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ON THE LOGARITHMIC DERIVATIVE OF THE EULER PRODUCT

FILIP SAIDAK

ABSTRACT. The purpose of this paper is to prove a couple of new lemmas concerning functions related to the Euler product

$$\prod_p \left(1 - \frac{1}{p^s}\right)^{-1} := \zeta(s), \quad (\Re(s) > 1).$$

Some of the results deal with extremal bounds for derivatives of the Riemann zeta function $\zeta(s)$, and have a potential to have applications in number theory and cryptography, especially when combined with other analytic and algebraic results.

1. The Euler product

In his *Variae observationes* [8] of 1737, for $s \in \mathbb{R}$, L. Euler defined

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}, \tag{1}$$

where the 'Euler product' (1) converges for $s > 1$. Taking logarithms of both sides of (1) Euler was able to show that:

$$\log \zeta(s) = \sum_p \log\left(1 - \frac{1}{p^s}\right)^{-1} = \sum_p \frac{1}{p^s} + \sum_{p,\, a \geq 2} \frac{1}{a p^{a s}} = \sum_p \frac{1}{p^s} + O(1),$$

which, from divergence of the harmonic series, implies $\sum_p 1/p \to \infty$.

In 1859 Riemann [14] considered $\zeta(s)$, defined in (1), as a function of a complex variable $s = \sigma + it$, and showed that for $\Re(s) > 1$ the Euler product

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(1) converges and defines $\zeta(s)$ as a holomorphic function of s. Evidently, since $n^{-s} = s\int_n^\infty t^{-s-1}dt$, a simple integration gives us

$$\zeta(s) = \sum_{n=1}^{\infty}\left(s\int_n^\infty \frac{dt}{t^{s+1}}\right) = \frac{1}{s-1} + 1 - s\int_1^\infty \frac{x-[x]}{x^{s+1}}\,dx\,, \tag{2}$$

where $[x]$ denotes the integer part of x. For $\sigma > 0$, (2) now clearly defines $\zeta(s)$ as a meromorphic function with a single pole of first order at $s = 1$ (with residue 1). Repeating this process, the analytic continuation of $\zeta(s)$ to the left half-plane is easily obtained.

Preceding Riemann's work there were the ground-breaking researches of P. G. L. Dirichlet [4] and P. L. Chebyshev [3]. The latter defined

$$\vartheta(x) := \sum_{p\le x}\log p \quad \text{and} \quad \psi(x) := \sum_{p^a\le x}\log p\,,$$

and was able to prove that $Ax < \psi(x) < Bx$, for two relatively tight constants $A < 1 < B$. An alternative definition of $\psi(x)$ is given via the von Mangoldt's function: $\psi(x) = \sum_{n\le x}\Lambda(n)$. In this notation, for $\Re(s) > 1$, by a logarithmic differentiation of (1) we have:

$$\begin{aligned}\frac{\zeta'(s)}{\zeta(s)} &= \left[\sum_p \log\left(1-\frac{1}{p^s}\right)\right]' = \sum_p \frac{\log p}{p^s}\left(1-\frac{1}{p^s}\right)\\ &= \int_1^\infty x^{-s}d\psi(x) = s\int_1^\infty \frac{\psi(x)}{x^{s+1}}\,dx = \sum_{n=2}^\infty \frac{\Lambda(n)}{n^s}\,,\end{aligned}$$

making clear the relevance[1] of a study of derivatives of $\zeta(s)$. In fact, it is through this form that we will have something to say about Euler products in general.

2. The functional equation

In his seminal paper [14] Riemann also proved a famous conjecture of Euler [9] from 1748, namely that $\zeta(s)$ satisfies the fundamental functional equation:

$$\zeta(s) = \left[2^s\pi^{s-1}\Gamma(1-s)\sin\frac{\pi s}{2}\right]\zeta(1-s)\,, \tag{3}$$

[1]**Note**: this shows that $\psi(x)$ is the sum of the first $[x]$ coefficients of the Dirichlet series of $-\zeta'(s)/\zeta(s)$.

where

$$\Gamma(s) := \int_0^\infty e^{-t} t^{s-1}\, dt\,.$$

This enables one to extend $\zeta(s)$, with the help of an analogue of (2), to the entire complex plane. All the zeros of $\zeta(s)$ introduced by the gamma factor in (3) (i.e., $s = -2, -4, \cdots$), are called *trivial*, the rest are known as the *non-trivial* zeros of $\zeta(s)$. The importance of (3) lies in the fact that it introduces the notion of symmetry into the picture.

