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MOMENT PROBLEM IN PARTIALLY ORDERED VECTOR SPACES

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ABSTRACT. We present a moment problem in the context of σ-complete partially ordered vector spaces. A generalization of the Hausdorff moment theorem is formulated and proved.

1. Introduction

If g is a real-valued function of bounded variation on the unit interval I of the real line, then the numbers

$$a_k = \int_0^1 t^k \, dg(t), \qquad k \in \mathbb{N}.$$

are called the (power) moments of g.

For every sequence of real numbers a_k, $k \in \mathbb{N}$, and every pair of non-negative integers n, k, set

$$\Delta^n a_k := \sum_{j=0}^{n} (-1)^j \binom{n}{j} a_{k+j} .$$

The sequence $(a_k, k \in \mathbb{N})$ is called completely monotone if $\Delta^n a_k \geq 0$ for all integers $n, k \geq 0$. For a sequence (a_k) of real numbers to be the moment sequence of some non-decreasing function g (this case being of particular interest), it is necessary and sufficient that (a_k) be completely monotone ([5], [7]). The result permits a generalization to the situation where (a_k) is a completely monotone sequence of elements of a vector lattice. The definition of a completely monotone sequence is the same. In the paper [4] it was considered the same problem in the

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case of a σ-complete, weakly σ-distributive vector lattice Y. In the paper [2] the Hausdorff moment problem was solved for an arbitrary complete vector lattice. On the other hand, there was used the representation of an order bounded linear operator on the space of continuous functions on the interval $[a,b]$ of real line into a complete vector lattice Y ([1]). The present paper solves the same problem for a σ-complete partially ordered vector space Y.

2. Monotone σ-complete partially ordered vector spaces

In the whole paper the symbol Y denotes a real monotone σ-complete partially ordered vector space. A real partially ordered vector space is said to be monotone σ-complete if any increasing sequence $(a_n)_{n=1}^{\infty}$, which is bounded above, has a least upper bound $\bigvee\limits_{n=1}^{\infty} a_n$, see [8].

Proposition 1.1. *Let Y be a real monotone σ-complete partially ordered vector space.*

Then

(i) *Any decreasing sequence $(a_n)_{n=1}^{\infty}$, which is bounded below, has a greatest lower bound $\bigwedge\limits_{n=1}^{\infty} a_n$ and $\bigvee\limits_{n=1}^{\infty}(-a_n) = -\bigwedge\limits_{n=1}^{\infty} a_n$.*

(ii) *Y is Archimedean, i.e., for any positive $y \in Y$ the sequence $(ny)_{n=1}^{\infty}$ is unbounded above and $\bigwedge\limits_{n=1}^{\infty} \frac{1}{n}y = 0$.*

(iii) *For any positive $y \in Y$ and any decreasing sequence $(\alpha_n)_{n=1}^{\infty}$ of reals with $\lim\limits_{n\to\infty} \alpha_n = 0$ we have $\bigwedge\limits_{n=1}^{\infty} \alpha_n y = 0$.*

(iv) *For any increasing (decreasing) sequences $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$, which are bounded above (below), we have $\bigvee\limits_{n=1}^{\infty}(a_n+b_n) = \bigvee\limits_{n=1}^{\infty} a_n + \bigvee\limits_{n=1}^{\infty} b_n$, $\left(\bigwedge\limits_{n=1}^{\infty}(a_n+b_n) = \bigwedge\limits_{n=1}^{\infty} a_n + \bigwedge\limits_{n=1}^{\infty} b_n\right)$.*

3. Riemann-Stieltjes integral

Take a continuous real function f, nondecreasing Y-valued function g defined on the interval $[a,b]$ and a partition

$$\Pi = \{a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b\}$$

of the interval $[a,b]$. Put

$$s(f,g,\Pi)=\sum_{j=1}^{n} m_j\bigl(g(x_j)-g(x_{j-1})\bigr)$$

and

$$S(f,g,\Pi)=\sum_{j=1}^{n} M_j\bigl(g(x_j)-g(x_{j-1})\bigr),$$

where $m_j=\min\limits_{x\in[x_{j-1},x_j]} f(x)$ and $M_j=\max\limits_{x\in[x_{j-1},x_j]} f(x)$.

