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PRIMITIVE SEQUENCES IN ARITHMETICAL SEMIGROUPS

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ABSTRACT. We extend the basic results of F. Behrend, S. Pillai, P. Erdős, A. Sárközy and E. Szemerédi on primitive sequences to arithmetical semigroups satisfying Axiom A. Such generalization allows to transfer the classical results from integers to other objects as algebraic integers, ideals of number fields, finite Abelian groups, etc.

1. Introduction

The history of the notion of the primitive sequence has its roots in the effort to tackle probably the oldest mathematical problem complex at all, the problems on perfect and related numbers (cf. [8, pp. 3–33]). Since any multiple of an abundant (i.e., $\sigma(n) > 2n$) or perfect number ($\sigma(n) = 2n$) is abundant, a non-deficient number ($\sigma(n) \geqslant 2n$) is called *primitive* if it is not a multiple of a smaller non-deficient number ([6]). For instance, any perfect number is a primitive non-deficient number.

Behrend [2] proved[1] that for the number $\mathcal{A}(n)$ of non-deficient numbers $\leqslant n$ we have $0.241n \leqslant \mathcal{A}(n) \leqslant 0.314n$ for all sufficiently large n. Davenport [5] proved[2] that $\mathcal{A}(n)/n$ tends to a limit as $n \to \infty$. An elementary proof of this fact was given by Erdős [10]. His proof is based on the observation that

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[1]In his first paper [1] he proved that $\mathcal{A}(n) < 0.47n$. It is interesting to note that papers were presented by Issai Schur (as a member of the Academy) and that the first paper was written by F. Behrend when he was a student of mathematics in Berlin. The impetus to the paper was given by the result that at most $1/4$ of all odd positive integers is non-deficient, a result which was a traditional part of Schur's lectures on number theory.

[2]and reproved by F. Behrend and S. Chowla independently of each other as stated in [5] (see also [9, p. 3] for other details). Actually they all proved a more general result, namely that the density of κ-abundant numbers exists, where a number is κ-abundant if $\sigma(n) \geqslant \kappa n$.

if $\sum 1/a_k$ converges then $\mathcal{A}(n)/n$ tends to a limit. He deduced the convergence from the result proved in this paper saying that the number of primitive non-deficient numbers not exceeding n is $o(n/\log^2 n)$.

Thus the sequence of primitive non-deficient numbers satisfies the following properties:

(1) no number of the sequence is divisible by another number of the sequence,

(2) the density of the sequence vanishes,

(3) the density of the set of all multiples of the elements of the sequence exists.

According to Behrend [3] it was H. Davenport who raised the following question[3]. Let

$$a_1 < a_2 < a_3 < \dots \tag{1}$$

be an increasing sequence of positive integers. What can be said about the density of the sequence possessing the property

P: no a_i divides an a_j except when $i = j$.

This seems to be the first appearance of the notion of the primitive sequence which afterwards are defined as finite or infinite sequences fulfilling property **P** above. Simple examples of primitive sequences are:

EXAMPLE 1. The set of integers m such that $n < m \leqslant 2n$ for fixed n.

EXAMPLE 2. The set of integers which have exactly k prime factors with factors counted according to their multiplicity.

It can be easily proved (cf. [3] or [16, p. 244]) that if (1) is a primitive sequence and $\mathcal{A}(x) = \sum\limits_{a_i \leqslant x} 1$ then

$$\limsup_{x\to\infty} \frac{\mathcal{A}(x)}{x} = d < \frac{1}{2}.$$

Besicovitch [4] (cf. also [16, p. 245]) proved that to any given $\varepsilon > 0$ there exists a primitive sequence with upper density $d > \frac{1}{2} - \varepsilon$. This shows that property (2) does not transfer to general primitive sequences as it was originally conjectured by Chowla, Davenport and Erdős (cf. [11]). On the other hand, Erdős proved in a letter to Davenport that we have (cf. [11], [16, p. 245])

$$\sum_{a_i \leqslant x} \frac{1}{a_i \log a_i} = \mathcal{O}(1). \tag{2}$$

[3] Besicovitch [4] attributes this problem also to S. Chowla. In [13] P. Erdős is also given as the third author of the problem.

for every primitive sequence (1). This shows that

$$\sum_{a_i \leqslant x} \frac{1}{a_i} = o(\log x),$$

or in other words, that the logarithmic density (and consequently also the lower asymptotic density) of a primitive sequence vanishes. F. Behrend proved a more stronger result, namely that[4]

$$\sum_{a_i \leqslant x} \frac{1}{a_i} = \mathcal{O}\left(\frac{\log x}{\sqrt{\log\log x}}\right). \tag{3}$$

S. Pillai [19] proved that this result is in certain sense the best possible: there exists a c so that for every x there is a primitive sequence $a_1 < \cdots < a_k \leqslant x$ such that

$$\sum_{a_i < x} \frac{1}{a_i} > c\frac{\log x}{\sqrt{\log\log x}}.$$

P. Erdős, A. Sárközy and E. Szemerédi [12] improved the estimate (3) for infinite primitive sequences showing: If (1) is an infinite primitive sequence then

$$\sum_{a_i \leqslant x} \frac{1}{a_i} = o\left(\frac{\log x}{\sqrt{\log\log x}}\right). \tag{4}$$

They also proved that this result is the best possible.[5]

Since this notion of the primitive sequence depends only on the multiplicative structure of the set of positive integers, it is very natural to reformulated this notion in terms of a theory depending only on the multiplication structure. In [20, 21] some of these results are extended via an analysis of the notion of the logarithmic density, which plays an important role in this subject. The aim of this note is to show that the ideas of the authors mentioned above can also be modified in the terms of the so called arithmetical semigroups which besides the set of positive integers include also other arithmetical and algebraic objects.

