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# BILATERAL QUASICONTINUITY IN TOPOLOGICAL SPACES

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ABSTRACT. A function $f\colon \mathbb{R} \to \mathbb{R}$ is said to be bilaterally quasicontinuous at a point $x$ if for every $\delta > 0$ and for every neighbourhood $V$ of $f(x)$ there exist open nonempty sets $G_1 \subset (x-\delta, x) \cap f^{-1}(V)$ and $G_2 \subset (x, x+\delta) \cap f^{-1}(V)$. Some possibilities to define a bilateral quasicontinuity for functions defined on metric or topological spaces are given.

A real function $f\colon \mathbb{R} \to \mathbb{R}$ is said to be left (right) hand sided quasicontinuous at a point $x \in \mathbb{R}$ if for every $\delta > 0$ and for every neighbourhood $V$ of $f(x)$ there exists an open nonempty set $G \subset (x-\delta, x) \cap f^{-1}(V)$ $(G \subset (x, x+\delta) \cap f^{-1}(V))$. A function $f$ is bilaterally quasicontinuous at $x$ if it is both left and right hand sided quasicontinuous at this point. A function is bilaterally quasicontinuous if it is such at each point ([7]).

In this paper we will give some possibilities to define a bilateral quasicontinuity for functions defined on metric or topological space.

For a subset $A$ of a topological space we denote by $\operatorname{Cl} A$ and $\operatorname{Int} A$ the closure and the interior of $A$, respectively. The letters $\mathbb{N}$, $\mathbb{Q}$ and $\mathbb{R}$ stand for the set of positive integers, rational and real numbers, respectively. If $(X, d)$ is a metric space then $S(a, \varepsilon) = \{y \in X \colon d(y, a) < \varepsilon\}$ denotes the open ball with the centre $a$ and the radius $\varepsilon$. For a set $A \subset X$ and a point $a \in X$, $\operatorname{dist}(a, A) = \inf\{d(a, b) \colon b \in A\}$.

**DEFINITION 1.** Let $X$ and $Y$ be topological spaces. We shall say that $f\colon X \to Y$ is *$B$-quasicontinuous at $x \in X$* if for every neighbourhood $V$ of $f(x)$ and for every open connected set $A$ such that $x \in \operatorname{Cl} A$ there exists an open nonempty set $G \subset A$ such that $f(G) \subset V$. A function $f$ is *$B$-quasicontinuous* if it is such at each point.

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**DEFINITION 2.** Let $(X, d)$ be a metric space and let $Y$ be a topological space. We shall say that a function $f\colon X \to Y$ is *$S$-quasicontinuous at* $x \in X$ if for every neighbourhood $V$ of $f(x)$ and for every $y \in X$, $y \neq x$, there exists an open nonempty set $G \subset S\bigl(y, d(x,y)\bigr)$ such that $f(G) \subset V$.

We shall say that a function $f\colon X \to Y$ is $T$-quasicontinuous at $x \in X$ if for every neighbourhood $V$ of $f(x)$ and for every $y \in X$ such that $y \neq x$ and $x \in \operatorname{Cl} S\bigl(y, d(x,y)\bigr)$ there exists an open nonempty set $G \subset S\bigl(y, d(x,y)\bigr)$ such that $f(G) \subset V$.

**THEOREM 1.** *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function. Then the following conditions are equivalent:*

a) *$f$ is bilaterally quasicontinuous at $x$;*

b) *$f$ is $B$-quasicontinuous at $x$;*

c) *$f$ is $S$-quasicontinuous at $x$;*

d) *$f$ is $T$-quasicontinuous at $x$.*

Proof. Obvious. □

The following proposition shows that the assumption of the connectedness of $A$ in Definition 1 cannot be omitted.

**PROPOSITION 1.** *Let $X$ be a topological space and let $Y$ be a regular space. Then a function $f\colon X \to Y$ is continuous if and only if for each $x \in X$, for each neighbourhood $V$ of $f(x)$ and for each open set $A$ with $x \in \operatorname{Cl} A$ there is an open nonempty set $G \subset A$ such that $f(G) \subset V$.*

Proof. Suppose that $f$ is not continuous at a point $x \in X$. Then there is a neighbourhood $W$ of $f(x)$ such that for each neighbourhood $U$ of $x$ there is $x_U \in U$ such that $f(x_U) \notin W$. Let $V$ be an open neighbourhood of $f(x)$ such that $\operatorname{Cl} V \subset W$. Then there is an open nonempty set $G_U \subset U$ such that $f(G_U) \subset Y \setminus \operatorname{Cl} V$. The set $H = \bigcup\{G_U : U \text{ is an open neighbourhood of } x\}$ is open and $x \in \operatorname{Cl} H$. However, $f(G_U) \not\subset V$ for each open nonempty $G \subset H$. The opposite implication is obvious. □

