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REMARKS ON STATISTICAL MAPS AND FUZZY (OPERATIONAL) RANDOM VARIABLES

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ABSTRACT. We study statistical maps, fuzzy random variables, observables, and some other basic notions of the fuzzy (operational) probability theory in the framework of a categorical duality theory for generallized measurable spaces and generalized fields of probability events.

1. Introduction

Statistical maps and related notions (fuzzy or operational random variables, probability or Markov kernels, effect valued measures, statistical functions, observables, dual statistical maps) play a crucial role in a recently developed fuzzy or operational probability theory, see Gudder [17], Bugajski, Hellwig and Stulpe [1], Bugajski [2], [3], Foulis [7]. Even though the domain and the range of a statistical map are the sets of probability measures, each standard (classical Kolmogorov) random variable f can be considered as a special case of a statistical map. Indeed, f is a measurable map sending elementary events into real numbers $\mathbb{R}$ and, consequently, sending each probability measure p (via the composition of the preimage map $f^{\leftarrow}$ and p) into the *distribution* p_f of f. Hence f yields a map D_f sending the probability measures on the original field of events into the measures on the Borel subsets of R and D_f is "measure preserving" in the sense that

$$\big(D_f(p)\big)(B) = p_f(B) = p\big(f^{\leftarrow}(B)\big) = (p \circ f^{\leftarrow})(B) \qquad \text{(mp)}$$

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for each probability p and each Borel set $B \subseteq R$. In general, a statistical map sends an elementary event, i.e., a point-measure, into a nondegenerated probability on the range field of events and then condition (mp) has an integral form (see condition (BG) in Section 2) and instead of crisp sets we have to measure measurable fuzzy sets.

We claim that the fuzzy or operational probability theory becomes more transparent if we look at condition (mp) from a broader perspective. Namely, if some basic categorical tools are used and the one-to-one correspondence between statistical maps and observables is described within the framework of a categorical duality for generalized measurable spaces and generalized fields of events, developed in Frič [12], [15] and Papčo [24], [25]. In turn, this leads to a natural generalization. On the one hand, the duality requires only elements of the category theory (cf. Mac Lane [20]). On the other hand, besides a powerful machinery of general constructions and theorems, the categorical approach provides criteria to select objects, morphisms, invariants and similar natural notions whenever there are several otherwise equivalent options. For example, from the categorical viewpoint the notion of a statistical map is more natural than the (otherwise equivalent) notion of a Markov kernel. Indeed, the domain and the range of a statistical map are of the same nature: the set of all probability measures on the respective measurable spaces. Hence statistical maps are morphisms of a natural category the objects of which are structured sets of the probability measures. Contrarywise, the domain and the range of a Markov kernel are of a different nature: the domain is the product of sets Ω (the elementary events of one field of events) and $\mathcal{B}(\Xi)$ (the target field of events), whereas the range is the unit interval $[0,1]$. Hence Markov kernels do not compose and violate the fundamental property of morphisms (cf. p. 697 in Gudder [17]). Further, the σ-additivity of probability measures is equivalent to the sequential continuity with respect to the pointwise convergence of characteristic functions (cf. Novák [23], Frič [14]) and, since the pointwise convergence is the categorical convergence in the powers of unit intervals $[0,1]^T$, the sequential continuity of all kinds of morphisms within the fuzzy probability theory becomes a natural requirement. Finally, the measurability of maps can be reformulated in terms of a composition—this makes the duality between statistical maps and observables rather simple and natural and some technical proofs of deep theorems and their generalization become straightforward. In particular, the measurability in terms of a composition leads to the understanding of the role and nature of the sequential continuity of probability measures and observables.

Some notions related to effect algebras and D-posets will be recalled throughout the paper. Otherwise we generally follow Dvurečenskij and Pulmanová [6] or the papers cited in the introduction.

Let $(\Omega, \mathcal{B}(\Omega))$, $(\Xi, \mathcal{B}(\Xi))$ be classical measurable spaces and let f be a map of Ω into Ξ. Traditionally, f is said to be measurable (more precisely $(\mathcal{B}(\Xi), \mathcal{B}(\Omega))$-measurable) if for each $B \in \mathcal{B}(\Xi)$ the preimage $f^{\leftarrow}(B)$ belongs to $\mathcal{B}(\Omega)$. Observe that the measurability of f can be expressed in terms of characteristic functions and their composition with f. Indeed, f is measurable iff for each $B \in \mathcal{B}(\Xi)$ the composition $\chi_B \circ f = \chi_{f^{\leftarrow}(B)}$ is a characteristic function of some set in $\mathcal{B}(\Omega)$. This reformulation is used to define measurability for generalized measurable maps (see Frič [12], Papčo [24]).

2. Measurability

In Frič [12], having in mind applications to probability, we have proposed categorical definitions of a measurable space and a measurable map. The parameters are a basic complete concrete category S (e.g., the MV-algebras) and a distinguished object C (e.g., the closed unit interval $I = [0,1]$) serving as a cogenerator (describes the logic of the events in question). For such spaces we have constructed a duality in which sequential convergence plays an important role. On the one hand, the duality has some common features with those in Rodabaugh [30] and Diers [5]. On the other hand, it is exactly the sequential convergence which makes possible to capture some finer invariants of probability, e.g., σ-additivity of measures and s-perfectness (or sobriety) of measurable spaces. In Papčo [24], the construction has been applied to the category od D-posets with $[0,1]$ as the cogenerator. The resulting duality covers the duality between statistical maps and observables. In this section we prove some auxiliary properties needed in the duality to which the next section is devoted.