From Hadamard's theory of entire functions we know that $\zeta(s)$ has infinitely many non-trivial zeros in the *critical strip* $0 \le \Re(s) \le 1$, and moreover, as von Mangoldt proved (using the argument principle):

$$\begin{aligned} N(T) &:= \#\{\rho\colon \zeta(\rho) = 0\,,\ 0 \le \Re(\rho) \le 1\,,\ 0 \le \Im(\rho) \le T\} \\ &= \frac{T}{2\pi}\log\left(\frac{T}{2\pi}\right) - \frac{T}{2\pi} + O(\log T)\,,\quad \text{as } T \to \infty\,. \end{aligned}$$

The celebrated Riemann Hypothesis, the most important unsolved problem of mathematics, can also be found in Riemann's [14]. It states that all non-trivial zeros ρ of $\zeta(s)$ satisfy $\Re(\rho) = 1/2$. In terms of the modulus of $\zeta(s)$, this is equivalent to: For $-1/2 \le \Delta \le 1/2$,

$$\left|\zeta\left(\frac{1}{2} + \Delta + it\right)\right| = 0 \quad \Longrightarrow \quad \Delta = 0\,. \tag{RH}$$

In our recent paper [15] we used the functional equation (3) in order to give a proof of the following theorem concerning the modulus of $\zeta(s)$ in the critical strip: For $0 \le \Delta \le \frac{1}{2}$, and all $t \ge 2\pi + 1$, we have

$$\left|\zeta\left(\frac{1}{2} - \Delta + it\right)\right| \ge \left|\zeta\left(\frac{1}{2} + \Delta + it\right)\right|. \tag{4}$$

Note. If, for $0 < \Delta \le \frac{1}{2}$, we could improve (4) to obtain

$$\left|\zeta\left(\frac{1}{2} - \Delta + it\right)\right| > \left|\zeta\left(\frac{1}{2} + \Delta + it\right)\right|, \tag{4'}$$

then the Riemann Hypothesis would follow. This is because a consequence of the functional equation (3) is that the non-trivial zeros either occur on $\sigma = \frac{1}{2}$, or in pairs $s = \frac{1}{2} \pm \Delta + it$, with $0 < \Delta \le \frac{1}{2}$. Unfortunately, as we noted in [15], the use of the functional equation as a main tool was the primary reason why it seemed impossible, without further information about $\zeta(s)$, to break that 'equal sign' barrier.

3. Main theorem

The situation is similar for the derivatives of $\zeta(s)$, but with an important difference, readily seen from the following theorem:

Theorem 3.1. *If, in the range $0 \le \Delta \le \frac{1}{2}$ and $t \ge 2\pi + 1$, one has*

$$\left|\zeta'\left(\frac{1}{2} - \Delta + it\right)\right| \ge \left|\zeta'\left(\frac{1}{2} + \Delta + it\right)\right|, \tag{5}$$

then the Riemann Hypothesis (RH) follows. The RH is also implied by:

$$\left|\frac{\partial}{\partial \Delta}\left|\zeta\left(\frac{1}{2} - \Delta + it\right)\right|\right| \ge \left|\frac{\partial}{\partial \Delta}\left|\zeta\left(\frac{1}{2} + \Delta + it\right)\right|\right|, \tag{5'}$$

as long as it holds true for $0 \le \Delta \le \frac{1}{2}$ and all $t \ge 2\pi + 1$.

Note. Unlike in the original case of $|\zeta(s)|$, for the moduli of the derivatives of $\zeta(s)$ we do not need strict inequalities in (5) and (5′) in order to deduce the RH. This apparent detail could prove to be of some importance, especially since (4) was unconditional.

In what follows we discuss several statements concerning $\zeta'(s)$ that imply the RH, all of them are related to (4) and (5). In addition to the standard use of (3), we obtain interesting information supplied by properties of the Euler product.

It is believed that the location of the non-trivial zeros of many functions much more general than $\zeta(s)$ is analogous, as long as these functions have both—Euler products and functional equations (in the form of generalizations of (1) and (3), respectively). Therefore, if one wishes to prove the Riemann Hypothesis, it seems natural to expect that one needs to make use of information concerning both of these fundamental objects.

4. Lemmas

Our aim is to prove the following result:

Lemma 4.1. *If for some $\varepsilon > 0$ we have*

$$\mathfrak{R}(\sigma) := \Re\left(\frac{\zeta'(\sigma + it)}{\zeta(\sigma + it)}\right) > -(1 - \varepsilon)\log\frac{t}{2\pi} + O\left(\frac{1}{t}\right), \tag{6}$$

for $t > 4\pi$ and $1/2 \le \sigma \le 1$, then the Riemann Hypothesis follows.

