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ON THE LATTICE OF ADDITIVE HEREDITARY PROPERTIES OF OBJECT SYSTEMS

PETER MIHÓK

ABSTRACT. The notion of an *object-system* over a concrete category $\mathbf{C}$ is introduced as a common generalization of graphs, hypergraphs, digraphs and other mathematical structures. Let $\mathbf{C}$ be a concrete category. A *simple finite object-system* over $\mathbf{C}$ is an ordered pair $S = (V, E)$, where V is a finite set and $E = \{A_1, A_2, \dots, A_m\}$ is a finite set of the objects of $\mathbf{C}$, such that the ground-set $V(A_i) \subseteq V$ of each object $A_i \in E$ is a finite set. Analogously as for graphs, we define *the additive hereditary property* of simple object-systems as any class of systems closed under disjoint unions, subsystems and isomorphisms of systems, respectively. The structure of the lattice $\mathbb{L}^a(\mathbf{C})$ of all additive hereditary properties of object-systems is investigated.

1. Motivation

Let us denote by $\mathcal{I}$ the class of all simple graphs. A graph property p is any nonempty proper isomorphism closed subclass of $\mathcal{I}$. A graph property is said to be hereditary if it is closed under taking subgraphs and additive if it is closed under disjoint union of graphs (see [7]). The lattice $\mathbb{L}^a$ of additive hereditary graph properties have been introduced in [17] and investigated in connection with generalized colorings of graphs; cf. Borowiecki and Mihók [6] and Borowiecki, Broere, Frick, Mihók and Semanišin ([7]). More results on the structure of $\mathbb{L}^a$ may be found in [2, 3, 9, 10]. The lattices of some specific varieties of graphs and digraphs has been studied by Haviar, Nedela and Monoszová ([12, 13]). The lattice $\mathbb{L}^a$ contains as a sublattice the *Colouring poset* C of all colour classes defined by graph homomorphisms (see, e.g., [19]). Some results on the lattice $\mathbb{L}^a$ can be easily generalized and proved for digraphs, hypergraphs, posets and other structures (see [15]). The aim of this paper is to introduce a general notion of *object-system* and to describe the ideas and reasons, which allows to prove different types of general statements.

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The object-systems are introduced in Section 2. The structure of the lattice of additive and hereditary properties of object-systems is investigated in Section 3. Some open problems are presented in Section 4.

2. Object-systems over a concrete category

To define a common generalization of graphs, digraphs and hypergraphs we will use basic notions of category theory. In general we follow the terminology and notation of [4, 22] (see also [24]). Let us recall, that *a category* $\mathbf{C}$ is a collection of *objects* and *arrows* called *morphisms.* The category $\mathbf{C}$ is said to be *concrete* if every object A in $\mathbf{C}$ is a set $V(A)$ with structure. A (homo)morphism between two objects A and B is a structure preserving mapping $\phi\colon V(A) \to V(B)$ and they have to satisfy the known axioms of the category theory. We will use the notions of *homomorphism, monomorphism, epimorphism, isomorphism and automorphisms* in a standard way, e.g., *isomorphisms* are structure preserving bijections between the ground-sets of objects and they will play an important role in our investigations.

The natural examples of concrete categories are: **Set** of sets, **FinSet** of finite sets, **Graph** of graphs, **Grp** of groups, **Poset** of partially ordered sets, etc., with *homomorphisms* of corresponding structures. The objects of mentioned categories form proper classes and such categories are called *large.* The categories whose collections of objects and arrows are both sets are called *small.*

Definition 1. Let $\mathbf{C}$ be a given concrete category. A *finite simple object-system* over $\mathbf{C}$ is an ordered pair $S = (V, E)$, where V is a finite set, the ground-set of the elements of S (we will call it also as "set of vertices of S" and denote it by $V(S)$), and $E = \{A_1, A_2, \dots, A_m\}$ is a finite system of objects of $\mathbf{C}$, such that the ground-set $V(A_i)$ of each object $A_i \in E$ is a finite set and $V \supseteq \bigcup_{i=1}^{m} V(A_i)$. Cardinality of $V(S)$ determines *the order* of the object-system S.