S. Bugajski considered statistical maps as formal representatives of fuzzy (operational) random variables. The main goal of this section is to show that statistical maps really generalize classical random variables as measure preserving measurable maps in the spirit of condition (mp). As a byproduct we get the implication that observables are sequentially continuous (with respect to the pointwise convergence).

In the literature different definitions of a statistical map can be found, e.g., see Bugajski [2], Gudder [17]. It follows from Corollary 5.3 in Gudder [17] that the two definitions are equivalent and we choose their join characterization given in Corollary 5.3 in Gudder [17] as a definition of a statistical map.

Let $(\Omega, \mathcal{B}(\Omega))$ be a classical measurable space. Throughout the paper we assume that all singletons are measurable, i.e., $\{\omega\} \in \mathcal{B}(\Omega)$ for all $\omega \in \Omega$, and $\mathcal{B}(\Omega)$ is reduced, i.e., if $\omega \neq \omega'$, then there exists $B \in \mathcal{B}(\Omega)$ such that $\omega \in B$, $\omega' \in \Omega \setminus B$.

Denote by $M_1^+(\Omega)$ the set of all σ-additive probability measures on $\mathcal{B}(\Omega)$. For each $\omega \in \Omega$, denote by δ_ω the Dirac measure (point-measure) concentrated at ω and denote by $\delta\Omega$ the set of all such measures. In what follows, for simplicity, we identify Ω and $\delta\Omega$ and, as well as, we identify a set $B \in \mathcal{B}(\Omega)$ and its characteristic function χ_B. Further, denote by $\mathcal{E}(\Omega)$ the set of all measurable functions of Ω into $I = [0,1]$. Then $\mathcal{E}(\Omega)$ carries the usual order, algebraic operations (defined pointwise), and the pointwise sequential convergence. In particular, $\mathcal{E}(\Omega)$ can be considered as an effect algebra or, equivalently, as a D-poset of fuzzy sets (cf. Foulis and Bennett [8], Kôpka and Chovanec [19], Dvurečenskij and Pulmannová [6]).

Definition 2.1. Let $(\Omega, \mathcal{B}(\Omega))$ and $(\Xi, \mathcal{B}(\Xi))$ be measurable spaces. Let A be a map of $M_1^+(\Omega)$ into $M_1^+(\Xi)$ such that, for each $B \in \mathcal{B}(\Xi)$, the assignment $\omega \mapsto \big(A(\delta_\omega)\big)(B)$ yields a measurable function of Ω into $[0,1]$ and

$$\big(A(\mu)\big)(B) = \int \big(A(\delta_\omega)\big)(B)\, d\mu \tag{BG}$$

for all $\mu \in M_1^+(\Omega)$ and all $B \in \mathcal{B}(\Xi)$. Then A is said to be a *statistical map in the sense of Bugajski and Gudder* or, simply, a *statistical map.*

Definition 2.2. Let $\mathcal{B}(\Omega)$ be a σ-field of subsets of Ω. For each $v \in \mathcal{E}(\Omega)$ define a function $\mathrm{ev}(v)$ of $M_1^+(\Omega)$ into $[0,1]$ by

$$\big(\mathrm{ev}(v)\big)(\mu) = \int v(\omega)\, d\mu\,, \qquad \mu \in M_1^+(\Omega)\,,$$

and call it the *evaluation* or the *prolongation* of v over $M_1^+(\Omega)$. The elements of the set $\mathrm{ev}\big(\mathcal{B}(\Omega)\big)$ of all evaluations of characteristic functions χ_B, $B \in \mathcal{B}(\Omega)$, are said to be *extended events.*

It is easy to see that $\mathrm{ev}\big(\mathcal{E}(\Omega)\big)$ and $\mathrm{ev}\big(\mathcal{B}(\Omega)\big)$ are effect algebras and hence D-posets. Moreover, since they are functions ranging in $[0,1]$, they are ID-objects and IE-objects in the sense of Papčo [24], [25]. Recall that an ID-poset is a family $\mathcal{X} \subseteq I^X$ of functions on a set X into I carrying the usual pointwise partial order and sequential convergence, containing the constant functions 0_X and 1_X, and closed under the usual difference $\ominus$ (a partial operation in $\mathcal{X}$: if $u, v \in \mathcal{X}$ and $v(x) \le u(x)$ for all $x \in X$, then $u \ominus v$ is defined by $(u \ominus v)(x) = u(x) - v(x)$, $x \in X$, and belongs to $\mathcal{X}$). Then $(X, \mathcal{X})$ is said to be an ID-*measurable* space. If $(Y, \mathcal{Y})$ is another ID-measurable space and f is a map of Y into X such that, for each $u \in \mathcal{X}$, $u \circ f$ belongs to $\mathcal{Y}$, then f is said to be $(\mathcal{X}, \mathcal{Y})$-*measurable. Remember*, if $\mathcal{X}$ and $\mathcal{Y}$ are characteristic functions of fields of sets, then f is $(\mathcal{X}, \mathcal{Y})$-measurable iff it is measurable in the usual sense.

