we say that a charge m is of *σ_φ-finite $(\mathcal{U},\mathcal{W})$-semivariation* if for every $U \in \mathcal{U}$, the charge m is of σ-finite $(\mathcal{U},\varphi(U))$-semivariation.

We say that a charge m is *of σ_Φ-finite $(\mathcal{U},\mathcal{W})$-semivariation* if there exists a function $\varphi \in \Phi$ such that for every $U \in \mathcal{U}$ the charge is of σ_φ-finite $(\mathcal{U},\mathcal{W})$-semivariation.

Let $W \in \mathcal{W}$. We say that a charge $\mu\colon \Sigma \to Y$ is a (W,σ)-additive vector measure, if μ is an Y_W-valued (countable additive) vector measure.

Definition 1. We say that a charge $\mu\colon \Sigma \to Y$ is a *$(\mathcal{W},\sigma)$-additive vector measure*, if there exists $W \in \mathcal{W}$ such that μ is a (W,σ)-additive vector measure.

Note that if $\mu\colon \Sigma \to Y$ is a (W,σ)-additive vector measure and $W \subset W_1$, $W, W_1 \in \mathcal{W}$, then μ is a (W_1,σ)-additive vector measure.

Let $W \in \mathcal{W}$. Let $\nu_n\colon \Sigma \to Y$, $n \in \mathbb{N}$, be a sequence of (W,σ)-additive vector measures. Recall the following notion. If for every $\varepsilon > 0$, $E \in \Sigma$, $p_W(\nu_n(E)) < \infty$ and $E_i \in \Sigma$, $E_i \cap E_j = \emptyset$, $i \neq j$, $i,j \in \mathbb{N}$, there exists $J_0 \in \mathbb{N}$, such that for every $J \geq J_0$, $p_W\left(\nu_n\left(\bigcup_{i=J+1}^{\infty} E_i \cap E\right)\right) < \varepsilon$ uniformly for every $n \in \mathbb{N}$, then we say that the sequence of measures ν_n, $n \in \mathbb{N}$, is *uniformly (W,σ)-additive on* Σ, cf. [8].

Note that if a sequence ν_n, $n \in \mathbb{N}$, of measures is uniformly (W,σ)-additive on Σ, $W \in \mathcal{W}$, then the sequence ν_n, $n \in \mathbb{N}$, of measures is uniformly (W_1,σ)-additive on Σ whenever $W_1 \supset W$, $W_1 \in \mathcal{W}$.

Definition 2. We say that the family of measures $\nu_n\colon \Sigma \to Y$, $n \in \mathbb{N}$, is *uniformly $(\mathcal{W},\sigma)$-additive on* Σ if there exists $W \in \mathcal{W}$ such that the family ν_n, $n \in \mathbb{N}$, of measures is uniformly (W,σ)-additive on Σ.

The following definition generalizes the notion of the σ-additivity of an operator valued measure in the strong operator topology in Banach spaces, cf. [1], to CBLCS.

Definition 3. Let $\varphi \in \Phi$. We say that a charge $m\colon \Delta \to L(X,Y)$ is a *σ_φ-additive measure* if m is of σ_φ-finite $(\mathcal{U},\mathcal{W})$-semivariation, and for every $A \in \Delta_{U,\varphi(U)}$, the charge $m(A \cap \cdot)x\colon \Sigma \to Y$ is a $(\varphi(U),\sigma)$-additive measure for every $x \in X_U$, $U \in \mathcal{U}$. We say that a charge $m\colon \Delta \to L(X,Y)$ is a *σ_Φ-additive measure* if there exists $\varphi \in \Phi$, such that m is a σ_φ-additive measure.

In what follows the measure m is supposed to be σ_Φ-additive.

If $\varphi \le \psi$, $\varphi, \psi \in \Phi$, and a charge $m\colon \Delta \to L(X,Y)$ is a σ_φ-additive measure, then m is a σ_ψ-additive measure. Indeed, the measure m is of σ_ψ-finite $(\mathcal{U},\mathcal{W})$-semivariation. The assertion that for every $A \in \Delta_{U,W}$, the charge $m(A\cap\cdot)x\colon \Sigma \to Y$ is a $\big(\psi(U),\sigma\big)$-additive measure for every $x \in X_U$, follows from the inequality $p_{\psi(U)}(y) \le p_{\varphi(U)}(y)$, $y \in Y$.