The class of all finite simple object-systems over $\mathbf{C}$ will be denoted by $\mathcal{I}(\mathbf{C})$. The unique object system of order 0 will be denoted by $K_0 = (\emptyset, \emptyset)$. For example, simple graphs can be viewed as object-systems over the concrete category of two-element sets with bijections as morphisms, digraphs as object-systems of the category of ordered two-tuples with order preserving mappings, hypergraps as finite set-systems over the category **FinSet** with total mappings as morphisms. Let us remark, that the L-structures (called also *Relational structures*) generalizing graphs, digraphs and k-uniform hypergraphs (introduced by Fraïssé in [11], cf. [5]), are object-systems too and our results can be applied to L-structures, too.

Definition 2. Let $S_1 = (V_1, E_1)$ and $S_2 = (V_2, E_2)$ be finite simple object-systems of a given concrete category $\mathbf{C}$. The systems S_1 and S_2 are said to be *isomorphic* if there is a pair of bijection:

$$\phi\colon V_1 \longleftrightarrow V_2\,; \qquad \psi\colon E_1 \longleftrightarrow E_2\,,$$

such that if $\psi(A_{1i}) = A_{2j}$ then $\phi/V(A_{1i})\colon V(A_{1i}) \leftrightarrow V(A_{2j})$ is an isomorphism of the objects $A_{1i} \in E_1$ and $A_{2j} \in E_2$ in the category $\mathbf{C}$.

More general, a *homomorphism* of a object-system S_1 to an object-system S_2 is an object-preserving mapping $\phi\colon V(S_1) \to V(S_2)$, i.e., for every object A_{1i} of the system S_1 there is an object A_{2j} of the system S_2 such that $\phi/V(A_{1i})\colon V(A_{1i}) \leftrightarrow V(A_{2j})$ is an isomorphism of the objects $A_{1i} \in E_1$ and $A_{2j} \in E_2$ of the category $\mathbf{C}$. In this case we say that S_1 is *homomorphic* to S_2 and we write $S_1 \to S_2$.

Let us denote by $\mathbf{SC}$ the concrete category of all object-systems with above defined homomorphisms. The category $\mathbf{SC}$ posses two important properties of graphs, digraphs and hypergraphs, namely, the category $\mathbf{SC}$ has all coproducts i.e., *disjoint unions* and for every subset U of the ground set $V(S)$ of any object-system S there is an *induced subsystem* $S[U]$ in $\mathbf{SC}$. Let us define these notions:

Definition 3. The *disjoint union of the object-systems* S_1 and S_2 is the object-system $S_1 + S_2 = (V_1 \cup V_2,\, E_1 \cup E_2)$, where we assume that $V_1 \cap V_2 = \emptyset$. It is easy to verify, that $S_1 + S_2$ is a coproduct in the category $\mathbf{SC}$.

Definition 4. The subsystem of S induced by the set $U \subseteq V(S)$, denoted $S[U]$, has $V\big(S[U]\big) = U$ and objects $E\big(S[U]\big) := \big\{A_i \in E(S)\colon V(A_i) \subseteq U\big\}$. S_2 is said to be *an induced-subsystem* of S_1 if it is isomorphic to $S_1[U]$ for some $U \subseteq V(S_1)$, we denote this fact by $S_2 \leq S_1$.

The object-system S_1 is *a subsystem* of the S_2 if $V_1 \subseteq V_2$ and $E_1 \subseteq E_2$.

Obviously, every induced-subsystem of the object-system S is its subsystem, but conversely it is not true. Using these definitions we can define, analogously as for graphs, that *an additive hereditary property* of simple object-systems over $\mathbf{C}$ is any subclass of $\mathcal{I}(\mathbf{C})$ closed under disjoint unions, subsystems and isomorphisms of object-systems, respectively. The additivity is closed to the notion of connectivity.

Definition 5. A system S of $\mathbf{SC}$ is said to be *connected* if it cannot be expressed as a disjoint union of two systems $S = S_1 + S_2$.