Definition 2.3. Let $(\Omega, \mathcal{B}(\Omega))$ and $(\Xi, \mathcal{B}(\Xi))$ be measurable spaces. Let A be a map of $M_1^+(\Omega)$ into $M_1^+(\Xi)$ such that for each $u \in \mathcal{E}(\Xi)$ there exists $v \in \mathcal{E}(\Omega)$ such that $\operatorname{ev}(u) \circ A = \operatorname{ev}(v)$. Then A is said to be $\bigl(\operatorname{ev}(\mathcal{E}(\Xi)), \operatorname{ev}(\mathcal{E}(\Omega))\bigr)$-*-measurable* or, simply, *measurable*.

Theorem 2.4. *Let $(\Omega, \mathcal{B}(\Omega))$ and $(\Xi, \mathcal{B}(\Xi))$ be measurable spaces and let A be a map of $M_1^+(\Omega)$ into $M_1^+(\Xi)$. Then A is a statistical map iff it is measurable.*

Proof. (i) Let A be measurable. Let $B \in \mathcal{B}(\Xi)$ and let $\mu \in M_1^+(\Omega)$. We have to prove that

$$\bigl(A(\mu)\bigr)(B) = \int \bigl(A(\delta_\omega)\bigr)(B)\, d\mu .$$

Let χ_B be the characteristic function of B and let $\bigl(\operatorname{ev}(\chi_B)\bigr)(\nu)$, $\nu \in M_1^+(\Xi)$, be its prolongation. We know that there exists $v \in \mathcal{E}(\Omega)$ such that $\operatorname{ev}(\chi_B) \circ A = \operatorname{ev}(v)$. But for $\omega \in \Omega$ we have $v(\omega) = \bigl(\operatorname{ev}(\chi_B)\bigr)\bigl(A(\delta_\omega)\bigr) = \bigl(A(\delta_\omega)\bigr)(B)$. Consequently, for each point-measure μ we have

$$\bigl(A(\mu)\bigr)(B) = \int \bigl(A(\delta_\omega)\bigr)(B)\, d\mu$$

(since the integral amounts to the value of the integrated function at the point representing the measure μ). For a measure μ which is not a point-measure at some $\omega \in \Omega$, according to the definition of a prolongation $\operatorname{ev}(v)$ of v we get

$$\bigl(A(\mu)\bigr)(B) = \bigl(\operatorname{ev}(\chi_B)\bigr)\bigl(A(\mu)\bigr) = \bigl(\operatorname{ev}(v)\bigr)(\mu) = \int \bigl(A(\delta_\omega)\bigr)(B)\, d\mu .$$

(ii) Let A be a statistical map. Let $u \in \mathcal{E}(\Xi)$ and let $\operatorname{ev}(u)$ be its prolongation over $M_1^+(\Xi)$. We have to prove that there exists $v \in \mathcal{E}(\Omega)$ such that

$$\operatorname{ev}(u) \circ A = \operatorname{ev}(v) .$$

First, let u be the characteristic function χ_B of some $B \in \mathcal{B}(\Xi)$. For $\mu \in M_1^+(\Omega)$ we have $\bigl(\operatorname{ev}(\chi_B)\bigr)\bigl(A(\mu)\bigr) = \bigl(A(\mu)\bigr)(B) = \int \bigl(A(\delta_\omega)\bigr)(B)\, d\mu$. Thus $\omega \mapsto \bigl(A(\delta_\omega)\bigr)(B)$, $\omega \in \Omega$, yields a measurable map $v \in \mathcal{E}(\Omega)$ such that $\bigl(\operatorname{ev}(\chi_B)\bigr)\bigl(A(\mu)\bigr) = \bigl(\operatorname{ev}(v)\bigr)(\mu)$. Hence

$$\operatorname{ev}(\chi_B) \circ A = \operatorname{ev}(v) .$$

Consequently, if u is a simple function of the type $\sum_{i=1}^{n} \lambda_i \chi_{B_i}$, where $0 \le \lambda_i \le 1$, $i = 1, 2, \ldots, n$, and $\{B_i \in \mathcal{B}(\Xi) : i = 1, 2, \ldots, n\}$ is a partition of Ξ, then $\operatorname{ev}(u) \circ A$ is a prolongation of some $v \in \mathcal{E}(\Omega)$. A straightforward verification (using the Lebesgue convergence theorem, cf. Riečan and Neubrunn [28]) shows that if $\langle u_n \rangle$ is a sequence of functions in $\mathcal{E}(\Xi)$ which pointwise converges to a function $u \in \mathcal{E}(\Xi)$ and there exists a sequence $\langle v_n \rangle$ in $\mathcal{E}(\Omega)$ such that $\mathrm{ev}(u_n) \circ A = \mathrm{ev}(v_n)$, then the sequence $\langle \mathrm{ev}(u_n) \rangle$ of prolongations pointwise converges to the prolongation $\mathrm{ev}(u)$, the sequence $\langle \mathrm{ev}(v_n) \rangle$ pointwise converges to some prolongation $\mathrm{ev}(v)$ of a function $v \in \mathcal{E}(\Omega)$, and $\mathrm{ev}(u) \circ A = \mathrm{ev}(v)$. This implies that the set of all $u \in \mathcal{E}(\Xi)$ such that $\mathrm{ev}(u) \circ A = \mathrm{ev}(v)$ for some $v \in \mathcal{E}(\Omega)$ is actually the whole $\mathcal{E}(\Xi)$. Thus A is measurable. $\square$