Note. We are unable to prove the inequality in Lemma 4.1 over the given[2] range, however, it can be shown that (6) is true at the end points of $[1/2, 1]$. In fact, very precise estimates can be proved:

Lemma 4.2. *For all $t > 4\pi$, we have*

$$\mathfrak{R}(1) := \Re\left(\frac{\zeta'(1+it)}{\zeta(1+it)}\right) > -\frac{1}{2}\log\frac{t}{2\pi} + O\left(\frac{1}{t}\right). \tag{7}$$

and, in the same range, if $\zeta(1/2+it) \neq 0$, then

$$\mathfrak{R}\left(\frac{1}{2}\right) := \Re\left(\frac{\zeta'(\frac{1}{2}+it)}{\zeta(\frac{1}{2}+it)}\right) > -\frac{1}{2}\log\frac{t}{2\pi} + O\left(\frac{1}{t}\right). \tag{7'}$$

P r o o f. Combining the above expression for $-\zeta'(s)/\zeta(s)$ with a logarithmically differentiated Hadamard's product formula (see [10]), v o n M a n g o l d t was able to obtain his well-known estimate:

$$\frac{\zeta'(s)}{\zeta(s)} = -\frac{1}{s-1} + \sum_{\rho}\frac{1}{s-\rho} - \frac{1}{2}\frac{\Gamma'(s/2+1)}{\Gamma(s/2+1)} + \frac{1}{2}\log\pi, \tag{8}$$

with ρ denoting the non-trivial zeros of $\zeta(s)$. Taking real parts of both sides, this immediately gives us:

$$\Re\left(\frac{\zeta'(s)}{\zeta(s)}\right) = \Re\left(\sum_{\rho}\frac{1}{s-\rho}\right) - \frac{1}{2}\log\frac{t}{2} + \frac{1}{2}\log\pi + O\left(\frac{1}{t}\right).$$

For $t > 4\pi$, (7) follows from the prime number theorem, i.e., the fact that for all non-trivial zeros ρ of $\zeta(s)$ we have $0 < \Re(\rho) < 1$ (see [10], [22]), since this implies that the sum on the right hand side of the last equation must be positive, its every term being greater than zero. On the other hand, from Corollary 1 of S p i r a [17], p. 327, we have

$$\Re\left(\frac{\zeta'(1/2+it)}{\zeta(1/2+it)}\right) > -\frac{1}{2}\log\frac{t}{2\pi} + \frac{1}{53t^2},$$

and the inequality $(7')$ immediately follows. □

Corollary 4.3. *For any fixed $t > 2\pi+1$, let $D(t)$ be the number of solutions of $\frac{\partial}{\partial\sigma}\zeta(\sigma+it) = 0$ in the range $0 \leq \sigma \leq 1/2$. Then $D(t)$ is always even as long as $\zeta(1/2+it) \neq 0$.*

Now we can prove Lemma 4.1 with the help of two technical lemmas and a third result that is its equivalent (and is of independent interest), linking Lemma 4.1 to Theorem 3.1 from the previous section:

[2] or in any other fixed interval, for that matter, although the argument in Lemma 4.2 generalizes to prove the bound at infinitely many points inside.

Lemma 4.4. (Zvengrowski, [25].) *For all $s = \sigma + it$ we have*

$$\Re\left(\cot\left(\frac{\pi s}{2}\right)\right) = O(e^{-t}) \,. \tag{9}$$

Proof. From elementary trigonometry $\sinh(x) = \frac{1}{2}(e^x - e^{-x})$ and $\cosh(x) = \frac{1}{2}(e^x + e^{-x})$, while $\cosh(x) = \sqrt{1 + \sinh^2(x)}$. Therefore

$$\begin{aligned}
\cot\left(\frac{\pi s}{2}\right) &= \frac{\cos(\frac{\pi\sigma}{2})\cosh(\frac{\pi t}{2}) - i\sin(\frac{\pi\sigma}{2})\sinh(\frac{\pi t}{2})}{\sin(\frac{\pi\sigma}{2})\cosh(\frac{\pi t}{2}) + i\cos(\frac{\pi\sigma}{2})\sinh(\frac{\pi t}{2})} \\
&= -\, i\, \frac{\cos(\frac{\pi\sigma}{2})\cosh(\frac{\pi t}{2}) - i\sin(\frac{\pi\sigma}{2})\sinh(\frac{\pi t}{2})}{\cos(\frac{\pi\sigma}{2})\sinh(\frac{\pi t}{2}) - i\sin(\frac{\pi\sigma}{2})\cosh(\frac{\pi t}{2})} \\
&= -\, i\, \frac{\cos(\frac{\pi\sigma}{2})\left[\sinh(\frac{\pi t}{2}) + O(1)\right] - i\sin(\frac{\pi\sigma}{2})\left[\cosh(\frac{\pi t}{2}) + O(1)\right]}{\cos(\frac{\pi\sigma}{2})\sinh(\frac{\pi t}{2}) - i\sin(\frac{\pi\sigma}{2})\cosh(\frac{\pi t}{2})} \\
&= -\, i\left(1 + \frac{O(1)}{\cos(\frac{\pi\sigma}{2})\sinh(\frac{\pi t}{2}) - i\sin(\frac{\pi\sigma}{2})\cosh(\frac{\pi t}{2})}\right),
\end{aligned}$$