This assertion does not hold pointwise (e.g., $X = Y = \mathbb{R}$ with the usual topology, $f(1/n) = n$ and $f(x) = 0$ otherwise) and it does not hold for arbitrary topological space $Y$ (e.g., $X = \mathbb{R}$ with the usual topology, $Y = \mathbb{R}$ with the topology $\mathcal{T}$, where $A \in \mathcal{T}$ iff $A = \emptyset$ or $\mathbb{R} \setminus A$ is nowhere dense and $f\colon X \to Y$ is the identity).

Recall that a function $f\colon X \to Y$ is quasicontinuous at $x$ if for each neighbourhood $V$ of $f(x)$ and for each neighbourhood $U$ of $x$ there exists an open nonempty set $G \subset U$ such that $f(G) \subset V$ ([11]). Denote by $C(f)$, $Q(f)$, $B(f)$, $S(f)$ and $T(f)$ the set of all continuity points, quasicontinuity points,

$B$-quasicontinuity points, $S$-quasicontinuity points and $T$-quasicontinuity points of a function $f\colon X \to Y$, respectively. If $X = Y = \mathbb{R}$ then $C(f) \subset B(f) = S(f) = T(f) \subset Q(f)$. Further denote by $\mathcal{C}$, $\mathcal{B}$, $\mathcal{S}$, $\mathcal{T}$ and $\mathcal{Q}$ the family of all continuous, $B$-quasicontinuous, $S$-quasicontinuous, $T$-quasicontinuous and quasicontinuous functions, respectively.

**DEFINITION 3.** We shall say that a metric space $(X,d)$ possesses PCB (property of closed balls) if $\operatorname{Cl} S(x,\varepsilon) = \{y \in X : d(x,y) \le \varepsilon\}$ for each $x \in X$ and each $\varepsilon > 0$.

**THEOREM 2.** *Let $Y$ be a topological space. Then:*

a) *If $X$ is a topological space then $C(f) \subset B(f) \not\subset Q(f)$.*

b) *If $X$ is a locally connected space then $B(f) \subset Q(f)$.*

c) *If $X$ is a metric space then $C(f) \not\subset S(f) \subset Q(f)$.*

d) *If $X$ is a metric space then $C(f) \subset T(f) \not\subset Q(f)$.*

e) *If a metric space $X$ possesses* PCB *then $S(f) = T(f)$.*

Proof. a) Let $V$ be a neighbourhood of $f(x)$ and let $A$ be an open connected set with $x \in \operatorname{Cl} A$. If $x \in C(f)$ then there is an open neighbourhood $U$ of $x$ such that $f(U) \subset V$. Then $G = A \cap U$ is an open nonempty subset of $A$ and $f(G) \subset V$, i.e., $x \in B(f)$. If $X = \mathbb{Q}$ with the usual topology then each function on $X$ is $B$-quasicontinuous.

b) Let $V$ be a neighbourhood of $f(x)$ and $U$ be a neighbourhood of $x$. Then there exists an open connected neighbourhood $T$ of $x$ such that $T \subset U$. Now there exists an open nonempty set $G \subset T$ such that $f(G) \subset V$.

c) Let $x \in S(f)$, let $V$ be a neighbourhood of $f(x)$ and let $U$ be a neighbourhood of $x$. Let $\varepsilon > 0$ be such that $S(x,\varepsilon) \subset U$. If $x$ is an isolated point of $X$ then evidently $x \in Q(f)$. Otherwise there is a $y \in S(x,\varepsilon/2)$, $y \ne x$. Now there is an open nonempty set $G \subset S\big(y, d(x,y)\big) \subset S(x,\varepsilon)$ such that $f(G) \subset V$, i.e., $x \in Q(f)$. If $X = \mathbb{N}$ with the usual metric and $f\colon X \to \mathbb{R}$, $f(u) = u$, then $C(f) = X$ and $S(f) = \emptyset$.

d) Let $x \in C(f)$, let $V$ be a neighbourhood of $f(x)$ and let $y \in X$, $y \ne x$, be such that $x \in \operatorname{Cl} S$, where $S = S\big(y, d(x,y)\big)$. Then there exists an open neighbourhood $U$ of $x$ such that $f(U) \subset V$. Now $U \cap S$ is an open nonempty subset of $S$, i.e., $x \in T(f)$.