2. Arithmetical semigroups

Let $\mathbb{G}$ denote a free commutative semigroup relative to a multiplication operation denoted by juxtaposition, with identity element $1_{\mathbb{G}}$ and with at most

[4] The proof of this result, independently discovered also by Erdős several moths later, contains, see below, the first application of Sperner's lemma to number theory (cf. [9, p. 214]).

[5] For other historical details we refer the reader to the sources like [16, Chap. V], [15], or [9, p. 3, 214].

countably many generators. Such a semigroup will be called *arithmetical semigroup* if in addition a real-valued *norm* $|\cdot|$ is defined on $\mathbb{G}$ such that

(1) $|1_{\mathbb{G}}| = 1$, $|a| > 1$ for all $a \in \mathbb{G}$,

(2) $|ab| = |a| \cdot |b|$ for all $a, n \in \mathbb{G}$,

(3) the total number
$$N_{\mathbb{G}}(x) = \sum_{\substack{|a| \leqslant x \\ a \in \mathbb{G}}} 1$$
of elements $a \in \mathbb{G}$ of norm not exceeding x is finite for each real x.

We shall denote by $\mathcal{P}_{\mathbb{G}}$ the set of generators of $\mathbb{G}$ and its elements will be called *primes*.

In applications of the notion of an arithmetical semigroup a significant role is played by arithmetical semigroups satisfying the so-called

Axiom A. *There exist positive constants A and δ and a constant η with $0 \leqslant \eta < \delta$, such that*
$$N_{\mathbb{G}}(x) = Ax^{\delta} + \mathcal{O}(x^{\eta}).$$

More details on the abstract axiomatic approach to some arithmetical problems via the notion of the arithmetical semigroup and especially the theory of arithmetical semigroups satisfying Axiom A can be found in [18].

The following results enable us to extend the proofs of the mentioned results on primitive sequences to primitive sequences defined in arithmetical semigroups satisfying Axiom A:

Lemma 1. ([18, p. 85]) *Let $\mathbb{G}$ be an arithmetical semigroup satisfying Axiom A. Then for $x \to \infty$ we have*

(i) $\sum\limits_{|a| \leqslant x} |a|^{-\delta} = \delta A \log x + \gamma_{\mathbb{G}} + \mathcal{O}(x^{\eta-\delta})$,
where $\gamma_{\mathbb{G}} = A + \delta \int\limits_1^{\infty} \big(N_{\mathbb{G}}(x) - Ay^{\delta}\big) y^{-\delta-1} \, \mathrm{d}y$;

(ii) *for* $\Re(z) = \eta$, $\sum\limits_{|a| \leqslant x} |a|^{-z} = \frac{\delta A}{\delta - z} x^{\delta - z} + \mathcal{O}(\log x)$;

(iii) *if* $\Re(z) \leqslant \delta$, $\Re(z) = \sigma \neq \eta$ *and* $z \neq \delta$, *then*
$\sum\limits_{|a| \leqslant x} |a|^{-z} = \frac{\delta A}{\delta - z} x^{\delta - z} + \alpha(z) + \mathcal{O}\big(x^{\eta - \sigma}\big)$, *where* $\alpha(z)$ *is a constant.*

Lemma 2. ([18, p. 165–166]) *If the arithmetical semigroup $\mathbb{G}$ satisfies Axiom A then*

(i) $\sum\limits_{\substack{|p| \leqslant x \\ p \in \mathcal{P}_{\mathbb{G}}}} \frac{1}{|p|^{\delta}} = \log\log x + B + \mathcal{O}\Big(\frac{1}{\log x}\Big)$ *for some constant B depending on parameters appearing in Axiom A,*

(ii) $\sum_{\substack{|p|\leqslant x \\ p\in\mathcal{P}_{\mathbb{G}}}} \frac{\log|p|}{|p|^{\delta}} = \log x + \mathcal{O}(1)$.

The *prime-divisor* function ω is defined by

$$\omega(n) = \begin{cases} 0 & \text{if } n = 1_{\mathbb{G}}\,, \\ r & \text{if } n = p_1^{\alpha_1}\cdots p_r^{\alpha_r}\,, \text{ where } p_i \in \mathcal{P}_{\mathbb{G}} \text{ are distinct and } \alpha_i > 0\,. \end{cases}$$

If the arithmetical semigroup $\mathbb{G}$ satisfies Axiom A then we have ([18, p. 138])

$$\limsup_{|n|\to\infty} \frac{\omega(n)\log\log|n|}{\log|n|} = \delta\,.$$

If $d(n)$ denotes the total number of divisors of $n \in \mathbb{G}$ and $\mathbb{G}$ satisfies Axiom A, then ([18, p. 92, 137])

$$\sum_{|n|\leqslant x} d(n) = Ax^{\delta}(\delta A\log x + 2\gamma_{\mathbb{G}} - A) + \mathcal{O}\big(x^{(\delta+\eta)/2}\big)\,, \tag{5}$$

and

$$\limsup_{|n|\to\infty} \frac{\big(\log d(n)\big)\log\log|n|}{\log|n|} = \delta\log 2\,.$$

The function $\omega(n)$ has the normal order $\log\log|n|$, and the function $\log d(n)$ has the normal order $\log 2\cdot\log\log|n|$ (cf. [18, p. 180–181]).