and since $\sinh(x) \sim \cosh(x) \sim e^x$, the bound (9) easily follows. □

Lemma 4.5. *For $s = \frac{1}{2} + \Delta + it = re^{i\theta}$, where $0 \le \Delta \le \frac{1}{2}$ and $t \ge 4\pi$, we have*

$$F(s) := \Re\left(\frac{\Gamma'(s)}{\Gamma(s)}\right) - \Re\left(\frac{\pi}{2}\cot\left(\frac{\pi s}{2}\right)\right) - \log(2\pi) \ge 0\,. \tag{10}$$

Proof. Clearly, the previous Lemma 4.4 shows that the second term of $F(s)$ in (10) is negligible, while from Lemma 6 in [15] and remarks made in [15], on p. 113, we see that

$$\Re\left(\frac{\Gamma'(s)}{\Gamma(s)}\right) \ge \log|s| - \Re\left(\frac{1}{2s}\right) = \log|s| - \Re\left(\frac{\sigma - it}{2|s|^2}\right) > \log(2\pi)\,,$$

since $t \ge 4\pi$ and $|s|^2 = O(t^2)$. This proves the inequality (10). □

Finally, we prove:

Lemma 4.6. *For all $t \ge 4\pi$ and $0 \le \Delta \le 1/2$, if $\zeta(1/2 \pm \Delta + it) \ne 0$, then we have*

$$\frac{\frac{\partial}{\partial\Delta}\left|\zeta\left(\frac{1}{2} - \Delta + it\right)\right|}{\left|\zeta\left(\frac{1}{2} - \Delta + it\right)\right|} = \frac{\frac{\partial}{\partial\Delta}\left|\zeta\left(\frac{1}{2} + \Delta + it\right)\right|}{\left|\zeta\left(\frac{1}{2} + \Delta + it\right)\right|} + \log\frac{t}{2\pi} + O\left(\frac{1}{t}\right), \tag{11}$$

and, in particular,

$$\frac{\frac{\partial}{\partial\Delta}\left|\zeta\left(\frac{1}{2}-\Delta+it\right)\right|}{\left|\zeta\left(\frac{1}{2}-\Delta+it\right)\right|} > \frac{\frac{\partial}{\partial\Delta}\left|\zeta\left(\frac{1}{2}+\Delta+it\right)\right|}{\left|\zeta\left(\frac{1}{2}+\Delta+it\right)\right|}. \tag{11'}$$

P r o o f. First, let us recall that, in general, for a complex valued function $w(\Delta)$, everywhere except at Δ with $w(\Delta)=0$, one has

$$\Re\left(\frac{w'(\Delta)}{w(\Delta)}\right) = \frac{1}{|w(\Delta)|}\frac{\partial}{\partial\Delta}\left|w(\Delta)\right|. \tag{12}$$

On the other hand, for $\zeta(s)$, the functional equation (3) states

$$\zeta(s) = 2^s\pi^{s-1}\zeta(1-s)\Gamma(1-s)\sin\left(\frac{\pi s}{2}\right)$$
$$\Downarrow$$
$$\log\zeta(s) = s\log 2 + (s-1)\log\pi + \log\zeta(1-s) + \log\Gamma(1-s) + \log\sin\left(\frac{\pi s}{2}\right)$$
$$\Downarrow$$
$$\frac{\zeta'(s)}{\zeta(s)} = \log(2\pi) - \frac{\zeta'(1-s)}{\zeta(1-s)} - \frac{\Gamma'(1-s)}{\Gamma(1-s)} + \frac{\pi}{2}\cot\left(\frac{\pi s}{2}\right),$$

by taking logarithms and differentiating, respectively. Note that for all $s\in\mathbb{C}$, we have $\Gamma(\overline{s}) = \overline{\Gamma(s)}$ and $\zeta(\overline{s}) = \overline{\zeta(s)}$, and if we let $s = \frac{1}{2}-\Delta+it$, then $1-s = \overline{\frac{1}{2}+\Delta+it}$. Therefore, we can write