Lemma 2.5. *Let $(\Omega, \mathcal{B}(\Omega))$ and $(\Xi, \mathcal{B}(\Xi))$ be measurable spaces and let A be a map of $M_1^+(\Omega)$ into $M_1^+(\Xi)$. For each $u \in \mathrm{ev}(\mathcal{E}(\Xi))$, define $A^{\triangleleft}(u) = u \circ A$.*

(i) Let A be a statistical map. Then $A^{\triangleleft}(u) \in \mathrm{ev}(\mathcal{E}(\Omega))$ and $A^{\triangleleft}$ as a map of $\mathrm{ev}(\mathcal{E}(\Xi))$ into $\mathrm{ev}(\mathcal{E}(\Omega))$ is a sequentially continuous D-poset homomorphism.

(ii) Let $A^{\triangleleft}$ be a sequentially continuous D-poset homomorphism of $\mathrm{ev}(\mathcal{E}(\Xi))$ into $\mathrm{ev}(\mathcal{E}(\Omega))$. Then A is a statistical map.

Proof. (i) Theorem 2.4 implies that $\mathrm{ev}(\mathcal{E}(\Xi)) \circ A \subseteq \mathrm{ev}(\mathcal{E}(\Omega))$. Observe that the order, the algebraic operations, and the sequential convergence in $\mathrm{ev}(\mathcal{E}(\Xi)) \subseteq I^{M_1^+(\Xi)}$ and in $\mathrm{ev}(\mathcal{E}(\Omega)) \subseteq I^{M_1^+(\Omega)}$ are defined pointwise. It follows directly from the definition of $A^{\triangleleft}$ that it is a D-poset homomorphism (e.g., if $v \le u$, then $A^{\triangleleft}(v) = (v \circ A) \le (u \circ A) = A^{\triangleleft}(u)$ and $A^{\triangleleft}(u \ominus v) = (u \ominus v) \circ A = (u \circ A) \ominus (v \circ a) = A^{\triangleleft}(u) \ominus A^{\triangleleft}(v)$). Similarly, if $\langle u_n \rangle$ converges pointwise in $\mathrm{ev}(\mathcal{E}(\Xi))$ to u, then $\langle A^{\triangleleft}(u_n) \rangle \equiv \langle u_n \circ A \rangle$ converges pointwise in $\mathrm{ev}(\mathcal{E}(\Omega))$ to $u \circ A = A^{\triangleleft}(u)$ and hence $A^{\triangleleft}$ is sequentially continuous.

(ii) The assertion is trivial. Indeed, since $A^{\triangleleft}$ is a D-poset homomorphism, it follows that $\mathrm{ev}(\mathcal{E}(\Xi)) \circ A \subseteq \mathrm{ev}(\mathcal{E}(\Omega))$, i.e., A is measurable. According to Theorem 2.4, A is a statistical map. $\square$

Besides statistical maps, in Gudder [17] and Bugajski [2], [3] the following notions have been studied:

- an *observable* on $(\Omega, \mathcal{B}(\Omega))$ with value space $(\Xi, \mathcal{B}(\Xi))$; it is a map h of $\mathcal{B}(\Xi)$ into $\mathcal{E}(\Omega)$ such that $h(\Xi) = 1_\Omega$ and if $\langle B_i \rangle$ a sequence of mutually disjoint sets in $\mathcal{B}(\Xi)$, then $h(\cup B_i) = \sum h(B_i)$, where the convergence in the summation is pointwise (see Gudder [17], p. 696);
- a *probability kernel* (also Markov kernel) on $(\Omega, \mathcal{B}(\Omega))$ with value space $(\Xi, \mathcal{B}(\Xi))$; it is a map K of $\Omega \times \mathcal{B}(\Xi)$ into $[0,1]$ which is measurable for each fixed second coordinate and a measure for each fixed first coordinate (see Gudder [17], p. 696);
- an *extension of an observable* h to a σ-morphism $\tilde{h}$; it is an effect algebra homomorphism of $\mathcal{E}(\Xi)$ into $\mathcal{E}(\Omega)$ which preserves joins of increasing sequences (see Gudder [17], p. 697);