Generalized Dobrakov integral in CBLCS**.** Let $f \in \mathcal{M}_\mathcal{U}$. For every $\hat{m}_{\mathcal{U},\mathcal{W}}$-null set N, the function $f\cdot\chi_N$ is said to be $\hat{m}_{\mathcal{U},\mathcal{W}}$-*null.* The family of all $\hat{m}_{\mathcal{U},\mathcal{W}}$-null functions will be denoted by $\mathcal{H}_{\mathcal{U},\mathcal{W}}$. For $f \in \mathcal{H}_{\mathcal{U},\mathcal{W}}$, define $\int_E f\chi_N\,dm = \int_{N\cap E} f_n\,dm = 0$, $E \in \Sigma$.

It is easy to see that the family $\mathcal{H}_{\mathcal{U},\mathcal{W}}$ is a vector space.

The description of the following integral we can find in [8].

Definition 4. Let $m\colon \Delta \to L(X,Y)$ be a σ_Φ-additive measure. A function $f \in \mathcal{M}_\mathcal{U}$ is said to be $\Delta_{\mathcal{U},\mathcal{W}}$- *integrable*, we write $f \in \mathcal{I}_{\mathcal{U},\mathcal{W}}$, if

(a) there exists a sequence $f_n \in \mathcal{S}_{\mathcal{U},\mathcal{W}}$, $n \in \mathbb{N}$, of functions, such that

$$\mathcal{U}\text{-}\lim_{n\to\infty} f_n = f\ \hat{m}_{\mathcal{U},\mathcal{W}}\text{-a.e.},$$

(b) the integrals $\int_E f_n\,dm$, $n \in \mathbb{N}$, are uniformly $(\mathcal{W},\sigma)$-additive measures on Σ.

The integral of the function $f \in \mathcal{I}_{\mathcal{U},\mathcal{W}}$ on a set $E \in \Sigma$, is defined by the equality

$$\int_E f\,dm = \mathcal{W}\text{-}\lim_{n\to\infty}\int_E f_n\,dm\,.$$

3. The generalized Kolmogoroff integral of the first type in complete bornological locally convex spaces

Definition of the generalized Kolmogoroff integral of the first type, properties. Let $E \in \Sigma$, $(U,W) \in \mathcal{U}\times\mathcal{W}$. By a countable $\Delta_{U,W}$-partition of the set E we will call a countable system $\omega(E) = (E_i)$ of pairwise disjoint sets $E_i \in \Delta_{U,W}$, such that $\bigcup_{i=1}^{\infty} E_i = E$. If $\omega_1(E) = (E_i)$ and $\omega_2(E) = (F_j)$ are two countable $\Delta_{U,W}$-partitions of the set E then we say that $\omega_2(E)$ is a refinement of the partition $\omega_1(E)$ if to every set F_j there exists E_i, such that $F_j \subset E_i$. In this case we write $\omega_1(E) \le \omega_2(E)$. The set $\Omega_{U,W}(E)$ of all countable $\Delta_{U,W}$-partitions of the set E with this order is a direction.

The first part of the following definition is only an application of the classical Kolmogoroff's definition for Banach spaces to the context we mention.

DEFINITION 5. Let $(U,W) \in \mathcal{U} \times \mathcal{W}$. A function $f\colon T \to X$ is said to be $K^I_{U,W}$-*integrable* on E, if $f\colon T \to X_U$ and $E \in \Sigma$, there exists $y \in Y_W$ and for every $\varepsilon > 0$ there exists a countable $\Delta_{U,W}$-partition $\omega_\varepsilon(E)$, such that for every countable partition $(E_i) = \omega(E) \geq \omega_\varepsilon(E)$ and arbitrary points $t_i \in E_i$ the series $\sum\limits_{i=1}^{\infty} m(E_i)f(t_i)$ converges unconditionally in Y_W and

$$p_W\left(\sum_{i=1}^{\infty} m(E_i)f(t_i) - y\right) < \varepsilon\,.$$

The value y is said to be a $K^I_{U,W}$-*integral* on E, we write $K^I_{U,W}(E,f,m) = y$.

We say that the function $f\colon T \to X$ is $K^I_{\mathcal{U},\mathcal{W}}$-*integrable* (generalized Kolmogoroff integrable in the CBLCS) if there exists $(U,W) \in \mathcal{U} \times \mathcal{W}$, such that f is an $K^I_{U,W}$-integrable function. We write $K^I_{\mathcal{U},\mathcal{W}}(E,f,m) = y$, or simply we write $K^I(E,f,m) = y$.

The following assertions can be proved directly from Definition 5.

LEMMA 1. *Let $(U,W) \in \mathcal{U}\times\mathcal{W}$. Let $f\colon T \to X$ be a $K^I_{U,W}$-integrable function on $E \in \sigma(\Delta_{U,W})$, let (E_i), $i \in \mathbb{N}$, be a countable $\Delta_{U,W}$-partition of the set E. Then f is $K^I_{U,W}$-integrable on every E_i and*

$$K^I(E,f,m) = \sum_{i=1}^{\infty} K^I(E,f_i,m)\,, \tag{2}$$

where the series converges unconditionally in Y_W.