Proposition 2.1. *There is a unique element $y\in Y$ such that*

$$s(f,g,\Pi')\le y\le S(f,g,\Pi)$$

for any partitions Π' and Π of the interval $[a,b]$.

Proof. The inequality $s(f,g,\Pi')\le S(f,g,\Pi)$ is obvious. For any natural n and $0\le i\le 2^n$ put $x_{n,i}=a+i(b-a)2^{-n}$ and $\Pi_n=\{x_{n,i}\}_{i=0}^{2^n}$. Obviously,

$$\bigvee_{n=1}^{\infty} s(f,g,\Pi_n)=\bigwedge_{n=1}^{\infty} S(f,g,\Pi_n).$$

Now, the element

$$y=\bigvee_{n=1}^{\infty} s(f,g,\Pi_n)=\bigwedge_{n=1}^{\infty} S(f,g,\Pi_n)$$

has the required properties.

The element y will be denoted by $\int\limits_a^b f(x)\,dg(x)$ and is said to be the Riemann–Stieltjes integral of f with respect to g on the interval $[a,b]$. □

4. Positive linear maps

The set of all real continuous functions on the interval $I=[a,b]$ is denoted by $C(I)$. A linear map $\Phi\colon C(I)\to Y$ is said to be positive if $\Phi(f)\ge 0$ for $f\ge 0$. Obviously, any nondecreasing function $g\colon I\to Y$ defines a positive linear map $\Phi\colon C(I)\to Y$ by the formula $\Phi(f)=\int\limits_a^b f(x)\,dg(x)$. We shall show that it is sufficient to consider only functions g which are left o-continuous at any $x\in(a,b)$, i.e., $\bigvee\limits_{n=1}^{\infty} g(x_n)=g(x)$ for any increasing sequence $(x_n)_{n=1}^{\infty}$ converging to $x\in(a,b)$.

Recall that Y-valued Baire measure m on a compact X is a Y-valued function defined on Baire subsets of X such that $m(\emptyset) = 0$, $m(A) \geq 0$ and $m\Bigl(\bigcup\limits_{j=1}^{\infty} A_j\Bigr) = \bigvee\limits_{n=1}^{\infty} \sum\limits_{j=1}^{n} m(A_j)$ for any sequence $(A_j)_{j=1}^{\infty}$ of pairwise disjoint Baire subsets of X. For more details see [8].

THEOREM 3.1. *The formulas*

$$\Phi(f) = \int_a^b f(x)\,dg(x)\,,$$

$$\Phi(f) = \int_I f(t)\,dm(t)\,, \quad \text{and}$$

$$g(t) = m\bigl([a,t)\bigr) \qquad \text{for } t \in [a,b) \text{ and } g(b) = m(I)$$

define one-to-one correspondence between the sets of all positive linear maps $\Phi\colon C(I) \to Y$, Y-valued Baire measures m on the interval $[a,b)$ and nondecreasing functions $g\colon [a,b] \to Y$ which are left o-continuous at any $x \in (a,b)$ and $g(a) = 0$.

P r o o f. Let $\Phi\colon C(I) \to Y$ be a positive linear map. By [8, p. 687], for such a map there is a unique Y-valued Baire measure m on I such that $\Phi(f) = \int\limits_I f(t)\,dm(t)$ for any $f \in C(I)$. Now, put $g(t) = m\bigl([a,t)\bigr)$, for $t \in [a,b)$ and $g(b) = m(I)$. Then $g(a) = 0$ and the function g is left o-continuous at any $t \in (a,b)$. Obviously,

$$s(f,g,\Pi') \leq \int_I f(t)\,dm(t) \leq S(f,g,\Pi)$$

for any partitions Π' and Π of the interval $[a,b]$. It means

$$\int_I f(t)\,dm(t) = \int_a^b f(t)\,dg(t) = \Phi(f)\,.$$

Let $g_1, g_2\colon C(I) \to Y$ be two nondecreasing functions such that

$$\int_a^b f(x)\,dg_1(x) = \int_a^b f(x)\,dg_2(x) \qquad \text{for any } f \in C(I)\,,$$