Let $X = \{x_i : i \in \mathbb{N}\}$ with the metric $d$, where $d(x_i, x_i) = 0$, $d(x_1, x_j) = 1/j$ for $j \ne 1$ and $d(x,y) = 1/i + 1/j$ for $i \ne j$, $i,j > 1$. If we put $f(x_i) = i$ then $T(f) \setminus Q(f) \ne \emptyset$.

e) Obvious. $\square$

Therefore it is reasonable to investigate $B$-quasicontinuity on locally connected spaces and $S$-quasicontinuity and $T$-quasicontinuity on metric spaces with PCB.

Now we will give a characterization of the sets $B(f)$ and $Q(f)$. Recall that a topological space is *almost resolvable* if it is a countable union of sets with empty interiors. Every first countable topological space without isolated points, locally compact Hausdorff topological space without isolated points or real linear topological space is almost resolvable ([1]). A topological space is *perfect* if every closed subset of this space is $G_\delta$ ([4]). A space is *perfect normal* if it is normal (need not be $T_1$) and perfect.

**THEOREM 3.** *Let $X$ be a locally connected perfectly normal almost resolvable topological space. Let $B \subset X$. Then $B = B(f)$ for some function $f\colon X \to \mathbb{R}$ if and only if the set $\operatorname{Cl} B \setminus B$ is of the first category.*

**THEOREM 3'.** *Let $(X, d)$ be a metric space with* PCB *without isolated points. Let $B \subset X$. Then $B = S(f)$ for some function $f\colon X \to \mathbb{R}$ if and only if the set $\operatorname{Cl} B \setminus B$ is of the first category.*

Proof of Theorems 3 and 3'. Necessity. By Theorem 2 we obtain $C(f) \subset B(f) \subset Q(f)$ and $C(f) \subset S(f) \subset Q(f)$. By [5], the set $\operatorname{Cl} Q(f) \setminus C(f)$ is of the first category and hence $\operatorname{Cl} B(f) \setminus B(f) \subset \operatorname{Cl} Q(f) \setminus C(f)$ is of the first category and similarly for $\operatorname{Cl} S(f) \setminus S(f)$.

Sufficiency. The set $\operatorname{Cl} B \setminus B$ is of the first category, therefore $\operatorname{Cl} B \setminus B = \bigcup\limits_{n=1}^{\infty} A_n$, where $A_n$ are nowhere dense sets. We can assume that $A_n \subset A_{n+1}$ for each $n \in \mathbb{N}$. Since $X$ is perfectly normal there is a continuous function $h\colon X \to [0, \infty)$ such that $h^{-1}(0) = \operatorname{Cl} B$. Since $X$ is almost resolvable (by remark above a metric space without isolated points is almost resolvable) we can write $X = \bigcup\limits_{m=1}^{\infty} X_m$, where $\operatorname{Int} X_m = \emptyset$ for each $m \in \mathbb{N}$. We can assume that $X_i \cap X_j = \emptyset$ for $i \neq j$. Put $A_0 = \emptyset$ and define a function $f\colon X \to \mathbb{R}$ as

$$f(x) = \begin{cases} 0 & \text{if } x \in B\,, \\ 1/k & \text{if } x \in A_k \setminus A_{k-1}\,, \\ 2^{-m} \cdot h(x) & \text{if } x \in (X \setminus \operatorname{Cl} B) \cap X_m\,. \end{cases}$$

We shall show that $B(f) = B$ (for Theorem 3) and $S(f) = B$ (for Theorem 3').

1. Let $x \in B$ and let $\varepsilon > 0$. Let $k \in \mathbb{N}$ be such that $1/k < \varepsilon$. Since $h$ is continuous at $x$ there exists an open neighbourhood $U$ of $x$ such that $0 \le h(y) < \varepsilon$ for each $y \in U$. Let $A$ be an open connected set with $x \in \operatorname{Cl} A$ (for Theorem 3) or let $A = S\big(z, d(x,z)\big)$, $z \neq x$ (for Theorem 3'). Then $A \cap U \neq \emptyset$

and since $A_k$ is nowhere dense, the set $G = (A \cap U) \setminus \operatorname{Cl} A_k$ is an open nonempty subset of $A$.

Let $y \in G$. If $y \in B$ then $|f(x) - f(y)| = 0$. If $y \in X \setminus \operatorname{Cl} B$ then $y \in X_m$ for some $m \in \mathbb{N}$ and hence $|f(x) - f(y)| = 2^{-m} h(y) < h(y) < \varepsilon$. If $y \in \operatorname{Cl} B \setminus B$ then $y \in A_j \setminus A_{j-1}$ for some $j \in \mathbb{N}$. However $y \notin \operatorname{Cl} A_k$ and hence $j > k$ and $|f(x) - f(y)| = f(y) = 1/j < \varepsilon$. Therefore $B \subset B(f)$ (Theorem 3) and $B \subset S(f)$ (Theorem 3').

