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CONVERGENCE OF SEQUENCES OF FUNCTIONS HAVING PROPERTY $\mathcal{M}$

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ABSTRACT. A function $f:\mathbb{R}\to\mathbb{R}$ has the property $\mathcal{M}$ if the restricted function $f/D_{\mathrm{ap}}(f)$ is continuous ($D_{\mathrm{ap}}(f)$ denotes the set of all approximate discontinuity points of f). In this article I investigate the uniform, pointwise and transfinite limits of sequences of functions with the property $\mathcal{M}$.

Let $\mathbb{R}$ be the set of all reals. Denote by μ the Lebesgue measure in $\mathbb{R}$ and by μ_e the outer Lebesgue measure in $\mathbb{R}$. For a set $A\subset\mathbb{R}$ and a point x we define the *upper* (*lower*) *outer density* $D_u(A,x)$ ($D_l(A,x)$) of the set A at the point x ([1, 6]) as

$$\limsup_{h\to 0+}\frac{\mu_e(A\cap[x-h,x+h])}{2h}$$

$$\left(\liminf_{h\to 0+}\frac{\mu_e(A\cap[x-h,x+h])}{2h},\ \text{respectively}\right).$$

A point x is said an *outer density point* (a *density point*) of a set A if $D_l(A,x)=1$ (if there is a Lebesgue measurable set $B\subset A$ such that $D_l(B,x)=1$).

The family T_d of all sets A for which the implication

$$x\in A \implies x \text{ is a density point of } A$$

holds, is a topology called the *density topology* ([1, 6]). The sets $A\in T_d$ are measurable ([1]).

Let T_e be the Euclidean topology in $\mathbb{R}$. A continuity of functions $f:\mathbb{R}\to\mathbb{R}$ treating as applications from $(\mathbb{R},T_d)$ to $(\mathbb{R},T_e)$ is said *approximate continuity* ([1]).

For a function $f:\mathbb{R}\to\mathbb{R}$ let $C(f)$ denote the set of all continuity points of f, let $A(f)$ denote the set of all approximate continuity points of f, and let $D(f)=\mathbb{R}\setminus C(f)$ and $D_{\mathrm{ap}}(f)=\mathbb{R}\setminus A(f)$.

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In [2] R. Pawlak introduced and investigated the following property of functions:

> A function $f\colon \mathbb{R} \to \mathbb{R}$ has the property $\mathcal{B}_1^{**}$ if the restricted function $f/D(f)$ is continuous.

In this paper I investigate a similar property $\mathcal{M}$ defined as follows:

> A function $f\colon \mathbb{R} \to \mathbb{R}$ has the property $\mathcal{M}$ ($f \in \mathcal{M}$) if the restricted function $f/D_{\mathrm{ap}}(f)$ is continuous.

Remark 1. If $f \in \mathcal{M}$ then f is measurable (in the sense of Lebesgue).

P r o o f. Denote by $\mathbb{Z}$ the set of all integers and by $\mathrm{int}_d(A)$ the density interior of A, i.e, the union of all subsets of A which belong to T_d. For each positive integer n and each integer $k \in \mathbb{Z}$ let

$$I_{k,n} = \left(\frac{k-1}{2^n}, \frac{k+1}{2^n}\right).$$

Since

$$A(f) = \bigcap_{n\geq 1} \bigcup_{k\in\mathbb{Z}} \mathrm{int}_d\big(f^{-1}(I_{k,n})\big)\,,$$

the set $A(f)$ is measurable. For each $a \in \mathbb{R}$ the set $\{x \in A(f)\colon f(x) < a\} \in T_d$, so it is measurable. Consequently, the restricted function $f/A(f)$ is measurable. The set $D_{\mathrm{ap}}(f) = \mathbb{R} \setminus A(f)$ is also measurable and the restricted function $f/D_{\mathrm{ap}}(f)$ is continuous, so it is measurable. Thus f is measurable and the proof is completed. $\square$

However, there are functions $f \in \mathcal{M}$ which do not have the property of Baire. For example, if $A \subset \mathbb{R}$ is a residual G_δ-set of measure zero then there is a decomposition of the set A in disjoint subsets $B, C \subset A$ without the Baire property. The function

$$f(x) = 1 \text{ on } B \quad \text{and} \quad f(x) = 0 \text{ otherwise on } \mathbb{R}$$

belongs to $\mathcal{M}$ but it does not have the Baire property.