$$g_1(a) = g_2(a) = 0\,,$$

and

$$\bigvee_{n=1}^{\infty} g_i(x_n) = g_i(x)$$

for $i = 1, 2$ and any increasing sequence $(x_n)_{n=1}^{\infty}$ converging to $x \in (a, b)$. Then

$$g_1(b) = \int_a^b 1\, dg_1(x) = \int_a^b 1\, dg_2(x) = g_2(b)\,.$$

Let $(x_n)_{n=1}^{\infty}$ be an increasing sequence converging to $x \in (a, b)$. Put

$$f_n(t) = \min\left(1, \max\left(0, \frac{x-t}{x-x_n}\right)\right).$$

Then

$$\bigvee_{n=1}^{\infty} \int_a^b f_n(x)\, dg_1(x) = \bigvee_{n=1}^{\infty} \int_a^b f_n(x)\, dg_2(x)$$

and

$$g_i(x) = \bigvee_{n=1}^{\infty} g_i(x_n) \le \bigvee_{n=1}^{\infty} \int_a^b f_n(x)\, dg_i(x) \le g_i(x) \qquad \text{for } i = 1, 2\,.$$

Hence, $g_1(x) = g_2(x)$. It completes the proof. □

Remark. The preceding theorem may be formulated and proved without the measure m. The function g may be defined by the formulas

$$g(a) = 0, \quad g(b) = \Phi(1) \qquad \text{and} \qquad g(x) = \bigvee_{n=1}^{\infty} \Phi(f_n) \quad \text{for } x \in (a, b)\,,$$

where f_n are the same functions as above.

5. Moment problem theorem

In this section we solve the moment problem on the interval $I = [0, 1]$. For a sequence $(y_k)_{k=0}^{\infty}$ of elements of Y we find a necessary and sufficient condition for the existence such a non-decreasing function $g\colon [0, 1] \to Y$ for which

$$\int_0^1 t^n\, dg(t) = y_n\,, \qquad y_n \in Y, \quad n = 0, 1, \ldots$$

We show that this condition is the complete monotonicity of the sequence $(y_k)_{k=0}^{\infty}$.

Let q_k and $q_k^{(n)}$ be the functions defined by

$$q_k(t) = t^k$$

and

$$q_k^{(n)}(t) = t^k(1-t)^n\,,$$

i.e.,

$$q_k^{(n)} = \sum_{j=0}^{n} \binom{n}{j} (-1)^j q_{j+k}\,.$$

For a function $f \in C(I)$ we denote by $B_n(f)$ the nth Bernstein polynomial of f, i.e.,

$$B_n(f) = \sum_{j=0}^{n} \binom{n}{j} f\left(\frac{j}{n}\right) q_j^{(n-j)}\,.$$

LEMMA 5.1.

$$B_n(q_k) = q_k \qquad \textit{for } k = 0, 1\,.$$

More generally, for integers $k \geq 1$

$$B_n(q_k) = \frac{1}{n^k} \sum_{i=1}^{k} a_{k,i}\, q_i \prod_{j=0}^{i-1} (n-j)\,,$$

where the coefficients $a_{k,i}$ *are defined as follows*

$$a_{k,1} = a_{k,k} = 1 \quad \textit{and} \quad a_{k+1,i} = i\, a_{k,i} + a_{k,i-1}$$

for all positive integers k *and* $i = 2, \dots, k$.

P r o o f . We have

$$B_n(q_k; t) = \sum_{j=0}^{n} \binom{n}{j} \left(\frac{j}{n}\right)^k t^j (1-t)^{n-j}\,.$$

Let $t \in I$ be fixed. For the evaluation of the sum

$$\sum_{j=0}^{n} \binom{n}{j} j^k t^j (1-t)^{n-j}$$

consider a function

$$F(s) = (te^s + 1 - t)^n = \sum_{j=0}^{n} \binom{n}{j} e^{js} t^j (1-t)^{n-j}\,.$$

Obviously,

$$F^{(k)}(0) = \sum_{j=0}^{n} \binom{n}{j} j^k t^j (1-t)^{n-j} .$$

The formula

$$F^{(k)}(s) = \sum_{i=1}^{k} a_{k,i} t^i e^{is} (te^s + 1 - t)^{n-i} \prod_{j=0}^{i-1} (n-j)$$

may be easily proved by induction. It proves that

$$\sum_{j=0}^{n} \binom{n}{j} j^k t^j (1-t)^{n-j} = \sum_{i=1}^{k} a_{k,i} t^i \prod_{j=0}^{i-1} (n-j) .$$