The next property of the zeta function $\zeta_{\mathbb{G}}$ of an arithmetical semigroup $\mathbb{G}$ extends the well-known property of the classical Riemann zeta-function:

Lemma 3. ([18, p. 84]) *Let $\mathbb{G}$ be an arithmetic semigroup satisfying Axiom A. Then*

$$\zeta_{\mathbb{G}}(z) = \sum_{n\in\mathbb{G}} \frac{1}{|n|^{z}}$$

is absolutely convergent for all z with $\Re(z) > \delta$ and divergent for all z with $\Re(z) \leqslant \delta$.

3. Primitive sequences

Unless contrary is stated we shall always suppose that the arithmetical semigroup under consideration satisfies Axiom A. Under this assumption we can define (cf. [20, 21]) using Lemma 1 (i) the lower $\underline{\ell}_{\mathbb{G}}$ and upper $\overline{\ell}_{\mathbb{G}}$ logarithmic density of a sequence $\mathcal{C} \subset \mathbb{G}$ by

$$\underline{\ell}_{\mathbb{G}}(\mathcal{C}) = \liminf_{x\to\infty} \frac{1}{\delta A\log x} \sum_{|a|\leqslant x} |a|^{-\delta} \quad \text{and} \quad \overline{\ell}_{\mathbb{G}}(\mathcal{C}) = \limsup_{x\to\infty} \frac{1}{\delta A\log x} \sum_{|a|\leqslant x} |a|^{-\delta}\,.$$

A subsequence $\{a_i\} \subset \mathbb{G}$ will be called *primitive* if no its term is divisible by another one. The next result extends the known one that the logarithmic density of a primitive sequence in $\mathbb{N}$ vanishes:

PROPOSITION 4. ([20, p. 1313]) *Let $\mathbb{G}$ satisfy Axiom A and $\mathcal{C} \subset \mathbb{G}$ be a sequence such that every element of $\mathcal{C}$ divides only finitely many elements of $\mathcal{C}$, then the logarithmic density of $\mathcal{C}$ vanishes.*

As mentioned above this result for primitive sequences also follows from Behrend's result having now the form:

THEOREM 5. *Let $\mathcal{C}$ be a primitive sequence in an arithmetical semigroup $\mathbb{G}$ satisfying Axiom A. Then*

$$\frac{1}{\log x} \sum_{\substack{a\in\mathcal{C} \\ |a|\leqslant x}} |a|^{-\delta} = \mathcal{O}\big((\log\log x)^{-1/2}\big) \tag{6}$$

with the involved constant depending only on the parameters appearing in Axiom A.

P r o o f . Let $r(n)$ denote the number of ordered representations of an element $n \in \mathbb{G}$ in the form $n = ab$ with $a \in \mathcal{C}$. Then Lemma 1 (ii) gives

$$\sum_{|n|\leqslant x} r(n) = \sum_{\substack{|ma|\leqslant x \\ a\in\mathcal{C}, m\in\mathbb{G}}} 1 = \sum_{\substack{a\in\mathcal{C} \\ |a|\leqslant x}} N_{\mathbb{G}}\left(\frac{x}{|a|}\right) = x^{\delta} \sum_{\substack{a\in\mathcal{C} \\ |a|\leqslant x}} |a|^{-\delta} + \mathcal{O}(x^{\delta}),$$

that is

$$\sum_{\substack{a\in\mathcal{C} \\ |a|\leqslant x}} |a|^{-\delta} = \frac{1}{x^{\delta}} \sum_{|n|\leqslant x} r(n) + \mathcal{O}(1). \tag{7}$$

On the other hand, let $s(n)$, $n \in \mathbb{G}$, denote the maximal cardinality of a set of divisors of n such that no element of the set divides another one. If $\mathcal{C}$ is a primitive sequence consisting only from squarefree elements of $\mathbb{G}$, then $r(n) \leqslant s(n)$ for $n \in \mathcal{C}$, and $s(n)$ can be estimated using Sperner's lemma and Stirling's formula as follows

$$s(n) \leqslant \binom{\omega(n)}{\frac{\omega(n)}{2}} \ll \frac{2^{\omega(n)}}{\sqrt{\omega(n)}}.$$

Then

$$\begin{aligned}\sum_{|n|\leqslant x} r(n) &\ll \sum_{\substack{|n|\leqslant x\\ \omega(n)\leqslant k}} \frac{2^{\omega(n)}}{\sqrt{\omega(n)}} + \sum_{\substack{|n|\leqslant x\\ \omega(n)>k}} \frac{2^{\omega(n)}}{\sqrt{\omega(n)}}\\ &\ll \frac{2^k}{\sqrt{k}}\, x^\delta + \frac{1}{\sqrt{k}} \sum_{|n|\leqslant x} 2^{\omega(n)}\\ &\leqslant \frac{2^k}{\sqrt{k}}\, x^\delta + \frac{1}{\sqrt{k}} \sum_{|n|\leqslant x} d(n)\,.\end{aligned}$$

Since $\sum\limits_{|n|\leqslant x} d(n) \ll x^\delta \log x$, the theorem follows for $k = \log\log x$ provided the elements of $\mathcal{C}$ are squarefree.