For each measurable function $f\colon \mathbb{R} \to \mathbb{R}$ there is a function $g\colon \mathbb{R} \to \mathbb{R}$ of Baire class 2 such that the set $\{x\colon f(x) \neq g(x)\}$ is of measure zero. In next example we show that there are functions $f \in \mathcal{M}$ such that for each function $h\colon \mathbb{R} \to \mathbb{R}$ of Baire class 1 the set $\{x\colon f(x) \neq h(x)\}$ is of positive measure.

Example 1. Let (I_n) be an enumeration of all open intervals with rational endpoints and let (A_n) be a sequence of pairwise disjoint nowhere dense perfect sets of positive measure such that

$$A_{2n-1} \cup A_{2n} \subset I_n \qquad \text{for } n = 1, 2, \dots$$

Put

$$f(x) = \begin{cases} 1 & \text{for } x \in \operatorname{int}_d(A_{2n-1}),\ n \geq 1, \\ -1 & \text{for } x \in \operatorname{int}_d(A_{2n}),\ n \geq 1, \\ 0 & \text{otherwise on } \mathbb{R}. \end{cases}$$

Then

$$A(f) = \bigcup_{n \geq 1} \left(\operatorname{int}_d(A_{2n-1}) \cup \operatorname{int}_d(A_{2n})\right) \cup \operatorname{int}_d\left(f^{-1}(0)\right)$$

and

$$f(x) = 0 \qquad \text{for } x \in D_{\mathrm{ap}}(f),$$

so $f \in \mathcal{M}$.

However, for each index n and for each set A of measure zero we have

$$\operatorname{int}_d(A_n) \setminus A \neq \emptyset, \qquad \text{so} \qquad f^{-1}(1) \cap I_n \neq \emptyset \neq f^{-1}(-1) \cap I_n.$$

Consequently, for each function $g\colon \mathbb{R} \to \mathbb{R}$ with $f/(\mathbb{R} \setminus A) = g/(\mathbb{R} \setminus A)$ we have $C(g) = \emptyset$ and a such g is not of the first Baire class.

In [3] H. Pawlak shows that the uniform limit of functions belonging to $\mathcal{B}_1^{**}$ need not be in this class.

In next example we show that there are uniformly convergent sequences of functions from $\mathcal{M}$ whose limits do not have the property $\mathcal{M}$.

Example 2. Enumerate all rationals in a sequence (w_n) such that $w_n \neq w_m$ for $n \neq m$ and put

$$f(x) = \begin{cases} \frac{1}{n} & \text{for } x = w_n,\ n \geq 1, \\ 0 & \text{otherwise on } \mathbb{R}. \end{cases}$$

Then $A(f) = \mathbb{R} \setminus \mathbb{Q}$, where $\mathbb{Q}$ denotes the set of all rationals, and $D_{\mathrm{ap}}(f) = \mathbb{Q}$. Since the restricted function $f/\mathbb{Q}$ is discontinuous at each point $x \in \mathbb{Q}$, the function f does not have the property $\mathcal{M}$.

However, f is the uniform limit of the sequence of functions

$$f_n(x) = \begin{cases} \frac{1}{k} & \text{for } x = w_k,\ k \leq n, \\ 0 & \text{otherwise on } \mathbb{R}, \end{cases}$$

which evidently belong to $\mathcal{M}$.

Theorem 1. *If a function $f\colon \mathbb{R} \to \mathbb{R}$ is the limit of a uniformly convergent sequence of functions $f_n \in \mathcal{M}$ then the set $D_{\mathrm{ap}}(f)$ is the union of sets A_n such that $A_n \subset A_{n+1}$ and the restricted functions f/A_n are continuous for $n \geq 1$.*

Proof. Fix $x \in \mathbb{R}$ and observe that if there is a subsequence (n_k) with $x \in A(f_{n_k})$ then from the uniform convergence of (f_n) it follows that $x \in A(f)$.