□

Lemma 5.2. *For any function $f \in C(I)$ there are decreasing and increasing sequences $(\psi_n)_{n=1}^{\infty}$ and $(\varphi_n)_{n=1}^{\infty}$ of polynomials converging uniformly to the function f.*

P r o o f. It is well known that $B_n(f)$ are polynomials which converge uniformly to the function f. So, for any natural n take N so large that $|B_N(f) - f| < 2^{-n}$ and put $f_n = B_N(f)$. Then $f_n - 2^{-n} < f < f_n + 2^{-n}$. Put $\psi_n = f_n + 3 \cdot 2^{-n}$ and $\varphi_n = f_n - 3 \cdot 2^{-n}$. We have

$$\psi_{n+1} = f_{n+1} + 3 \cdot 2^{-n-1} < f + 4 \cdot 2^{-n-1} < f_n + 2^{-n} + 4 \cdot 2^{-n-1} = f_n + 3 \cdot 2^{-n} = \psi_n$$

and

$$\varphi_{n+1} = f_{n+1} - 3 \cdot 2^{-n-1} > f - 4 \cdot 2^{-n-1} > f_n - 2^{-n} - 4 \cdot 2^{-n-1} = f_n - 3 \cdot 2^{-n} = \varphi_n .$$

So, we obtain the required sequences of polynomials. □

For a sequence $(y_k)_{k=0}^{\infty}$ of elements of Y put

$$\Delta^n y_k := \sum_{j=0}^{n} (-1)^j \binom{n}{j} y_{k+j} ,$$

or equivalently

$$\Delta^0 y_k = y_k , \quad \Delta^{n+1} y_k = \Delta^n y_k - \Delta^n y_{k+1} .$$

Definition 5.1. A sequence $(y_k)_{k=0}^{\infty}$ of elements of Y is called *completely monotone* if

$$\Delta^n y_k \geq 0 \qquad \text{for } n, k = 0, 1, 2, \dots$$

We shall derive a concrete result relating to the power moment problem in the interval $[0,1]$. We use similar ideas as in paper [3] for the proof of the Herglotz theorem.

THEOREM 5.1. *Let $(y_k)_{k=0}^{\infty}$ be a sequence of elements of Y. A non-decreasing function $g\colon [0,1] \to Y$ such that*

$$\int_0^1 t^n \, dg(t) = y_n \,, \qquad y_n \in Y, \quad n = 0,1,2,\dots$$

exists if and only if the sequence $(y_k)_{k=0}^{\infty}$ is completely monotone.

Proof. The necessity. Let the moment problem (1) be solved. Denote by Φ the linear operator on the space $C[0,1] \to Y$ generated by a non-decreasing function $g\colon [0,1] \to Y$. Then

$$\Phi(q_n) = y_n \,.$$

Since

$$0 \le q_k^{(n)} = \sum_{j=0}^{n} \binom{n}{j} (-1)^j q_{j+k} \,,$$

we have

$$0 \le \Phi(q_k^{(n)}) = \sum_{j=0}^{n} \binom{n}{j} (-1)^j \Phi(q_{j+k}) = \sum_{j=0}^{n} (-1)^j \binom{n}{j} y_{k+j} = \Delta^n y_k \,.$$

It means that the sequence $(y_k)_{k=0}^{\infty}$ is completely monotone.

The sufficiency. By Theorem 3.1, it suffices to construct a positive linear map $\Phi\colon C(I) \to Y$ such that

$$\Phi(q_n) = y_n \,, \quad n = 0,1,\dots$$

So, we put

$$\Phi(p) = \alpha_0 y_0 + \alpha_1 y_1 + \dots + \alpha_n y_n \,.$$

for

$$p = \alpha_0 q_0 + \alpha_1 q_1 + \dots + \alpha_n q_n \,.$$

Since the functions q_n are linearly independent, the definition of Φ is correct. The defined operator Φ is clearly additive and homogeneous. We have to prove that complete monotonicity of the sequence $(y_k)_{k=0}^{\infty}$ implies positiveness of the map Φ (defined on the vector space of all polynomials). We have