- a *dual statistical map*; it is a sequentially continuous (with respect to the pointwise convergence) affine map B of $\mathcal{E}(\Xi)$ into $\mathcal{E}(\Omega)$ such that $B(1_\Xi) = 1_\Omega$ (see Bugajski [2], p. 331);
- a *statistical function* on $(\Omega, \mathcal{B}(\Omega))$ with the outcome space $(\Xi, \mathcal{B}(\Xi))$; it is a map ϕ of Ω into $M_1^+(\Xi)$ such that for each $B \in \mathcal{B}(\Xi)$ the function $\phi(\omega)(B)$ belongs to $\mathcal{E}(\Omega)$ (see Bugajski [2], p. 331);
- an $\mathcal{E}(\Omega)$ *valued measure* on $\mathcal{B}(\Xi)$; the same as observable in the sense of S. Gudder (see Bugajski [2], p. 331);

Although defined in different terms and, perhaps, having different domains and/or ranges, such maps fall into two groups. Statistical maps and statistical functions belong to the first group; they map certain probability measures on $(\Omega, \mathcal{B}(\Omega))$ into probability measures on $(\Xi, \mathcal{B}(\Xi))$. All other belong to the second group; they map certain elements (functions, generalized events) of $\mathcal{E}(\Xi)$ into $\mathcal{E}(\Omega)$. Statistical maps and statistical functions are in one-to-one correspondence (Theorem 4.2 in Bugajski [2]). All maps in the second group are in one-to-one correspondence with the dual statistical maps (cf. Gudder [17], pp. 696–697, Theorem 4.5 in Bugajski [2]). The maps in the first group capture one aspect of a classical random variable: it is a channel through which the probability measures on $(\Omega, \mathcal{B}(\Omega))$ are transformed into probability measures on $(\Xi, \mathcal{B}(\Xi))$. The channel is rather *indirectly* described in terms of measurability of the random variable in question. The maps in the second group capture the other aspect of a classical random variable: it induces a (sequentially continuous) homomorphism of the range field of events $\mathcal{B}(\Xi)$ into the domain field of events $\mathcal{B}(\Omega)$. This is a quite *explicit* description of the channel: the homomorphism composed with a probability measure on the original field of probability events transforms probability measures on the original field of probability events into probability measures on the new field of probability events. The two aspects are dual and it is our task to find a suitable category (objects and morphisms) in which all that can be described. We claim that the category ID introduced in Papčo [24] suites the purpose. The objects are D-posets of fuzzy sets (or subeffect algebras of powers I^T of the closed unit intervals) and morphisms are sequentially continuous (with respect to the pointwise sequential convergence) D-poset homomorphisms.

We end this section with some observations concerning the mutual relationships between the monotone sequential convergence and the pointwise sequential convergence. Let $(\Omega, \mathcal{B}(\Omega))$ be a measurable space. Then $\mathcal{E}(\Omega) \subseteq I^\Omega$ is an effect algebra, but it is also a D-poset or a Łukasiewicz tribe. It is known that $\mathcal{E}(\Omega)$ is a closed subset of I^Ω with respect to the pointwise sequential convergence. Further, (cf. Novák [23]) each (σ-additive) probability measure μ on $\mathcal{B}(\Omega)$ is sequentially continuous with respect to the pointwise sequential convergence (of characteristic functions). Clearly, μ can be characterized as a nonnegative, normed, subtractive ($B \subseteq A$ implies $\mu(A \setminus B) = \mu(A) - \mu(B)$) function of $\mathcal{B}(\Omega)$ into $[0,1]$ which is sequentially continuous with respect to the pointwise sequential convergence. Due to the Lebesgue convergence theorem, the extension of μ over $\mathcal{E}(\Omega)$ via $\mu(f) = \int f(\omega)\,d\mu$, called in Gudder [17] the probability of f, is also sequentially continuous with respect to the pointwise sequential convergence. If a sequence $\langle u_n \rangle$ converges in $\mathcal{E}(\Omega)$ to u, then in $\mathcal{E}(\Omega)$ there are sequences $\langle v_n \rangle$, $\langle w_n \rangle$ such that $w_n \leq u_n \leq v_n$, $n = 1, 2, \dots$, and $\langle w_n \rangle$ converges upward to u and $\langle v_n \rangle$ converges downward to u. This of course means that if $(\Xi, \mathcal{B}(\Xi))$ is another measurable space and h is an effect algebra σ-homomorphism of $\mathcal{E}(\Xi)$ into $\mathcal{E}(\Omega)$, then h is sequentially continuous with respect to the pointwise sequential convergence.

OBSERVATION 2.6. (*i*) *A map $\mu\colon \mathcal{B}(\Omega) \to [0,1]$ is a probability measure iff it is a sequentially continuous D-poset homomorphism (equivalently, effect algebra homomorphism).*

(*ii*) *Each probability measure μ on $\mathcal{B}(\Omega)$ can be uniquely extended to a sequentially continuous D-poset homomorphism (equivalently, effect algebra homomorphism) $\tilde{\mu}$ over $\mathcal{E}(\Omega)$: $\tilde{\mu}(u) = \int u(\omega)\,d\mu$, $u \in \mathcal{E}(\Omega)$.*

(*iii*) *A map h of $\mathcal{E}(\Xi)$ into $\mathcal{E}(\Omega)$ is a dual statistical map iff it is a sequentially continuous D-poset homomorphism (equivalently, effect algebra homomorphism).*

3. Duality

It is known that effect algebras and D-posets are equivalent structures, more exactly, the corresponding categories are isomorphic. As stated in Foulis [7], a requirement that positive (normed) measures determine the partial order of the effect algebra in question is the *sine qua non* condition of algebraic measure theory (for effect algebras, hence for D-posets). Such structures are studied in Frič [15] and each of them can be represented as a system of fuzzy subsets of a class of measures determining the partial order. The representation is not unique and some important measure-theoretic properties depend on the choice of the class (see Frič [9], [11], [12], [13]). Remember, there are effect algebras admitting no states, cf. Riečanová [29].