So if $x \in D_{\mathrm{ap}}(f)$ then there is an index $n(x)$ such that $x \in D_{\mathrm{ap}}(f_n)$ for $n \geq n(x)$. For $n \geq 1$ let

$$B_n = \bigcap_{k \geq n} D_{\mathrm{ap}}(f_k).$$

Then

$$B_n \subset B_{n+1} \quad \text{for } n \geq 1, \qquad \text{and} \qquad D_{\mathrm{ap}}(f) \subset \bigcup_{n \geq 1} B_n .$$

The restricted functions f_n/B_k, $n \geq k$ and $k \geq 1$, are continuous and the sequence (f_n) uniformly converges to f, so the restricted functions f/B_k are continuous for $k \geq 1$. Now for $n \geq 1$ we put $A_n = B_n \cap D_{\mathrm{ap}}(f)$ and observe that the sequence (A_n) satisfies all requirements. $\square$

Theorem 2. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function such that $\mu(D_{\mathrm{ap}}(f)) = 0$. Suppose that there is a sequence (A_n) such that*

(1) *$A_n \subset A_{n+1}$ for $n \geq 1$ and $D_{\mathrm{ap}}(f) = \bigcup_{n \geq 1} A_n$;*

(2) *the restricted functions f/A_n are continuous for $n \geq 1$;*

(3) *at each point $x \in \mathbb{R} \setminus A_1$ there is the finite approximative limit $f_{\mathrm{ap}}(x)$ of f at x;*

(4) *if $a_n = \sup\{|f(x) - f_{\mathrm{ap}}(x)| : x \in D_{\mathrm{ap}}(f) \setminus A_n\}$ for $n \geq 1$ then $\lim_{n\to\infty} a_n = 0$.*

Then there is a sequence of functions $f_n \in \mathcal{M}$, $n \geq 1$, which uniformly converges to f.

P r o o f. If there is an index k such that $D_{\mathrm{ap}}(f) = A_k$ then $f \in \mathcal{M}$ and we can put $f_n = f$ for $n \geq 1$. So we suppose that $A_{n+1} \setminus A_n \neq \emptyset$ for infinitely many indices n and for $n \geq 1$ we put

$$f_n(x) = \begin{cases} f(x) & \text{for } x \in A(f), \\ f(x) & \text{for } x \in A_n, \\ f'_{\mathrm{ap}}(x) & \text{for } x \in D_{\mathrm{ap}}(f) \setminus A_n . \end{cases}$$

Then for $n \geq 1$ we obtain

$$A(f_n) \supset A(f) \cup \big(D_{\mathrm{ap}}(f) \setminus A_n\big)$$

and the restricted function $f_n/D_{\mathrm{ap}}(f_n) = f/D_{\mathrm{ap}}(f_n)$ is continuous. So $f_n \in \mathcal{M}$ for $n \geq 1$. Since $|f_n - f| \leq a_n$, by (4) the sequence (f_n) uniformly converges to f. $\square$

THEOREM 3. *Suppose that a sequence of functions $f_n \in \mathcal{M}$ pointwise converges to a function $f\colon \mathbb{R} \to \mathbb{R}$. Then there are sets $A, A_1, A_2, \dots$ such that:*

(1) $A \supset \bigcap_{n\geq 1} A(f_n)$ *and for each point $x \in A$ for infinitely many indices $n_i(x)$ we have $x \in A\big(f_{n_i(x)}\big)$;*

(2) $A_n \subset A_{n+1} \cap D_{\mathrm{ap}}(f_n)$ *for $n \geq 1$ and*
$$\mathbb{R} \setminus A = \bigcup_{n\geq 1} A_n\,;$$

(3) *the restricted functions f/A_n are the limits of sequences of continuous functions f_k/A_n, where $k \geq n$.*

P r o o f. Since $f_n \in \mathcal{M}$, the restricted functions $f_n/D_{\mathrm{ap}}(f_n)$, $n \geq 1$, are continuous. For $n \geq 1$ let
$$A_n = \bigcap_{k\geq n} D_{\mathrm{ap}}(f_k)\,.$$
Then conditions (1), (2) and (3) are evidently satisfied and the proof is completed. □

Observe that the pointwise limits of sequences of functions from $\mathcal{M}$ are measurable. So they are almost everywhere equal to some functions of Baire class 2.

For two functions $f, g\colon \mathbb{R} \to \mathbb{R}$ denote by $(f \neq g)$ the set $\big\{x \in \mathbb{R}\colon f(x) \neq g(x)\big\}$.