$$\Phi\big(q_k^{(s)}\big) = \Delta^s y_k \,, \qquad k, s = 0,1,\dots$$

Take any polynomial $p(t) = \alpha_0 + \alpha_1 t + \dots + \alpha_m t^m$ such that $p(t) \ge 0$ for any $t \in I$. Consider the sequence $(p_n)_{n=0}^{\infty}$ of the Bernstein polynomials of p, i.e.,

$$p_n(t) = B_n(p;t) = \sum_{k=0}^{n} \binom{n}{k} p\left(\frac{k}{n}\right) t^k (1-t)^{n-k} .$$

We have

$$\Phi(p_n) = \sum_{k=0}^{n} \binom{n}{k} p\left(\frac{k}{n}\right) \Delta^{n-k} y_k \geq 0\,.$$

We want to prove

$$\Phi(p) \geq 0\,.$$

Since $\Phi(p_n) \geq 0$, we have

$$\Phi(p) \geq \Phi(p - p_n)\,.$$

By Lemma 5.1,

$$\begin{aligned} p - p_n &= \sum_{k=2}^{m} \alpha_k \left(q_k - \frac{1}{n^k} \sum_{i=1}^{k} a_{k,i}\, q_i \prod_{j=0}^{i-1} (n-j) \right) \\ &= \sum_{k=2}^{m} \alpha_k\, q_k \left(1 - \frac{1}{n^k} \prod_{j=0}^{k-1} (n-j) \right) - \sum_{k=2}^{m} \alpha_k \frac{1}{n^k} \sum_{i=1}^{k-1} a_{k,i}\, q_i \prod_{j=0}^{i-1} (n-j)\,. \end{aligned}$$

It means

$$\Phi(p - p_n) = \sum_{k=2}^{m} \alpha_k\, y_k \left(1 - \prod_{j=0}^{k-1} \left(1 - \frac{j}{n} \right) \right) - \sum_{k=2}^{m} \alpha_k \frac{1}{n^k} \sum_{i=1}^{k-1} a_{k,i}\, y_i \prod_{j=0}^{i-1} (n-j)\,.$$

We denote by

$$\alpha = \max\{|\alpha_2|, \ldots, |\alpha_m|\}$$

and

$$A_m = \max\{a_{k,i} \colon 1 \leq k \leq m,\, 1 \leq i \leq k\}\,.$$

$$\begin{aligned} \Phi(p - p_n) &\geq -\alpha \left(1 - \prod_{j=0}^{m-1} \left(1 - \frac{j}{n} \right) \right) \sum_{k=2}^{m} y_k - \alpha A_m \sum_{k=2}^{m} \frac{1}{n^k} \sum_{i=1}^{k-1} y_i\, n^i \\ &= -\alpha \left(1 - \prod_{j=0}^{m-1} \left(1 - \frac{j}{n} \right) \right) \sum_{k=2}^{m} y_k - \alpha A_m \sum_{i=1}^{m-1} y_i \sum_{k=i+1}^{m} \frac{1}{n^{k-i}}\,. \end{aligned}$$

Note that

$$\prod_{j=0}^{m-1} \left(1 - \frac{j}{n} \right) \geq 1 - \sum_{j=0}^{m-1} \frac{j}{n} = 1 - \frac{(m-1)m}{2n} > 1 - \frac{m^2}{n}\,,$$

$$\sum_{k=i+1}^{m} \frac{1}{n^{k-i}} \leq \frac{m-1}{n} < \frac{m}{n}\,,$$

$$\sum_{i=1}^{m-1} y_i \leq (m-1)\, y_0 \leq m\, y_0\,,$$

and

$$\sum_{k=2}^{m} y_k \le (m-1)y_0 \le m y_0 \,.$$

Therefore

$$\Phi(p) \ge \Phi(p-p_n) \ge -\frac{m^2\alpha}{n}(m+A_m)\,y_0 \,.$$

Finally,

$$\Phi(p) \ge \bigvee_{n=1}^{\infty} -\frac{m^2\alpha}{n}(m+A_m)\,y_0 = -m^2\alpha\big(m+A_m\big) \bigwedge_{n=1}^{\infty} \frac{y_0}{n} = 0 \,.$$