If the elements of $\mathcal{C}$ are not all squarefree then we can write them in the form $a = k^2 a^{(k)}$ where $k \in \mathbb{G}$ is the square factor of a with the greatest possible norm, and

$$\sum_{\substack{a\in\mathcal{C}\\ |a|\leqslant x}} |a|^{-\delta} = \sum_{k\in\mathbb{G}} |k|^{-2\delta} \sum_{\substack{a\in\mathcal{C}\\ |a^{(k)}|\leqslant x/|k|^2}} \left|a^{(k)}\right|^\delta \leqslant \sum_{k\in\mathbb{G}} |k|^{-2\delta} \sum_{\substack{a\in\mathcal{C}\\ |a^{(k)}|\leqslant x}} \left|a^{(k)}\right|^\delta .$$

To the inner sum we can apply the result proved above for the squarefree case, while the sum $\sum\limits_{k\in\mathbb{G}} |k|^{-2\delta}$ is convergent due to Lemma 3. □

Lemma 2 allows us to adapt the argument used in [16] to prove the following extension of Pillai's theorem:

Theorem 6. *Let $\mathcal{C}$ be a primitive sequence in an arithmetical semigroup $\mathbb{G}$ satisfying Axiom A. Then there exists an absolute constant $c = c(A,\delta,\eta)$ such that for every sufficiently large x there exists a primitive sequence*

$$|a_1| \leqslant |a_2| \leqslant \cdots \leqslant |a_k| \leqslant x\,, \qquad a_i \in \mathbb{G}\,, \tag{8}$$

depending on x for which

$$\frac{1}{\log x} \sum_{i=1}^{k} |a_i|^{-\delta} > \frac{c}{(\log\log x)^{1/2}}\,. \tag{9}$$

P r o o f. Lemma 2 (ii) shows that there are constants $\alpha' > 0$ and c' such that for $x > c'$ we have

$$\sum_{|p|\leqslant x} |p|^{-\delta} \geqslant \log\log x - \alpha'\,. \tag{10}$$

Let $\alpha > \alpha'$, $\beta > 1$, and $\beta' > 1$ be such that

$$\alpha - \alpha' > \beta\beta' \log 2 . \tag{11}$$

Let $c_1 > c'$ be such that the following two inequalities are satisfied for $x > c_1$

$$\sum_{|p|\leqslant x} \frac{\log|p|}{|p|^\delta} \leqslant \beta \log x \tag{12}$$

and

$$\log\log x > \frac{\alpha\beta'}{\beta' - 1} . \tag{13}$$

We shall construct a finite sequence using the sequences of the type described in Example 2. Let $u_i = u_i^{(r)}$ be the sequence of elements each having exactly r prime factors counted with multiplicity where r will depend on x in such a way that

$$r \leqslant \beta'(\log\log x - \alpha) . \tag{14}$$

Let

$$\lambda_r(x) = \sum_{|u_i^{(r)}|\leqslant x} \bigl|u_i^{(r)}\bigr|^{-\delta} .$$

We prove by induction on r that

$$\lambda_r(x) \geqslant \frac{(\log\log x - \alpha)^r}{r!} \tag{15}$$

for $x > c_1^{2^{r-1}}$. This inequality is true for $r = 1$ due to (10) and the condition $\alpha > \alpha'$. An element $u^{(r+1)} = p_1^{\alpha_1} \dots p_s^{\alpha_s}$ with $\sum\limits_{i=1}^{s} \alpha_i = r + 1$ and s distinct prime factors can be expressed in the form $p \cdot u^{(r)}$ with $p|u^{(r)}$ in at least s distinct ways, e.g., $p_i\Bigl|\frac{u^{(r+1)}}{p_j}$ with $j \neq i$ and $i = 1, \dots, s$. Consequently, if $r \geqslant 1$ then

$$(r+1)\lambda_{r+1}(x) > \lambda_{r+1}(x) \geqslant \sum_{|pu^{(r)}|\leqslant x} \frac{1}{|pu^{(r)}|^\delta} \geqslant \sum_{|p|\leqslant\sqrt{x}} \frac{1}{|p|^\delta} \cdot \lambda_r\Bigl(\frac{x}{|p|}\Bigr) .$$

If $x > c_1^{2^r}$ and $|p| \leqslant \sqrt{x}$ then $x/|p| > c_1^{2^{r-1}}$ and the induction hypothesis shows that

$$(r+1)\lambda_{r+1}(x) \geqslant \frac{1}{r!} \sum_{|p|\leqslant\sqrt{x}} \frac{1}{|p|^\delta} \Bigl(\log\log\frac{x}{|p|} - \alpha\Bigr)^r = \frac{(\log\log x - \alpha)^r}{r!} t_r(x) ,$$

where

$$t_r(x) = \sum_{|p|\leqslant\sqrt{x}} \frac{1}{|p|^\delta}\left(1 + \frac{\log\left(1 - \frac{\log|p|}{\log x}\right)}{\log\log x - \alpha}\right)^r . \tag{16}$$

Relation (15) will be proved if we prove that

$$t_r(x) \geqslant \frac{\log\log x - \alpha}{r!} \qquad \text{for } x > c_1^{2^{r-1}} .$$

If $|p| \leqslant \sqrt{x}$ then $(\log|p|)/\log x \leqslant 1/2$. Since $-\log(1-x)$ is convex and increasing for $0 \leqslant x \leqslant 1/2$,

$$0 < -\log\left(1 - \frac{\log|p|}{\log x}\right) \leqslant (2\log 2)\frac{\log|p|}{\log x} .$$