Categorical aspects od D-posets and effect algebras of fuzzy subsets are studied in Papčo [24], [25]. Let us recall some notions. Let I be the interval $[0,1]$ carrying the usual algebraic operations, order, and sequential convergence. Let X be a set, let I^X be the power carrying the pointwise operations, order, and sequential convergence, and let $\mathcal{X} \subseteq I^X$ be a reduced (points of X are separated by $\mathcal{X}$) D-poset of fuzzy subsets of X. Such D-posets form the category

ID, the morphisms of which are the sequentially continuous D-poset homomorphisms. Each $x \in X$ represents an evaluation morphism of $\mathcal{X}$ into I, defined by $\mathrm{ev}_x(u) = u(x), u \in \mathcal{X}$. Morphisms of $\mathcal{X}$ into I are called *states* and if each state is an evaluation at some point $x \in X$ (it is uniquely determined if $\mathcal{X}$ is reduced and we assume only reduced systems), then $\mathcal{X}$ is said to be *sober*. Pairs $(X, \mathcal{X})$ are called ID-*measurable spaces* and form the category MID, the morphisms of which are measurable maps (see the paragraph before Definition 2.3). Let SID and MSID be the (full) subcategories of sober (reduced) objects of ID and MID, respectively.

Theorem 1.12 in Papčo [24] states that ID and MSID are dual: the opposite (i.e., dual) category $\mathrm{ID}^{\mathrm{op}}$ and the category MSID are naturally equivalent (cf. Mac Lane [20]). The duality can be decomposed: ID is naturally equivalent to its subcategory SID and the categories SID and MSID are dually isomorphic. Indeed, for each ID-poset $\mathcal{X} \subseteq I^X$ there exists an isomorphic sober ID-poset $\mathcal{X}^* \subseteq I^{X^*}$which can be described as follows. Consider X as the fixed morphisms of $\mathcal{X}$ into I and let X^* be the set of all morphisms of $\mathcal{X}$ into I. Let $\mathcal{X}^* \subseteq I^{X^*}$be the evaluation of $\mathcal{X}$, i.e., the elements of $\mathcal{X}^*$ are of the form $u^* = \mathrm{ev}(u) = \{h(u) \colon h \in \hom(\mathcal{X}, I)\}$, $u \in \mathcal{X}$. Then $\mathcal{X}$ and $\mathcal{X}^*$ are isomorphic ID-posets. The rest is a categorical routine. The duality between SID and MSID is more relevant for us. Lemma 2.5 states that each statistical map A of $M_1^+(\Omega)$ into $M_1^+(\Xi)$ generates a unique sequentially continuous D-poset homomorphism $A^{\triangleleft}$ of $\mathrm{ev}(\mathcal{E}(\Xi))$ into $\mathrm{ev}(\mathcal{E}(\Omega))$. The natural question is whether for each sequentially continuous D-poset homomorphism h of $\mathrm{ev}(\mathcal{E}(\Xi))$ into $\mathrm{ev}(\mathcal{E}(\Omega))$ there exists a unique statistical map A such that $h = A^{\triangleleft}$. Lemma 1.10 in Papčo [24] leads to a positive answer. To support our claim that deep theorems in Bugajski [2], [3] have simple categorical proofs, we present the lemma together with the proof.

Lemma 3.1. *Let h be a sequentially continuous D-poset homomorphism of a sober ID-poset $\mathcal{X} \subseteq I^X$ into an ID-poset $\mathcal{Y} \subseteq I^Y$. Then there exists a unique $(\mathcal{X}, \mathcal{Y})$-measurable map A of Y into X such that $h = A^{\triangleleft}$.*

Proof. For each $y \in Y$, the composition $\mathrm{ev}_y \circ h$ is a sequentially continuous D-poset homomorphism of $\mathcal{X}$ into I. Since $\mathcal{X}$ is sober (and reduced), there exists a unique $x \in X$ such that $\mathrm{ev}_x = \mathrm{ev}_y \circ h$. Put $A(y) = x$. This defines a map A of Y into X. A straightforward calculation shows that $h = A^{\triangleleft}$. Indeed, we have to prove that $u \circ A = h(u)$ for all $u \in \mathcal{X}$. So, let $u \in \mathcal{X}$ and $y \in Y$. Then $\big(h(u)\big)(y) = \mathrm{ev}_y\big(h(u)\big) = (\mathrm{ev}_y \circ h)(u) = \mathrm{ev}_x(u) = \mathrm{ev}_{A(y)}(u) = (u \circ A)(y)$. □

Remember (cf. Papčo [24]), the assumption that $\mathcal{X}$ is sober cannot be leaved out.