So, Φ is positive. Now, take an arbitrary $f \in C(I)$. By Lemma 5.2, there exist nonincreasing and nondecreasing sequences $(\psi_n)_{n=1}^{\infty}$ and $(\varphi_n)_{n=1}^{\infty}$ of polynomials converging uniformly to the function f. We have

$$\varphi_n \le \varphi_{n+1} \le f \le \psi_{m+1} \le \psi_m$$

for any integers m and n. Therefore

$$\Phi(\varphi_n) \le \Phi(\varphi_{n+1}) \le \Phi(\psi_{m+1}) \le \Phi(\psi_m)$$

which implies $\Phi(\varphi_n) \le \Phi(\psi_m)$ for any integers m and n. It means

$$\bigvee_{n=1}^{\infty} \Phi(\varphi_n) \le \Phi(\psi_m) \qquad \text{and} \qquad \bigvee_{n=1}^{\infty} \Phi(\varphi_n) \le \bigwedge_{m=1}^{\infty} \Phi(\psi_m) \,.$$

Now, consider the sequence $(\psi_n - \varphi_n)_{n=1}^{\infty}$ which is nonincreasing and converges uniformly to zero. Put

$$\alpha_n = \sup_{t\in[0,1]} \big(\psi_n(t) - \varphi_n(t)\big) \,.$$

Then

$$0 = \bigwedge_{n=1}^{\infty} \alpha_n \Phi(1) = \bigwedge_{n=1}^{\infty} \Phi(\alpha_n) \ge \bigwedge_{n=1}^{\infty} \Phi(\psi_n - \varphi_n) = \bigwedge_{n=1}^{\infty} \Phi(\psi_n) - \bigvee_{n=1}^{\infty} \Phi(\varphi_n)$$

which means

$$\bigvee_{n=1}^{\infty} \Phi(\varphi_n) \ge \bigwedge_{n=1}^{\infty} \Phi(\psi_n) \,.$$

Therefore

$$\bigvee_{n=1}^{\infty} \Phi(\varphi_n) = \bigwedge_{n=1}^{\infty} \Phi(\psi_n) \,.$$

So, neither $\bigvee_{n=1}^{\infty} \Phi(\varphi_n)$ nor $\bigwedge_{n=1}^{\infty} \Phi(\psi_n)$ depend on the choice of nonincreasing and nondecreasing sequences $(\varphi_n)_{n=1}^{\infty}$ and $(\psi_n)_{n=1}^{\infty}$ of polynomials converging uniformly to the function f. Put

$$\Phi(f) = \bigvee_{n=1}^{\infty} \Phi(\varphi_n) = \bigwedge_{n=1}^{\infty} \Phi(\psi_n)\,.$$

Then we have the required positive linear map $\Phi\colon C(I) \to Y$. $\square$

Remark. We would like to mention that the moment theorem can be applied to obtain a new proof of the spectral theorem for Hermitian operators. Cf. [6], where the moment theorem is considered in ordered locally convex spaces with some properties.

REFERENCES

[1] CRISTESCU, R.: *Ordered Vector Spaces and Linear Operators*, Abacus Press, Kent, 1976.

[2] DEBIEVE, C.—DUCHOŇ, M.—RIEČAN, B.: *Moment problem in some ordered vector spaces*, Tatra Mt. Math. Publ. **12** (1997), 247–251.

[3] DUCHOŇ, M.—MALIČKÝ, P.: *On Herglotz theorem in partially ordered vector spaces*, Soft Comput. **4** (2000), 5–8.

[4] DUCHOŇ, M.—RIEČAN, B.: *Moment problem in vector lattices*, Novi Sad J. Math. **26** (1996), 53–61.

[5] HAUSDORFF, F.: *Momentprobleme für ein endliches Interval*, Math. Z. **16** (1923), 220–248.

[6] SCHAEFFER, H. H.: *A generalized moment problem*, Math. Ann. **146** (1962), 325–330.

[7] WIDDER, D. V.: *The Laplace Transform*, Princeton University Press, Princeton, 1946.

[8] WRIGHT, J. D. M.: *Measures with values in a partially ordered vector space*, Proc. London Math. Soc. **25** (1972), 675–688.

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