2. Let $x \in X \setminus \operatorname{Cl} B$. Then $x \in X_m$ for some $m \in \mathbb{N}$. Put $\alpha = \frac{3h(x)}{2^m \cdot 8} > 0$. If $X$ is locally connected then there exists an open connected neighbourhood $A$ of $x$ such that $A \subset X \setminus \operatorname{Cl} B$ and $|h(x) - h(y)| < \frac{h(x)}{4}$ for $y \in A$. If $X$ is a metric space with PCB without isolated points then there exists $\delta > 0$ such that $S(x,\delta) \subset X \setminus \operatorname{Cl} B$ and $|h(x) - h(y)| < \frac{h(x)}{4}$ for $y \in S(x,\delta)$. Since $x$ is not an isolated point there exists $z \in S(x,\delta/2)$, $z \neq x$. Then $A = S\big(z, d(x,z)\big) \subset S(x,\delta)$.

We shall show that $x \notin B(f)$ (for Theorem 3) and $x \notin S(f)$ (for Theorem 3'). Let $G \subset A$ be an open nonempty set. Since $\operatorname{Int} X_m = \emptyset$ there exists $k \neq m$ such that $G \cap X_k \neq \emptyset$. Choose $u \in G \cap X_k$. Then $f(u) = 2^{-k} h(u)$. If $m > k$ then $2^{m-k} h(u) \geq 2h(u) > h(x)$ and hence

$$|f(u) - f(x)| = \frac{1}{2^m}\big(2^{m-k} h(u) - h(x)\big) > \frac{1}{2^m}\left(\frac{3h(x)}{2} - h(x)\right) = \frac{h(x)}{2 \cdot 2^m} > \alpha\,.$$

If $m < k$ then $2^{m-k} h(u) \leq 2^{-1} h(u) < h(x)$ and hence

$$|f(u) - f(x)| = 2^{-m}\big(h(x) - 2^{m-k} h(u)\big) > 2^{-m}\left(h(x) - \frac{5h(x)}{8}\right) = \frac{3h(x)}{2^m \cdot 8} = \alpha\,.$$

3. Let $x \in \operatorname{Cl} B \setminus B$. Then $x \in A_k \setminus A_{k-1}$ for some $k \in \mathbb{N}$. Let $\alpha > 0$ be such that $|1/k - 1/j| > \alpha$ for each $j \neq k$. If $X$ is a locally connected topological space then there exists an open connected neighbourhood $A$ of $x$ such that $0 \leq h(y) < \alpha$ for each $y \in A$. If $X$ is a metric space with PCB without isolated points then there exists $\delta > 0$ such that $0 \leq h(y) < \alpha$ for each $y \in S(x,\delta)$. Now there exists $z \in S(x,\delta/2)$, $z \neq x$, and $0 \leq h(y) < \alpha$ for each $y \in A = S\big(z, d(x,z)\big)$.

We shall show that $x \notin B(f)$ (for Theorem 3) and $x \notin S(f)$ (for Theorem 3'). Let $G \subset A$ be an open nonempty set. Since $A_k$ is nowhere dense there exists $u \in G \setminus \operatorname{Cl} A_k$. If $u \in B$ then $|f(u) - f(x)| = 1/k > \alpha$. If $u \in X \setminus \operatorname{Cl} B$ then $f(u) = 2^{-m} h(u)$ for some $m \in \mathbb{N}$ and $f(u) < h(u) < \alpha < f(x)$. Therefore $|f(u) - f(x)| = 1/k - f(u) > 1/k - \alpha > 1/(2k) > \alpha$. If $u \in \operatorname{Cl} B \setminus B$ then $u \in A_j \setminus A_{j-1}$ for some $j \in \mathbb{N}$. Since $u \notin \operatorname{Cl} A_k$ we have $j \neq k$ and hence $|f(u) - f(x)| = |1/j - 1/k| > \alpha$. □

**Remark 1.** In [2], the quadruple $\big(C(f),\ B(f),\ Q(f),\ \operatorname{Cl} Q(f)\big)$ was characterized for $X = Y = \mathbb{R}$. In the proof, the fact that the set $Q(f) \setminus B(f)$ is countable, is essential. This fact is not true for $X = \mathbb{R}^2$ (e.g., if $f(x,y) = 1$ for $x \geq 0$ and $f(x,y) = 0$ for $x < 0$ then the set $Q(f) \setminus B(f) = Q(f) \setminus S(f)$ is uncountable.