LEMMA 2. *Let $f \in \overline{\sigma(\Delta_{U,W},X)}$, where $\overline{\sigma(\Delta_{U,W},X)}$ is the closure of the set $\sigma(\Delta_{U,W},X)$ with respect to the seminorm $\|\cdot\|_{T,U}$ in the Banach space of all U-bounded functions on T, let $E \in \Sigma$ and $\hat{m}_{U,W}(E) < \infty$.*

Then f is $K^I_{U,W}$-integrable on E, the function $f\chi_E$ is $\Delta_{U,W}$-integrable and $K^I(E,f,m) = \int\limits_E f\,dm$.

LEMMA 3. *Let $(U,W) \in \mathcal{U} \times \mathcal{W}$. Let $f\colon T \to X$ be a $\Delta_{U,W}$-measurable function, let $F = \{t \in T\colon f(t) \neq 0\}$. Let there exist sets $F_k \in \sigma(\Delta_{U,W})$, $k \in \mathbb{N}$, such that $F_k \nearrow F$ and $f\chi_{F_k}$ is a $\Delta_{U,W}$-integrable for $k \in \mathbb{N}$. Then the function f is $\Delta_{U,W}$-integrable if and only if there exists a σ-additive vector measure $\eta\colon \Sigma \to Y_W$, such that $\eta(E \cap F_k) = \int\limits_E f\chi_{F_k}\,dm$ for every $k \in \mathbb{N}$ and every $E \in \Sigma$. In this case $\eta(E) = \int\limits_E f\,dm$ for every $E \in \Sigma$.*

Equivalence theorem.

Theorem 1. *A $\Delta_{\mathcal{U},\mathcal{W}}$-measurable function $f\colon T \to X$ is $\Delta_{\mathcal{U},\mathcal{W}}$-integrable on Σ if and only if it is $K^I_{\mathcal{U},\mathcal{W}}$-integrable on Σ. In this case,*

$$\int_E f\,dm = K^I(E,f,m) \tag{3}$$

for every set $E \in \Sigma$.

P r o o f. We have to prove that for $(U,W) \in \mathcal{U} \times \mathcal{W}$, a $\Delta_{U,W}$-measurable function $f\colon T \to X$ is $\Delta_{U,W}$-integrable on Σ if and only if it is $K^I_{U,W}$-integrable on Σ. In this case,

$$\int_E f\,dm = K^I_{U,W}(E,f,m) \tag{4}$$

for every set $E \in \Sigma$.

Let $f\colon T \to X$ be a $\Delta_{U,W}$-measurable function. Take a sequence $f_n\colon T \to X$, $n \in \mathbb{N}$, of $\Delta_{U,W}$-simple functions, such that $\lim_{n\to\infty} p_U\big(f_n(t) - f(t)\big) = 0$ and $p_U\big(f_n(t)\big) \nearrow p_U\big(f(t)\big)$ for every $t \in T$. As we noted before, $U \in \mathcal{U}$ can be chosen such that X_U is a separable Banach space. Then by Theorem 13 (1) in [2] there exists a σ-additive measure $\eta^F_{U,W}\colon F \cap \sigma(\Delta_{U,W}) \to [0,1]$, such that $N \in F \cap \sigma(\Delta_{U,W})$ and $\eta^F_{U,W}(N) = 0$ implies $\hat{m}_{U,W}(N) = 0$. By the Egoroff–Luzin Theorem, cf. [1], there exists a set $N \in F \cap \sigma(\Delta_{U,W})$ and a sequence of sets $F_k \in F \cap \Delta_{U,W}$, $k \in \mathbb{N}$, such that $\eta^F_{U,W}(N) = 0$, $F_k \nearrow F \setminus N$, and the sequence f_n, $n \in \mathbb{N}$, converges uniformly to the function f on every set F_k, $k \in \mathbb{N}$. Clearly we can change the function f to the function $f\chi_{F\setminus N}$. Since the semivariation $\hat{m}_{U,W}$ is σ-finite on T, without loss of generality we let $\hat{m}_{U,W}(F_k) < \infty$ for every $k \in \mathbb{N}$. Then for every $k \in \mathbb{N}$, the function $f\chi_{F_k}$ and also $f\chi_{F_{k+1}\setminus F_k}$ are $\Delta_{U,W}$-integrable. By Lemma 1, they are $K^I_{U,W}$-integrable on F and both these integrals are equal on every set $E \in \Sigma$, $\hat{m}_{U,W}(E) < \infty$.