Therefore using (14) we get

$$\left(1 + \frac{\log\left(1 - \frac{\log|p|}{\log x}\right)}{\log\log x - \alpha}\right)^r \geqslant \left(1 - \frac{(2\log 2)\frac{\log|p|}{\log x}}{\log\log x - \alpha}\right)^r \geqslant 1 - \frac{(2r\log 2)\frac{\log|p|}{\log x}}{\log\log x - \alpha}$$
$$\geqslant 1 - \frac{(2\beta'\log 2)\log|p|}{\log x} .$$

Using (16) we get the inequality

$$t_r(x) \geqslant \sum_{|p|\leqslant\sqrt{(x)}} \frac{1}{|p|^\delta} - \frac{2\beta'\log 2}{\log x} \sum_{|p|\leqslant\sqrt{x}} \frac{\log|p|}{|p|^\delta} .$$

Since $x > c_1^{2^r} \geqslant c_1^2$, i.e., $\sqrt{x} \geqslant c_1$, the inequalities (10), (12) and (11) then imply

$$t_r(x) \geqslant \log\log x - \alpha' - \beta\beta'\log 2 \geqslant \log\log x - \alpha ,$$

and (15) is proved.

The final part of the proof can be now done mutatis mutandis as in [16, p. 251]. ☐

The following combinatorial result was proved in Lemma 2 of [12] which we reproduce here for the convenience of the reader:

Lemma 7. *Let $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$ where $\mathcal{S}_1 \cap \mathcal{S}_2 = \emptyset$ and $\operatorname{card}(\mathcal{S}_1) = k$, $\operatorname{card}(\mathcal{S}_2) = \ell$. Assume $\ell \geqslant k$. Let $\mathcal{A}_i \subset \mathcal{S}$ for $1 \leqslant i \leqslant r$ with*

$$r > c\frac{2^{k+\ell}}{\sqrt{k+\ell}} \tag{17}$$

be subsets of $\mathcal{S}$ no one of which contains any other. Denote by $\mathcal{B}_1, \dots, \mathcal{B}_t$ all the (distinct) subsets of $\mathcal{S}$ of the form

$$\mathcal{A}_i \cup \mathcal{R} , \qquad 1 \leqslant i \leqslant r,\ \mathcal{R} \subset \mathcal{S}_2 , \tag{18}$$

where in (18) $\mathcal{R}$ runs through all the 2^ℓ subsets of $\mathcal{S}_2$. Then $t > c_1 2^{k+\ell}$.

THEOREM 8. *Let $\mathcal{A}$ be a infinite primitive sequence in $\mathbb{G}$. Then*

$$\sum_{\substack{a\in\mathcal{A}\\ |a|\leqslant x}} |a|^{-\delta} = o\left(\frac{\log x}{\sqrt{\log\log x}}\right). \tag{19}$$

To prove Theorem 8 we shall need the following extension of Lemma 1 of [12]:

LEMMA 9. *Let $u < w \leqslant y$, where w is sufficiently large compared to u. Let $\mathcal{A}$ be a finite primitive sequence of squarefree elements of an arithmetical semigroup $\mathbb{G}$ such that*

i) *if $a \in \mathcal{A}$ then $u < |a| < w$, and*

ii) $\sum_{a\in\mathcal{A}} |a|^{-\delta} > c\dfrac{\log w}{\sqrt{\log\log w}}$.

Further, let $\mathcal{B}$ be a sequence of elements of $\mathbb{G}$ for which

iii) *$|b| \leqslant y$ for every $b \in \mathcal{B}$, and*

iv) *every element of $\mathcal{B}$ can be written in the form aq where $|q| \leqslant y/|a|$ and the norm of every prime factor of q is $> u$.*

Then

$$\sum_{b\in\mathcal{B}} |b|^{-\delta} > c_2 \log y,$$

where the constant c_2 depends on c, but does not depend on the sequences $\mathcal{A}$ and $\mathcal{B}$.

P r o o f. Axiom A implies that for suitable $0 < c_3 < c_4$ we have

$$c_3 x^{\delta} < N_{\mathbb{G}}(x) < c_4 x^{\delta}.$$

Here the constants depend only on the parameters of Axiom A. In what follows we shall restrict our attention to those w for which with given c we have

$$c_3 \cdot c > c_4 \frac{(\log\log w)^{1/2}}{(\log w)^{1-\log 2}}.$$

We prove Lemma in two steps. In the first one assume that $y = w$. Let

$$d_{\mathcal{A}}(n) = \sum_{\substack{a\in\mathcal{A}\\ a|n}} 1 \qquad \text{and} \qquad d_{\mathcal{B}}(n) = \sum_{\substack{b\in\mathcal{B}\\ b|n}} 1.$$

Axiom A implies that

$$\sum_{|n|\leqslant w} d_{\mathcal{B}}(n) = \sum_{\substack{b\in\mathcal{B}\\ |b|\leqslant y}} N_{\mathbb{G}}\left(\frac{w}{|b|}\right) < c_4 w^{\delta} \sum_{\substack{b\in\mathcal{B}\\ |b|\leqslant y}} |b|^{-\delta}.$$