Definition 3.2. Let $(\Omega, \mathcal{B}(\Omega))$ be a measurable space and let $\mathcal{E}(\Omega)$ be the effect algebra of measurable functions of Ω into $[0,1]$. Then the elements of $\mathrm{ev}(\mathcal{E}(\Omega)) \subseteq I^{M_1^+(\Omega)}$ are said to be *extended fuzzy events.*

Denote EFID the (full) subcategory of ID consisting of objects of the form $\mathrm{ev}(\mathcal{E}(\Omega))$ and MEFID the (full) subcategory of EFID-measurable spaces (i.e., pairs $(M_1^+(\Omega), \mathrm{ev}(\mathcal{E}(\Omega))))$.

Corollary 3.3. *(i) There is a one-to-one correspondence between statistical maps and dual statistical maps.*

(ii) The categories EFID and MEFID are dually isomorphic.

Proof. It is easy to see that ID-posets of the form $\mathrm{ev}(\mathcal{E}(\Omega)) \subseteq I^{M_1^+(\Omega)}$ are sober (and reduced). According to Theorem 2.4, a map A of $M_1^+(\Omega)$ into $M_1^+(\Xi)$ is a statistical map iff it is $(\mathrm{ev}(\mathcal{E}(\Xi)), \mathrm{ev}(\mathcal{E}(\Omega)))$-measurable. Since dual statistical maps are sequentially continuous D-poset homomorphisms, the assertion follows from the dual isomorphism between the categories SID and MSID (cf. Theorem 1.11 in Papčo [24]). $\square$

4. Generalizations

In this section we mention some consequences of the dual isomorphism between EFID and MEFID and relate them to the corresponding results in Gudder [17], Bugajski [2], and Foulis [7].

Let $(\Omega, \mathcal{B}(\Omega))$ be a measurable space. Let $\delta(M_1^+(\Omega)) \subseteq M_1^+(\Omega)$ be the set of all two-valued probability measures. In accordance with Frič [9], [11], [12], if $\delta(M_1^+(\Omega)) = \Omega$, i.e., if all two-valued probability measures are fixed, then $(\Omega, \mathcal{B}(\Omega))$ is said to be *s-perfect.*

Let $(\Omega, \mathcal{B}(\Omega))$ and $(\Xi, \mathcal{B}(\Xi))$ be measurable spaces. Let A be a statistical map of $M_1^+(\Omega)$ into $M_1^+(\Xi)$ and let h be a sequentially continuous D-poset homomorphism of $\mathrm{ev}(\mathcal{E}(\Xi))$ into $\mathrm{ev}(\mathcal{E}(\Omega))$ such that $h = A^{\triangleleft}$. Let $(\Xi, \mathcal{B}(\Xi))$ be s-perfect and let δh be the "δ-restriction" of h defined as follows: if h maps $\mathrm{ev}(u) \in \mathrm{ev}(\mathcal{E}(\Xi))$ to $\mathrm{ev}(v) \in \mathrm{ev}(\mathcal{E}(\Omega))$, then δh maps $u \in \mathcal{E}(\Xi)$ to $v \in \mathcal{E}(\Omega)$.

Lemma 4.1. *Let $(\Xi, \mathcal{B}(\Xi))$ be s-perfect. Then the following are equivalent:*

(i) A maps Ω into $\delta(M_1^+(\Xi)) = \Xi$;

(ii) h maps $\mathrm{ev}(\mathcal{B}(\Xi))$ into $\mathrm{ev}(\mathcal{B}(\Omega))$;

(iii) δh maps $\mathcal{B}(\Xi)$ into $\mathcal{B}(\Omega)$.

Proof. The assertion is a straightforward consequence of the measurability of A, i.e., $h(\mathrm{ev}(u)) = \mathrm{ev}(u) \circ A$ for all $u \in \mathcal{E}(\Xi)$. $\square$

Let $g\colon \mathcal{B}(\Xi) \to \mathcal{E}(\Omega)$ be an observable in the sense of S. Gudder and let $\tilde{g}\colon \mathcal{E}(\Xi) \to \mathcal{E}(\Omega)$ be its unique extension to a σ-morphism over $\mathcal{E}(\Xi)$. On p. 697 in Gudder [17] it is stated that: (i) if $\big(\Xi, \mathcal{B}(\Xi)\big)$ is a Polish measurable space, then g is crisp (maps $\mathcal{B}(\Xi)$ into $\mathcal{B}(\Omega)$) iff there exists a (classical) measurable map $f\colon \Omega \to \Xi$ such that $g(B) = \chi_{f^{\leftarrow}(B)}$; (ii) if $f\colon \Omega \to \Xi$ is a measurable map and g_f is the generated crisp observable, then $\widetilde{g_f}(u) = u \circ f$, $u \in \mathcal{E}(\Xi)$. But the duality (cf. Lemma 4.1) gives more. First, (i) is credited to Bugajski, Hellwig and Stulpe [1] (Theorem 1), where instead of the assumption that "$\big(\Xi, \mathcal{B}(\Xi)\big)$ is a Polish measurable space" it is assumed that "all point-measures on $\big(\Xi, \mathcal{B}(\Xi)\big)$ are fixed" (i.e., $\big(\Xi, \mathcal{B}(\Xi)\big)$ is s-perfect). Second, using the same argument as in Lemma 3.5 in Frič [12], it can be shown that s-perfectness is a necessary and sufficient assumption. But for s-perfect spaces (i) is obvious. Similarly, (ii) follows from the duality: each sequentially continuous D-poset homomorphism (crisp or fuzzy) is of the form $A^{\triangleleft}$ for some statistical map A.