**Problem.** Characterize the triplet $\big(C(f), B(f), Q(f)\big)$ or $\big(C(f), S(f), Q(f)\big)$ at least for $X = \mathbb{R}^2$.

Recall that a function $f\colon X \to Y$ has a closed graph if the set $G(f) = \big\{\big(x, f(x)\big) : x \in X\big\}$ is a closed subset of $X \times Y$. A function $f\colon X \to Y$ is $c$-continuous if for each open neighbourhood $V$ of $f(x)$ such that $Y \setminus V$ is compact there is an open neighbourhood $U$ of $x$ such that $f(U) \subset V$. A function is $c$-continuous if it is such at each point ([6]). It is known that if $Y$ is Hausdorff (but not arbitrary) then continuous functions have a closed graph. Every continuous function is $c$-continuous and every function with closed graph is $c$-continuous. If $Y$ is locally compact Hausdorff then $c$-continuous functions have a closed graph. The assumptions on $Y$ cannot be omitted ([10]). Quasicontinuous functions with closed graph need not be continuous. However, bilaterally quasicontinuous functions $f\colon \mathbb{R} \to \mathbb{R}$ with closed graph are continuous ([3]). We will generalize this result for $B$-quasicontinuity and $S$-quasicontinuity.

**LEMMA 1.** *Let $X$ and $Y$ be topological spaces. Let $f\colon X \to Y$ be a $c$-continuous function. If $f$ is not continuous at $x \in X$ then for each open neighbourhood $U$ of $x$ and for each neighbourhood $V$ of $f(x)$ such that $\operatorname{Cl} V$ is compact the set $U \setminus f^{-1}(\operatorname{Cl} V)$ is nonempty.*

P r o o f . Suppose that $U \setminus f^{-1}(\operatorname{Cl} V) = \emptyset$ for some open neighbourhood $U$ of $x$ and some open neighbourhood $V$ of $f(x)$ with compact closure. Since $f$ is $c$-continuous the set $f^{-1}(\operatorname{Cl} V)$ is closed. Since $f$ is not continuous at $x$ there exists an open neighbourhood $W$ of $f(x)$ such that for each neighbourhood $T$ of $x$ there is $x_T \in T$ with $f(x_T) \notin W$. The set $S = W \cup (Y \setminus \operatorname{Cl} V)$ is an open neighbourhood of $f(x)$ such that $Y \setminus S = \operatorname{Cl} V \setminus W$ is compact. Hence $H = f^{-1}(S) \cap U$ is a neighbourhood of $x$. Then $f(H) \subset f(U) \subset \operatorname{Cl} V$ and thus $f(H) \subset W$. However, $f(x_H) \notin W$, a contradiction. □

**THEOREM 4.** *Let $X$ be a locally connected topological space and let $Y$ be a locally compact topological space. Then $f\colon X \to Y$ is continuous if and only if $f$ is $B$-quasicontinuous and $c$-continuous.*

P r o o f . Necessity is obvious. Sufficiency. Suppose that $f$ is not continuous at $x$ for some $x \in X$. Let $U$ be a connected open neighbourhood of $x$ and let $V$ be an open neighbourhood of $f(x)$ such that $\operatorname{Cl} V$ is compact. Then according to Lemma 1, the set $U \setminus f^{-1}(\operatorname{Cl} V)$ is nonempty.

Let $T$ be a component in $U \setminus f^{-1}(\operatorname{Cl} V)$. Since $X$ is locally connected, $T$ is open. Since $U$ is connected we obtain $U \cap \operatorname{Cl} T \neq U \cap T$. Choose $t \in U \cap (\operatorname{Cl} T \setminus T)$. Then $t \in f^{-1}(\operatorname{Cl} V)$ (otherwise, $t$ is a member of some open component and hence $t \notin \operatorname{Cl} T$).

Therefore $T$ is an open connected set. Since $f(T) \subset Y \setminus \operatorname{Cl} V$ the $B$-quasicontinuity of $f$ at $t$ implies $f(t) \notin V$. Therefore, for each open connected neighbourhood $U$ of $x$ there is $t_U \in U$ such that $f(t_U) \in \operatorname{Cl} V \setminus V$. Now $P = V \cup (Y \setminus \operatorname{Cl} V)$ is an open neighbourhood of $f(x)$ such that $Y \setminus P = \operatorname{Cl} V \setminus V$ is compact. The $c$-continuous yields that $f^{-1}(P)$ is a neighbourhood of $x$. Let $D \subset f^{-1}(P)$ be an open connected neighbourhood of $x$. Then $f(D) \subset V \cup (Y \setminus \operatorname{Cl} V)$ and therefore $f(D) \cap (\operatorname{Cl} V \setminus V) = \emptyset$. However, $f(t_D) \in f(D) \cap (\operatorname{Cl} V \setminus V)$, a contradiction. □