Therefore it is sufficient (we assume $y = w$) to prove the inequality

$$\sum_{|n| \leqslant w} d_{\mathcal{B}}(n) > c_5 w^{\delta} \log w\,, \tag{20}$$

where c_5 depends on c from ii), but not on the sequences $\mathcal{A}$ and $\mathcal{B}$. Let $\mathcal{D}$ be the sequence of elements $n \in \mathbb{G}$ such that $|n| \leqslant w$ and

$$\omega(n) > \log\log w\,, \quad d_{\mathcal{A}}(n) > \frac{c_6 2^{\omega(n)}}{\sqrt{\omega(n)}}\,, \tag{21}$$

where c_6 is a sufficiently small constant to be specified later. Then

$$\sum_{n \in \mathcal{D}} d_{\mathcal{A}}(n) = \sum_{|n| \leqslant w} d_{\mathcal{A}}(n) - {\sum}' d_{\mathcal{A}}(n) - {\sum}'' d_{\mathcal{A}}(n)\,, \tag{22}$$

where the summation $\sum'$ runs over those n with $|n| \leqslant w$ for which

$$\omega(n) \leqslant \log\log w\,, \tag{23}$$

while $\sum''$ over those n with $|n| \leqslant w$ for which

$$\omega(n) > \log\log w \qquad \text{and} \qquad d_{\mathcal{A}}(n) \leqslant c_6 \frac{2^{\omega(n)}}{\sqrt{\omega(n)}}\,. \tag{24}$$

Relation ii) yields

$$\sum_{|n| \leqslant w} d_{\mathcal{A}}(n) = \sum_{a \in \mathcal{A}} N_{\mathbb{G}}\left(\frac{w}{|a|}\right) > c_3 w^{\delta} \sum_{a \in \mathcal{A}} \frac{1}{|a|^{\delta}} > c_7 w^{\delta} \frac{\log w}{\sqrt{\log\log w}}\,, \tag{25}$$

where $c_7 = c_3 c$ with c appearing in ii).

For the second sum in (22) note that all the a_i's are squarefree and that the number of squarefree divisors of an element n is $2^{\omega(n)}$. Then inequality (23) and Axiom A imply that

$${\sum}' d_{\mathcal{A}}(n) \leqslant c_4 w^{\delta} 2^{\log\log w}\,. \tag{26}$$

For those n over which the sum $\sum''$ runs we get from (24) that

$$d_{\mathcal{A}}(n) < c_6 2^{\omega(n)} / \sqrt{\log\log w}$$

and consequently

$${\sum}'' d_{\mathcal{A}}(n) < c_6 \sum_{|n| \leqslant w} \frac{2^{\omega(n)}}{\sqrt{\log\log w}} < c_6 \frac{\sum_{|d| \leqslant w} N_{\mathbb{G}}(w/|d|)}{\sqrt{\log\log w}} < c_4 c_5\, w^{\delta} \frac{\log w}{\sqrt{\log\log w}}\,.$$

Plugging this into (22) we get

$$\sum_{n\in\mathcal{D}} d_{\mathcal{A}}(n) > \left(c_3 c - c_4 \frac{\sqrt{\log\log w}}{(\log w)^{1-\log 2}} - c_4 c_5\right) \frac{w^{\delta}\log w}{\sqrt{\log\log w}}$$

which for sufficiently small c_5 yields

$$\sum_{n\in\mathcal{D}} d_{\mathcal{A}}(n) > c_8 w^{\delta} \frac{\log w}{\sqrt{\log\log w}}$$

for some $c_8 > 0$.

To finish the proof of (20) it is enough to prove that for every $n \in \mathcal{D}$ we have

$$d_{\mathcal{B}}(n) > c_9 d_{\mathcal{A}}(n)\sqrt{\omega(n)} > c_9 d_{\mathcal{A}}(n)\sqrt{\log\log w}\,.$$

The last inequality follows from (21) using the inequality $\omega(n) > \log\log w$ for every $n \in \mathcal{D}$. In the proof of the first inequality Lemma 7 will be instrumental.

Given a fixed $n \in \mathcal{D}$, let

$$\mathcal{S}_1 = \{p \in \mathcal{P}_{\mathbb{G}} \colon p|n,\ |p| \leqslant u\}, \quad \mathcal{S}_2 = \{q \in \mathcal{P}_{\mathbb{G}} \colon\ q|n,\ u < |q| \leqslant w\}\,.$$

Clearly $\mathcal{S}_1 \cap \mathcal{S}_2 = \emptyset$. Let $\operatorname{card}(\mathcal{S}_i) = r_i$, $i = 1, 2$. Then $r_1 < c_4 u^{\delta}$, and (21) shows that

$$\omega(n) = r_1 + r_2 > \log\log w\,.$$

Consequently, for $w > \exp\bigl(\exp(2c_4 u^{\delta})\bigr)$ we get

$$r_2 > \log\log w - c_4 u^{\delta} > 0.5\log\log w > c_4 u^{\delta} > r_1$$

as the assumptions of Lemma 7 require.

Let $a_1, \dots, a_t$, $t = d_{\mathcal{A}}(n)$, be the all divisors of n amongst the elements of $\mathcal{A}$. To each a_i assign the set

$$\mathcal{S}^{(i)} = \{p\colon\ p \in \mathcal{S}_1 \cup \mathcal{S}_2,\ p|a_i\}\,.$$

Since the elements $a \in \mathcal{A}$ are squarefree and form a primitive sequence, no of the sets $\mathcal{S}^{(i)}$, $1 \leqslant i \leqslant t$, is a subset of another one. The relation (21) shows that the number of sets $\mathcal{S}^{(i)}$ is at least $c_6 2^{r_1+r_2}/\sqrt{r_1 + r_2}$. To every set of the type $\mathcal{S}^{(i)} \cup \mathcal{R}$ with $\mathcal{R} \subset \mathcal{S}_2$ there corresponds the element

$$b = \prod_{p\in\mathcal{S}^{(i)}\cup\mathcal{R}} p\,,$$

for which we have

j) $b|n$ and $|b| \leqslant w$, for $\prod_{p\in\mathcal{S}_1} p \prod_{q\in\mathcal{S}_2} q$ divides n and $|n| \leqslant w$, jj) if $q^{(i)} = \prod_{q\in\mathcal{R}\setminus\mathcal{S}^{(i)}} q$, then j) shows that $|q^{(i)}| \leqslant w/|a_i|$ while if $q \in \mathcal{R}\setminus\mathcal{S}^{(i)}$ then $|q| > u$.