Observe that several assertions in Bugajski [2] are simple corollaries of the duality. Indeed, from "$h = A^{\triangleleft}$" we get the sequential continuity of dual statistical maps (Corollary 3.3) and various one-to-one correspondences between notions equivalent to statistical maps and dual statistical maps (Theorem 4.3, etc.).

On p. 450 in Foulis [7], a Δ-*observable* is defined as an affine mapping of a full set Δ of (σ-additive) probability measures on an effect algebra L admitting an order determining set of (σ-additive) probability measures into the set of all (σ-additive) probability measures on the Borel sets $\mathcal{B}(R)$ of the real line. The definition seems to be too general. Indeed, to get any form of a duality, some measurability condition is needed. Otherwise, for example, if $L = 2$, then $M_1^+(2)$ consists of the only measure and if we map it to a nontrivial measure on $\mathcal{B}(R)$, then this is not a statistical map and it does not generate any D-poset homomorphism going the opposite direction.

In the last section we mention a generalization for which the duality between generalized random variables and generalized observables holds.

5. Examples

Interesting examples and applications of genuine fuzzy random variables can be found in Bugajski [3]; Proposition 3.2 associates to each fuzzy random variable $H\colon M_1^+(\Omega) \to M_1^+(\Xi)$ a standard random variable $F\colon \Omega' \to \Xi$ which generates on the value space $\big(\Xi, \mathcal{B}(\Xi)\big)$ the same probability distributions as the original fuzzy random variable.

In Frič [16] a sort of dual problem is dealt with. We start with a standard random variable and ask whether there is a fuzzy random variable generating on the value space the same distributions as the original one. The example in Frič [16] models a digital display consisting of segments and a number of objects drawn at random. Each object is coded via a certain subset of segments and, due to some random failure, different subsets of segments can appear on the display. This means that, to each resulting subset of segments, only a probability distribution that a particular object has been drawn can be assigned —this is how to each new elementary event (i.e., the subset of segments) a probability measure is assigned in a consistent way. The result is a fuzzy random variable and the measures are calculated by means of conditional probabilities. The interested reader is referred to Frič [16] for details.

6. Concluding remarks

We are aware of the possible criticism by experts on quantum probability, but in view of the preceding arguments, we propose the following

DEFINITION 6.1. Let $\mathcal{X} \subseteq I^X$ be an ID-poset. The elements of $\mathcal{X}$ are said to be *generalized events*. Each ID-morphism of $\mathcal{X}$ into I is said to be a *state*. Each ID-morphism of $\mathcal{X} \subseteq I^X$ into $\mathcal{Y} \subseteq I^Y$ is said to be a *generalized observable*. Each $(\mathcal{X}, \mathcal{Y})$-measurable map of Y into X is said to be a *generalized random variable*.

Obviously, for suitable subcategories of ID and MID the generalized probability notions reduce to the usual ones. E.g., consider fields of sets, MV-algebras, EFID-objects. For sober objects we get a duality between generalized observables and generalized fuzzy random variables. Observe that Theorem 7.4.2 in Dvurečenskij and Pulmannová [6] becomes a special case: it is the duality between MV-observables into a tribe of fuzzy sets (an MV-observable on a σ-complete MV-algebra M is an MV-σ-homomorphism of $\mathcal{E}(R)$ into M) and suitable measurable functions. The key fact is that $\mathcal{E}(R)$ is sober (cf. Lemma 3.7, Theorem 3.8 and Corollary 3.9 in Frič [12]).

We believe that some constructions from Frič [10], [13], Jurečková [18], Chovanec and Kôpka [4], Mundici [21], Mundici and Riečan [22], Petrovičová [26], [27], can be generalized in the realm of generalized observables and random variables.

We summarize our arguments as follows: 1. The categorical methods help us to put various approaches to the foundation of probability into a perspective. Various related structures lead to different categories and the relationships between categories lead to a simplification.

2. The category ID has the advantage that both the probability measures (states) and fuzzy random variables become morphisms. Further, ID is isomor-

phic to the category of effect algebras admitting sufficiently many states and the states completely determine the structure of (generalized) events.

3. There are at least two good reasons to define the generalized probability measure as sequentially continuous: – the Lebesgue convergence theorem becomes an axiom, – the pointwise convergence in the powers I^X is the categorical product convergence, i.e., the finest one making all projections sequentially continuous, hence each projection is a probability and the probabilities determine the initial structure. This makes possible to use the powerful machinery of categorical products (dually coproducts).

4. The duality between generalized random variables and generalized observables has a transparent form and it generalizes the known dualities for random variables and observables.

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