**Corollary 1.** *Let $X$ be a locally connected topological space and let $Y$ be a locally compact Hausdorff topological space. Then $f\colon X \to Y$ is continuous if and only if $f$ is $B$-quasicontinuous and it has a closed graph.*

**Theorem 5.** *Let $(X, d)$ be a locally compact metric space and let $Y$ be a locally compact topological space. If $f\colon X \to Y$ is $S$-quasicontinuous and $c$-continuous then it is continuous.*

Proof. Suppose that $f$ is not continuous at $x$. Let $U$ be an open neighbourhood of $x$ and let $V$ be an open neighbourhood of $f(x)$ such that $\operatorname{Cl} U$ and $\operatorname{Cl} V$ are compact. Let $\sigma > 0$ be such that $S(x, \sigma) \subset U$.

Let $\eta$ be such that $0 < \eta < \sigma$. Then, according to Lemma 1, $D = S(x, \eta/2) \setminus f^{-1}(\operatorname{Cl} V) \neq \emptyset$. Choose $y \in D$. Since $f$ is $c$-continuous the set $f^{-1}(\operatorname{Cl} V)$ is closed and hence $D$ is open. This implies that the set $L = \{\varepsilon > 0\colon S(y, \varepsilon) \subset S(x, \eta) \setminus f^{-1}(\operatorname{Cl} V)\}$ is nonempty and bounded. Denote by $\beta = \sup L$ and $S = S(y, \beta)$. Then we obtain $S = \bigcup\limits_{0<\varepsilon<\beta} S(y, \varepsilon) \subset S(x, \eta) \setminus f^{-1}(\operatorname{Cl} V)$, i.e., $\beta \in L$.

Since $x \in f^{-1}(\operatorname{Cl} V)$ for each $\varepsilon \in L$ we have $x \notin S(y, \varepsilon)$, i.e., $d(x, y) \geq \varepsilon$ and hence $\beta \leq d(x, y) < \eta/2$.

If $d(z, y) < \varepsilon < \eta/2$ then $d(z, x) \leq d(z, y) + d(y, x) < \varepsilon + \eta/2 < \eta$, i.e., $S(y, \varepsilon) \subset S(x, \eta)$ for each $0 < \varepsilon < \eta/2$. Since $\beta \in L$ so $S(y, \varepsilon) \cap f^{-1}(\operatorname{Cl} V) \neq \emptyset$ for each $\varepsilon$ with $\beta < \varepsilon < \eta/2$. Therefore $\{\{t \in X\colon d(y, t) \leq \varepsilon\} \cap f^{-1}(\operatorname{Cl} V)\}_{\beta<\varepsilon<\eta/2}$ is a family of closed subsets of $\operatorname{Cl} U$ with the finite intersection property. Denote by $T = \{t \in X\colon d(y, t) \leq \beta\}$. Since $\operatorname{Cl} U$ is compact we obtain

$$T \cap f^{-1}(\operatorname{Cl} V) = \bigcap_{\beta<\varepsilon<\eta/2} \{t \in X\colon d(y, t) \leq \varepsilon\} \cap f^{-1}(\operatorname{Cl} V) \neq \emptyset .$$

Now let $t \in T \cap f^{-1}(\operatorname{Cl} V)$. Then $t \in T \setminus S$ and hence $d(y, t) = \beta$. Since $S \subset S(x, \eta) \setminus f^{-1}(\operatorname{Cl} V)$ we obtain $f(S) \subset Y \setminus \operatorname{Cl} V$. If we assume $f(t) \in V$ then there exists a nonempty open set $G \subset S\big(y, d(y, t)\big) = S$ such that $f(G) \subset V$, a contradiction. Hence $f(t) \notin V$.

Therefore for each $\eta > 0$ there exists $t_\eta \in S(x, \eta)$ such that $f(t_\eta) \in \operatorname{Cl} V \setminus V$.