Therefore to each set of the type $\mathcal{S}^{(i)} \cup \mathcal{R}$ there corresponds an element $n \in \mathcal{D}$ belonging to the set $\mathcal{B}$, and to distinct sets there correspond distinct divisors. Lemma 7 shows that there is at least $c_1 2^{r_1+r_2}$ of such sets, or in other words

$$d_{\mathcal{B}}(n) > c_1 2^{\omega(n)}.$$

On the other hand, the elements $a_1, \dots, a_t$ form a primitive sequences of divisors of n and as it was proved in Theorem 5 we have

$$2^{\omega(n)} \gg s(n)\sqrt{\omega(n)},$$

where $s(n)$ denotes the maximal cardinality of a primitive set of divisors of n. Since $s(n) \geqslant d_{\mathcal{A}}(n)$, the Lemma follows in the case $y = w$.

The general case $w < y$ can be reduces to just proved one as follows:

Let $\mathbb{G}\langle w\rangle$ denote the set of all elements of $\mathbb{G}$ possessing only prime divisors p of norm $> w$ together with the identity element $1_{\mathbb{G}}$. Then $\mathbb{G}\langle w\rangle$ is an arithmetical semigroup (cf. [18, p. 77]) for which

$$N_{\mathbb{G}\langle w\rangle}(x) = A \prod_{|p|\leqslant w} (1-|p|^{-\delta})x^{\delta} + \mathcal{O}(x^{\eta}).$$

Lemma 1 implies

$$\sum_{\substack{t\in\mathbb{G}\langle w\rangle\\ |t|<y/w}} |t|^{-\delta} > c_{10} \prod_{|p|\leqslant w} (1-|p|^{-\delta}) \log(y/w).$$

The analog of Merten's formula (cf. [18, p. 17]) gives

$$\prod_{|p|\leqslant w} (1-|p|^{-\delta}) > \frac{c_{11}}{\log w}.$$

Consequently,

$$\sum_{\substack{t\in\mathbb{G}\langle w\rangle\\ |t|<y/w}} |t|^{-\delta} > c_{12}\frac{\log(y/w)}{\log w} > c_{13}\frac{\log y}{\log w}.$$

If b with $|b| \leqslant w$ satisfies iv) and for $t \in \mathbb{G}\langle w\rangle$ we have $|t| < y/w$, then also bt satisfies iv). Therefore

$$\sum_{|b|\leqslant w} |b|^{-\delta} \geqslant \sum_{|b|\leqslant w} |b|^{-\delta} \sum_{\substack{t\in\mathbb{G}\langle w\rangle\\ |t|<y/w}} |t|^{-\delta},$$

and the proof follows since from the first part of the proof we know that $\sum_{|b|\leqslant w} |b|^{-\delta} \gg \log w$. $\square$

Proof of Theorem 8. We shall proceed by contradiction. Suppose there is a primitive sequence $\mathcal{C}$ of elements for which (19) fails. We can assume that the elements of this sequence are squarefree. To see this put $\mathcal{C} = \bigcup_{k\in\mathbb{G}} \mathcal{C}^{(k)}$, where $\mathcal{C}^{(k)}$ is the sequence of all those elements of $\mathcal{C}$ which the greatest square divisor is k^2. If each of these subsequences $\mathcal{C}^{(k)}$, $k \in \mathbb{G}$, satisfies (19), then arguing as at the end of the proof of Theorem 5 we get that also $\mathcal{C}$ fulfils (19), what contradicts the assumption.

Thus we can suppose that $\mathcal{C}$ is a primitive sequence of squarefree elements for which (19) fails. Then there exists an infinite sequence of real numbers $1 < x_1 < x_2 < \dots$ and a constant c_1 such that

$$\sum_{\substack{a\in\mathcal{C}\\ x_{i-1}<|a|<x_i}} |a|^{-\delta} > c\frac{\log x_i}{\sqrt{\log\log x_i}}\,.$$

Let t be such that $tc_2 > M$ where c_2 is the constant from the assertion of Lemma 9 and M is a constant for which

$$\sum_{\substack{n\in\mathbb{G}\\ |n|\leqslant x}} |n|^{-\delta} < M\log x\,.$$

Let $y = x_t$. For $1 \leqslant i \leqslant t$ let $\mathcal{B}^{(i)}$ be the sequence of the elements of the form aq with

$$x_{i-1} < |a| < x_i\,, \quad a \in \mathcal{C}, \quad \text{and} \quad q \in \mathbb{G}\langle x_{i-1}\rangle\,, \quad |q| \leqslant x_i/|a|\,,$$

where $\mathbb{G}\langle x_{i-1}\rangle$ denotes the set of elements of $\mathbb{G}$ having only prime factors of norm $> x_{i-1}$.