THEOREM 4. *If for a function $f\colon \mathbb{R} \to \mathbb{R}$ there is a function $g\colon \mathbb{R} \to \mathbb{R}$ of Baire class 2 such that the restricted function $f/(f \neq g)$ is the limit of a sequence of continuous functions $g_n\colon (f \neq g) \to \mathbb{R}$ then there are functions $f_n \in \mathcal{M}$ with $f = \lim_{n\to\infty} f_n$.*

P r o o f. Since g is of Baire class 2, by P r e i s s' theorem from [4] there is a sequence of approximately continuous functions $h_n\colon \mathbb{R} \to \mathbb{R}$ such that $f = \lim_{n\to\infty} h_n$. For $n \geq 1$ let
$$f_n(x) = \begin{cases} g_n(x) & \text{for } x \in (f \neq g)\,,\\ h_n(x) & \text{otherwise on } \mathbb{R}\,. \end{cases}$$
Then $f_n \in \mathcal{M}$ for $n \geq 1$ and $f = \lim_{n\to\infty} f_n$.

Now let $\omega_e 1$ denote the first uncountable ordinal number, let $f_\alpha\colon \mathbb{R} \to \mathbb{R}$, where $\alpha < \omega_e 1$, be a transfinite sequence of functions. We will say that a transfinite sequence $(f_\alpha)_{\alpha<\omega_e 1}$ converges to a function $f\colon \mathbb{R} \to \mathbb{R}$ ($\lim_{\alpha<\omega_e 1} f_\alpha = f$) if for each point $x \in \mathbb{R}$ there is a countable ordinal $\alpha(x)$ such that for each countable ordinal $\alpha > \alpha(x)$ the equality $f_\alpha(x) = f(x)$ is true ([5]). □

THEOREM 5. *Assume that the continuum hypothesis HC holds. For each function $f\colon \mathbb{R} \to \mathbb{R}$ there is a transfinite sequence of functions $f_\alpha\colon \mathbb{R} \to \mathbb{R}$ having the property $\mathcal{M}'$ such that $\lim_{\alpha < \omega_e 1} f_\alpha = f$.*

P r o o f. By HC there is a transfinite sequence $(a_\alpha)_{\alpha<\omega_e 1}$ of all reals such that $a_\alpha \neq a_\beta$ for $\alpha \neq \beta$. So, for $\alpha < \omega_e 1$, the sets

$$A_\alpha = \{a_\beta \colon \beta \le \alpha\}$$

are countable. Fix $\alpha < \omega_e 1$ and for all points $x \in A_\alpha$ find the set $B(x) \in T_d$ containing x and such that

$$B(x) \cap B(y) = \emptyset \qquad \text{for } x \neq y\,,\ x, y \in A_\alpha\,.$$

For $\alpha < \omega_e 1$ let

$$f_\alpha(t) = \begin{cases} f(x) & \text{for } t \in B(x),\ x \in A_\alpha\,, \\ 0 & \text{otherwise on } \mathbb{R}\,. \end{cases}$$

Since $f_\alpha / D_{\mathrm{ap}}(f_\alpha) = 0$, all $f_\alpha \in \mathcal{M}$. For $\alpha < \omega_e 1$ and $x \in A_\alpha$ the equalities $f_\alpha(x) = f(x)$ are true, so

$$\lim_{\alpha<\omega_e 1} f_\alpha = f\,.$$

□

REFERENCES

[1] BRUCKNER, A., M.: *Differentiation of Real Functions*, Lectures Notes in Math., Vol. 659, Springer-Verlag, Berlin, 1978.

[2] PAWLAK, R. J.: *On some class of functions intermediate between the family of continuous functions and the class $\mathcal{B}_1^*$*, Tatra Mt. Math. Publ. **19** (2000), 135–149.

[3] PAWLAK, H.: *On the set $\mathcal{B}_1^{**}$ in the space $\mathcal{B}_1$ of functions*, Tatra Mt. Math. Publ. **19** (2000), 263–271.

[4] PREISS, D.: *Limits of approximately continuous functions*, Czechoslovak Math. J. **96** (1971), 371–372.

[5] SIERPIŃSKI, W.: *Sur les suites transfinies convergentes de fonctions de Baire*, Fund. Math. **1** (1920), 132–141.

[6] TALL, F. D.: *The density topology*, Pacific J. Math. **62** (1976), 275–284.

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