Then $P = V \cup (Y \setminus \operatorname{Cl} V)$ is an open neighbourhood of $x$. Let $\delta > 0$ be such that $S(x,\delta) \subset f^{-1}(P)$. Then $f\bigl(S(x,\delta)\bigr) \subset V \cup (Y \setminus \operatorname{Cl} V)$, i.e., $f\bigl(S(x,\delta)\bigr) \cap (\operatorname{Cl} V \setminus V) = \emptyset$. However, $f(t_\delta) \in \operatorname{Cl} V \setminus V$, a contradiction. ☐

**COROLLARY 2.** *Let $(X,d)$ be a locally compact metric space with* PCB *and let $Y$ be a locally compact topological space. Then $f\colon X \to Y$ is continuous if and only if it is $S$-quasicontinuous and $c$-continuous.*

**COROLLARY 3.** *Let $(X,d)$ be a locally compact metric space with* PCB *and let $Y$ be a Hausdorff locally compact topological space. Then $f\colon X \to Y$ is continuous if and only if it is $S$-quasicontinuous with closed graph.*

**Remark 2.** Theorem 5 does not hold for $T$-quasicontinuity. If $X$ and $f$ are from the proof of Theorem 2 d) then $X$ is a compact metric space, $Y$ is a locally compact space and $f$ is $T$-quasicontinuous and $c$-continuous but not continuous.

**Remark 3.** Theorems 4 and 5 do not hold pointwise (if $X = Y = \mathbb{R}$ and $f(1/n) = n$ and $f(x) = 0$ otherwise then $f$ is $c$-continuous and bilaterally quasicontinuous at 0 but $f$ is not continuous at 0). In Theorems 4 and 5, local compactness of $Y$ cannot be replaced by the condition "the family of compact neighbourhoods of each point is a base for its neighbourhood system". If $X = \mathbb{R}$ with the usual metric and $Y = \mathbb{N}$ with the topology $\mathcal{T}$, where $A \in \mathcal{T}$ iff $A = \emptyset$ or $1 \in A$, then $f\colon X \to Y$, $f(1/n) = n$ and $f(x) = 1$ otherwise, is $B$-quasicontinuous, $S$-quasicontinuous and $c$-continuous but it is not continuous.

In [8] P. S. Keller introduced the concept of dense mapping property DMP. A function $f\colon X \to Y$, where $X$ and $Y$ are topological spaces, has DMP if for every subset $D$ of $X$ with $\operatorname{Cl} D$ connected, $f(\operatorname{Cl} D) \subset \operatorname{Cl} f(D)$. He proved that the characterization of the omega-limit sets for Darboux functions $f\colon \mathbb{R} \to \mathbb{R}$ with DMP is the same as for continuous functions ([8]), R. A. Mimna in [9] has showed that a function $f\colon \mathbb{R} \to \mathbb{R}$ has DMP if and only if it is bilaterally quasicontinuous. However, this possible generalization of the bilateral quasicontinuity is only global, not "pointwise".

**PROPOSITION 2.** *Let $X$ and $Y$ be topological spaces. If $f\colon X \to Y$ has* DMP *then $f$ is $B$-quasicontinuous.*

Proof. Suppose that $f$ is not $B$-quasicontinuous at $x \in X$. Then there exists an open neighbourhood $V$ of $f(x)$ and an open connected set $U$ with $x \in \operatorname{Cl} U$ such that for each nonempty open $G \subset U$ there exists $y_G \in G$ with $f(y_G) \notin V$. Put $D = \{y \in U : f(y) \notin V\}$. Then $f(D) \cap V = \emptyset$, i.e., $f(D) \subset Y \setminus V$ and also $\operatorname{Cl} f(D) \subset Y \setminus V$. However $D$ is dense in $U$, therefore $\operatorname{Cl} D = \operatorname{Cl} U$. Since $\operatorname{Cl} D$ is connected we obtain $f(\operatorname{Cl} D) \subset \operatorname{Cl} f(D)$. However $x \in \operatorname{Cl} D$ and thus $f(x) \in \operatorname{Cl} f(D) \cap V$, a contradiction. ☐

**Remark 4.** If $X = \mathbb{R}^2$ and $Y = \mathbb{R}$ (with the usual topology), $f\colon \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = \sin 1/x$ if $x \neq 0$, $f(0,y) = 0$ if $y$ is rational and $f(0,y) = 1$ if $y$ is irrational, then $f$ is $B$-quasicontinuous but not DMP.

Now we will show that separably $S$-quasicontinuous functions are $S$-quasicontinuous but separately $B$-quasicontinuous functions need not be $B$-quasicontinuous. If $(X, d_X)$ and $(Y, d_Y)$ are metric spaces then $(X \times Y, d)$ is a metric space with the metric $d\big((x_1, y_1), (x_2, y_2)\big) = \sqrt{d_X^2(x_1, x_2) + d_Y^2(y_1, y_2)}$. For a function $f\colon X \times Y \to Z$ the symbols $f_x$ and $f^y$ denote its $x$-section and $y$-section, respectively; i.e., $f_x$ is a function defined on $Y$ such that $f_x(y) = f(x,y)$ and similarly $f^y$.