The sets $\mathcal{B}^{(i)}$ are disjoint in pairs. In the opposite case $aq_i = a'q_j$ with $a, a' \in \mathcal{C}$, $q_i \in \mathbb{G}\langle x_{i-1}\rangle$, $q_j \in \mathbb{G}\langle x_{i-1}\rangle$ and $x_{i-1} < |a| < x_i$, $x_{j-1} < |a'| < x_j$. Then $a|a'q_j$, and we can suppose that $j > i$. Since every prime factor of q_j has norm $> x_{j-1} \geqslant x_i$, the elements a and q_j are coprime and consequently $a|a'$. But this contradicts the assumption that $\mathcal{C}$ is a primitive sequence because $|a'| > x_{j-1} \geqslant x_i > |a|$, i.e., $a' \neq a$. Hence the sets $\mathcal{B}^{(i)}$ are disjoint in pairs.

Every element of $\mathcal{B}^{(i)}$ with $i \leqslant t$ has the norm $< y$. Applying Lemma 9 to the set $\mathcal{B}^{(i)}$ we get

$$M\log y > \sum_{|n|<y} |n|^{-\delta} \geqslant \sum_{i=1}^{t} \sum_{b\in\mathcal{B}^{(i)}} |b|^{-\delta} \geqslant tc_2\log y > M\log y\,,$$

a contradiction, and the proof is finished. $\square$

4. Applications

The set $\mathbb{N}$ of positive integers forms a most natural prototype of an arithmetical semigroup satisfying Axiom A. In this case $N_{\mathbb{N}}(x) = x + O(1)$, and the proved results coincide with those mentioned in the introduction. There are some other classes of arithmetical semigroups satisfying Axiom A. The first such example, the semigroup $\mathbb{G}\langle w\rangle$, we encountered in the proof of Lemma 9. Some other classes of examples are listed in the following ones:

EXAMPLE 3. If $\mathbb{G} = G_K$, the semigroup of all non-zero integral ideals in a given algebraic number field K of degree $n = [K:\mathbb{Q}]$ over rationals $\mathbb{Q}$ with the usual norm function $|\mathfrak{a}| = \operatorname{card}(\mathcal{O}_K/\mathfrak{a})$, then according to Weber [22] we have

$$N_K(x) = A_K x + \mathcal{O}\big(x^{\frac{n-1}{n+1}}\big),$$

where A_K can be explicitly given. For instance, for Gaussian integers we have $A_K = \pi/4$, and $A_K = \big(1/\sqrt{2}\big)\log\big(1+\sqrt{2}\big)$ for $K = \mathbb{Q}\big(\sqrt{2}\big)$, etc.

It is interesting to note that in the case of ideals in a number field, the divisibility relation $\mathfrak{a}|\mathfrak{b}$ between two ideals $\mathfrak{a}$ and $\mathfrak{b}$ is equivalent to the set-theoretic inclusion $\mathfrak{a} \supset \mathfrak{b}$. Then Behrend's Theorem 5 gets the form:

COROLLARY 10. *Let $\mathcal{C}$ be a sequence of ideals of an algebraic number field K possessing the property that no ideals of $\mathcal{C}$ contains another one. Then*

$$\sum_{\substack{\mathfrak{a}\in\mathcal{C}\\ |\mathfrak{a}|\leqslant x}} |\mathfrak{a}|^{-1} = \mathcal{O}\left(\frac{\log x}{\sqrt{\log\log x}}\right),$$

where the involved constant depends only on K.

If K is a complex Euclidean quadratic field (i.e., $K = \mathbb{Q}(\sqrt{m})$ for $m = -1$, -2, -3, -7, -11) then the norm is the square of the usual absolute value and we get the following analog of the original Behrend's result (note that the elements of a primitive sequence cannot be associate):

COROLLARY 11. *Let $\mathcal{C}$ be a primitive sequence of integers in the complex Euclidean quadratic field $\mathbb{Q}(\sqrt{m})$. Then*

$$\frac{1}{\log x}\sum_{\substack{a\in\mathcal{C}\\ |a|\leqslant\sqrt{x}}} |a|^{-2} = \mathcal{O}\big((\log\log x)^{-1/2}\big), \tag{27}$$

where the involved constant depends only on m.

In a similar spirit we can also interpret the results of Theorems 6 and 8. However, there are some other examples of arithmetical semigroups satisfying Axiom A which deserve the attention:

Example 4. Let $\mathbb{G} = \mathbf{A}$ be the category of all finite Abelian groups with the usual direct product operation and the norm $|A| = \mathrm{card}(A)$ for $A \in \mathbf{A}$. The Fundamental Theorem on finite Abelian groups shows that $\mathbf{A}$ is really free and that the generators are the cyclic groups of prime-power order. The fact that this arithmetical semigroup satisfies Axiom A follows from an older result of Erdős and Szekeres that

$$N_{\mathbf{A}}(x) = \alpha x + \mathcal{O}(\sqrt{x}),$$

where

$$\alpha = \prod_{j=1}^{\infty} \zeta_{\mathbb{N}}(js)$$

with $\zeta_{\mathbb{N}} = \zeta$ denoting the classical Riemann zeta function.

In fact, the category $\mathbf{A}$ is one of the infinitely many subcategories of the category $\mathbf{F}$ of all finite modules over the domain $\mathcal{O}_K$ of all algebraic integers in a given algebraic number field K which satisfies Axiom A (for more details consult [18, p. 20, 116]).

The following interesting characterization was proved by Kertézs [17]: every subgroup of a general group G is its direct factor if and only if G is the direct product of cyclic groups of prime order, i.e., if it is of squarefree order (and clearly Abelian).

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