Suppose first that the function $f\chi_{F\setminus N}$ is $K^I_{U,W}$-integrable on $F \setminus N$. By the assertion of Lemma 1 (1) the $K^I_{U,W}$-integral $E \to K^I(E,f,m)$, $E \in \Sigma$, is a σ-additive Y_W-valued vector measure, $f\chi_{F\setminus N}$ is a $\Delta_{U,W}$-integrable and (4) holds by Lemma 2.

Conversely, let the function $f\chi_{F\setminus N}$ be $\Delta_{U,W}$-integrable. Take $\varepsilon > 0$, let $k \in \mathbb{N}$, is fixed. Then from the $K^I_{U,W}$-integrability of the function $f\chi_{F_{k+1}\setminus F_k}$ on $F_{k+1}\setminus F_k$, there exists a finite $\Delta_{U,W}$-partition $\omega_\varepsilon(F_{k+1}\setminus F_k)$, such that for every countable $\Delta_{U,W}$-partition $(E_{k,i}) = \omega_\varepsilon(F_{k+1} \setminus F_k) \geq \omega_\varepsilon(F_{k+1} \setminus F_k)$, arbitrary points $t_{k,i} \in E_{k,i}$ and every $E \in \Sigma$, the series $\sum\limits_{i=1}^{\infty} m(E \cap E_{k,i}) f(t_{k,i})$ converges unconditionally in Y_W and it is true

$$p_W\left(\sum_{i=1}^{\infty} m(E\cap E_{k,i})f(t_{k,i}) - \int\limits_{E\cap(F_{k+1}\setminus F_k)} f\,dm\right) \le \frac{\varepsilon}{2^{(k+1)}}. \tag{5}$$

Set $\omega_\varepsilon(F\setminus N) = \sum\limits_{i=1}^{\infty} \omega_\varepsilon(F_{k+1}\setminus N)$ and take a countable $\Delta_{U,W}$-partition $\omega(F\setminus N) \ge \omega_\varepsilon(F\setminus N)$. Then $\omega(F\setminus N) = \sum\limits_{k=1}^{\infty} \omega(F_{k+1}\setminus F_k)$, where $\omega(F_{k+1}\setminus F_k) = (E_{k,i}) \ge \omega_\varepsilon(F_{k+1}\setminus F_k)$ is a countable $\Delta_{U,W}$-partition for every $k\in\mathbb{N}$. Take arbitrary points $t_{k,i}\in E_{k,i}$. We assert that the series $\sum\limits_{k=1}^{\infty}\sum\limits_{i=1}^{\infty} m(E_{k,i})f(t_{k,i})$ converges unconditionally in Y_W. By [3], Lemma 4, this is equivalent to two assertions:

(a) For every $k\in\mathbb{N}$, the series $\sum\limits_{i=1}^{\infty} m(E_{k,i})f(t_{k,i})$ is unconditionally convergent in Y_W, which follows from the $K^I_{U,W}$-integrability of the function f on $F_{k+1}\setminus F_k$,

(b) For every sequence of subsets $I_k = \mathbb{N}$, $k\in\mathbb{N}$, the series

$$\sum_{k=1}^{\infty}\sum_{i\in I_k} m(E_{k,i})f(t_{k,i})$$

is unconditionally convergent in Y_W. This holds since for a fixed I_k, $k\in\mathbb{N}$, with respect to (5) we have:

$$p_W\left(\sum_{i\in I_k} m(E_{k,i})f(t_{k,i}) - y_k\right) < \varepsilon 2^{-(k+1)},$$

where

$$y_k = \int\limits_A f\,dm, \qquad A = \left(\bigcup_{i\in I_k} E_{k,i}\right)\cap(F_{k+1}\setminus F_k),$$

for every $k\in\mathbb{N}$, and the series $\sum\limits_{k=1}^{\infty} y_k$ is unconditionally convergent in Y_W which follows from the σ-additivity of the integral

$$E \to \int\limits_E f\chi_{F\setminus N}\,dm, \qquad E\in\Sigma.$$

By (5) we have

$$p_W\left(\sum_{k=1}^{\infty}\sum_{i=1}^{\infty} m(E_{k,i})f(t_{k,i}) - \int\limits_{F\setminus N} f\,dm\right) < \varepsilon.$$

Thus the function $f\chi_{F\setminus N}$ is $K^I_{U,W}$-integrable on $F\setminus N$ by Definition 4 and (4) holds. Therefore the function $f\chi_{F\setminus N}$ is K^I-integrable. $\square$

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