**Theorem 6.** *Let $(X, d_X)$ and $(Y, d_Y)$ be Baire locally separable metric spaces and let $Z$ be a regular space. If a function $f\colon X \times Y \to Z$ is separately $S$-quasicontinuous then $f$ is $S$-quasicontinuous.*

Proof. Suppose that $f$ is not $S$-quasicontinuous. Then there are points $(a,b), (c,d) \in X \times Y$, $(a,b) \neq (c,d)$, and an open neighbourhood $V$ of $f(a,b)$ such that for each open nonempty set $G \subset S$ there is a point $z_G \in G$ with $f(z_G) \notin V$, where $S = S\big((c,d),\, d\big((a,b),\, (c,d)\big)\big)$. Therefore we have

$$\operatorname{Int} f^{-1}(V) \cap S = \emptyset .$$

If $a \neq c$ and $t \in S\big(c, d_X(a,c)\big)$ then $d\big((t,b),(c,d)\big) = \sqrt{d_X^2(t,c) + d_Y^2(b,d)} < \sqrt{d_X^2(a,c) + d_Y^2(b,d)} = d\big((a,b),(c,d)\big)$, i.e., $(t,b) \in S$. Similarly we can prove that $\{a\} \times S\big(d, d_Y(b,d)\big) \subset S$ for $b \neq d$. Since $(a,b) \neq (c,d)$, at least one of these inclusions must hold. Assume, e.g., that $S\big(c, d_X(a,c)\big) \times \{b\} \subset S$.

Let $T$ be an open neighbourhood of $f(a,b)$ such that $f(a,b) \in T \subset \operatorname{Cl} T \subset V$. Since $f^b$ is $S$-quasicontinuous at $a$ there exists an open nonempty set $H \subset S\big(c, d_X(a,c)\big)$ such that $f^b(H) \subset T$. Let $P \subset Y$ be a separable neighbourhood of $b$. Let $U \subset H$ and $W \subset P$ be open sets such that $b \in W$ and $U \times W \subset (H \times Y) \cap S$. Let $(W_n)_n$ be a countable base in $W$. Denote by

$$U_n = \big\{x \in U \colon W_n \subset \operatorname{Int}\, (f_x)^{-1}(T)\big\} .$$

We shall show that $U = \bigcup\limits_{n=1}^{\infty} U_n$. Let $x \in U$. Then $f^b(x) \in T$. Since, according to Theorem 2c), $f_x$ is quasicontinuous at $b$, there exists an open nonempty set $L \subset W$ such that $f_x(L) \subset T$ and therefore $\operatorname{Int}\, (f_x)^{-1}(T) \cap W \neq \emptyset$. Hence there is $n \in \mathbb{N}$ with $W_n \subset \operatorname{Int}\, (f_x)^{-1}(T)$, i.e., $x \in U_n$. The opposite inclusion is obvious.

Now we shall prove that $U_n$ is nowhere dense in $U$ for each $n \in \mathbb{N}$. Let $D \subset U$ be an open nonempty set. Then $D \times W_n$ is an open nonempty subset of $S$. Hence $(D \times W_n) \cap \operatorname{Int} f^{-1}(V) = \emptyset$. Therefore there exists $(u,v) \in D \times W_n$ such that $f(u,v) \notin V$. Let $M$ be a neighbourhood of $f(u,v)$ such that $M \cap T = \emptyset$. Since $f^v$ is quasicontinuous at $u$ there exists an open nonempty set $E \subset D$ such that $f^v(x) = f(x,v) \in M$ for each $x \in E$. Therefore for each $x \in E$ we have $f(x,v) \notin T$, i.e., $v \notin (f_x)^{-1}(T)$. This yields $x \notin U_n$ for each $x \in E$ and therefore $E \cap U_n = \emptyset$; i.e., $U_n$ is nowhere dense in $U$. However then $U$ is an open nonempty set of the first category, a contradiction. □

**THEOREM 7.** *There is a separately continuous function $f\colon \mathbb{R}^2 \to \mathbb{R}$ which is not $B$-quasicontinuous.*

Proof. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = \frac{xy}{x^2+y^2}$ for $(x,y) \neq (0,0)$ and $f(0,0) = 0$. Then $f$ is separately continuous. If $A = \{(x,y) \in \mathbb{R}^2 \colon 0 < x < y < 2x\}$ then $A$ is open connected, $(0,0) \in \operatorname{Cl} A$ but $f(x,y) > 1/5$ for each $(x,y) \in A$, i.e., $f$ is not $B$-quasicontinuous. □

**THEOREM 8.** *If $X = \mathbb{R}^2$ then the following proper inclusions hold*

$$\mathcal{C} \subset \mathrm{DMP} \subset \mathcal{B} \subset \mathcal{S} = \mathcal{T}\,.$$

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