$$\frac{\zeta'\left(\frac{1}{2}-\Delta+it\right)}{\zeta\left(\frac{1}{2}-\Delta+it\right)} = \log(2\pi) - \frac{\overline{\zeta'\left(\frac{1}{2}+\Delta+it\right)}}{\overline{\zeta\left(\frac{1}{2}+\Delta+it\right)}} - \frac{\overline{\Gamma'\left(\frac{1}{2}+\Delta+it\right)}}{\overline{\Gamma\left(\frac{1}{2}+\Delta+it\right)}} + \frac{\pi}{2}\cot\left(\frac{\pi s}{2}\right). \tag{13}$$

Now, in the view of (12), one has

$$\Re\left(\frac{\zeta'\left(\frac{1}{2}-\Delta+it\right)}{\zeta\left(\frac{1}{2}-\Delta+it\right)}\right) = -\frac{\frac{\partial}{\partial\Delta}\left|\zeta\left(\frac{1}{2}-\Delta+it\right)\right|}{\left|\zeta\left(\frac{1}{2}-\Delta+it\right)\right|}$$

and

$$\Re\left(\frac{\overline{\zeta'\left(\frac{1}{2}+\Delta+it\right)}}{\overline{\zeta\left(\frac{1}{2}+\Delta+it\right)}}\right) = \frac{\frac{\partial}{\partial\Delta}\left|\zeta\left(\frac{1}{2}+\Delta+it\right)\right|}{\left|\zeta\left(\frac{1}{2}+\Delta+it\right)\right|},$$

provided $\left|\zeta(1/2\pm\Delta+it)\right| \neq 0$. Hence, comparing the real parts of the quantities in (13) we immediately obtain

$$\frac{\frac{\partial}{\partial\Delta}\left|\zeta\left(\frac{1}{2}-\Delta+it\right)\right|}{\left|\zeta\left(\frac{1}{2}-\Delta+it\right)\right|} = F\left(\tfrac{1}{2}+\Delta+it\right) + \frac{\frac{\partial}{\partial\Delta}\left|\zeta\left(\frac{1}{2}+\Delta+it\right)\right|}{\left|\zeta\left(\frac{1}{2}+\Delta+it\right)\right|}, \tag{14}$$

which, by Lemma 4.5, obviously proves both (11) and (11′). $\square$

It is clear that Lemma 4.1 can now be obtained upon rewriting (11)—the main estimate of Lemma 4.6—via (12). As far as Theorem 3.1 is concerned, its main result follows from the inequality $(11')$, when it is combined with our theorem (4).

5. Remarks

Remark 1. The above inequality $(11')$ is unconditional, and when it is added to (4) it falls short of the RH only in the case when both $\frac{\partial}{\partial\Delta}\left|\zeta(1/2-\Delta+it)\right| < 0$ and $\frac{\partial}{\partial\Delta}\left|\zeta(1/2+\Delta+it)\right| < 0$. Unfortunately, this is exactly the case where the real difficulty of the problem lies.

Remark 2. The most natural way of looking at the extreme behaviour of the growth of functions such as $\zeta'(s)/\zeta(s)$ is trough the theory of the maximal modulus principle. However, the best unconditional results in the field are at least one order weaker than what would be useful to us. The form in which (6) should be considered is

$$\left|\zeta'(\sigma+it)\right| > (1-\varepsilon)\left(\log\frac{t}{2\pi}\right)\left|\zeta(\sigma+it)\right|.$$

Remark 3. Various results concerning the connection between higher derivatives of $\zeta(s)$ and the Riemann Hypothesis can be obtained as well. The growth aspects are of a great interest here, too. Of course, $\zeta'(s), \zeta''(s), \cdots$ have many non-trivial zeros inside the critical strip (see [18], [20], [23] and [24]). However, it is possible to prove new results concerning their location. For example, we can show that the RH follows from the fact that the zeros of the second partial derivatives, at a given level, can never be located to the left of zeros of the first partial derivatives with respect to the real coordinates. Precise formulation and detailed explanation of similar results will be given elsewhere.

Remark 4. The method we have employed can be used to give new proofs of some standard results in the field, theorems of Speiser and Spira being two examples. Everything else discussed in this paper can also be extended to the theory of Dirichlet L-functions. Here the main difficulty in proving analogs of our lemmas is usually found in the technical nature of results arising from more complex Euler product and functional equation estimates.

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Centre for Information,
Security and Cryptography
Department of Mathematics and Statistics
University of Calgary, Alberta
CANADA
E-mail: cisac@math.ucalgary.ca