The connected object-systems can be characterized equivalently in the following way: S is connected if for any two object systems S_1, S_2 there holds:

$S \leq S_1 + S_2$ if and only if either $S \leq S_1$ or $S \leq S_2$ (cf. [21]). Every object-system can be expressed uniquely (apart from the order) as the disjoint union of its "connected components". The additivity of the property p implies that if every connected component of the object-system S has property p, then S itself posses p, too. This fact will be used later in the proof of our main Theorem. Let us remark, that connectivity of the object-system can be defined, as in Graph Theory, using paths, but we omit this approach and details here.

Colour classes of graphs, defined by homomorphism, form important examples of additive and hereditary graph properties (see [19, 20, 16]). We can generalize them to so-called hom-properties (colour classes) of object-systems over $\mathbf{C}$ in the following way. Let us denote by $\to S = \{S_i\colon S_i \to S\}$ the set of all object-systems over $\mathbf{C}$ which are homomorphic to S. It is easy to see that $\to S$ is an additive hereditary property for each $S \in \mathcal{I}(\mathbf{C})$, however it can be shown that not every additive and hereditary property is a colour class (for example acyclic graphs does not form a colour class).

3. Lattices of additive hereditary properties of object-systems

In this section we will prove, that the set of all additive hereditary properties of simple object-systems over $\mathbf{C}$ partially ordered by set inclusion, denoted by $\mathbb{L}^a(\mathbf{C})$ forms a complete and distributive lattice. To avoid formal difficulties, we will assume in what follows, that the elements $V(A)$ of each objects A of $\mathbf{C}$ are labelled by labels taken from a given infinite set $\mathcal{V}$ of cardinality α. Hence, any considered concrete category $\mathbf{C}$ will be small and the objects of $\mathbf{C}$ are "labelled" objects. We also assume, that for each object A of $\mathbf{C}$, after relabelling of the elements of $V(A)$ by labels from a set $V^* \subset \mathcal{V}$ the obtained "new" object A^* with $V(A^*) = V^*$ belongs to $\mathbf{C}$ too and it is always isomorphic to A. This requirements are natural, they are satisfied, e.g., if the concrete category $\mathbf{C}$ is any small full subcategory of the above mentioned categories **Set**, **FinSet**, **Graph**, **Poset**, etc.

We are going to prove our basic Theorem.

Theorem 1. *Let* $\mathbb{L}^a(\mathbf{C})$ *be the set of all additive hereditary properties of finite simple object-systems over* $\mathbf{C}$, *partially ordered by set-inclusion. Then* $\mathbb{L}^a(\mathbf{C})$ *is a complete and distributive lattice.*

Proof. The proof is analogous to the proof of the same result for graphs. Let us recall it briefly: since the property of subsets of the set $\mathcal{I}(\mathbf{C})$ of all finite simple object-systems over $\mathbf{C}$ "to be hereditary" is a "closure property", the set of all additive hereditary properties of object-systems over $\mathbf{C}$ is a complete lattice (see [4]). Let us denote by $\mathcal{I}(\mathbf{C})^{\mathrm{conn}}$ the set of all connected systems and for a property $p \in \mathbb{L}^a(\mathbf{C})$ by p^{conn} the set of all connected systems of property p. Let us consider the set M of *cuts*- downward closed subsets- in the poset $\mathcal{I}(\mathbf{C})^{\mathrm{conn}}$ partially ordered by the relation "to be a subsystem". It is well-known that "the set of all cuts M partially ordered by set-inclusion" forms a distributive lattice. It is also very easy to verify, that the mapping $f\colon \mathbb{L}^a(\mathbf{C}) \to M$ defined by $f(p) = p^{\mathrm{conn}}$ is an isomorphism of $\mathbb{L}^a(\mathbf{C})$ onto M, so that the lattice $\mathbb{L}^a(\mathbf{C})$ is distributive. □

Analogously as in [15], it can be proved that the lattice $\mathbb{L}^a(\mathbf{C})$ is completely distributive. By the method used in [15], it can be also shown that the Jordan-Dedekind condition for infinite chains fails to be valid in $L^a(\mathbf{C})$. Namely, that there exist elements $p_1, p_2 \in L^a(\mathbf{C})$ with $p_1 < p_2$ and maximal chains C_1, C_2 of the interval $[p_1, p_2]$ such that C_1 is denumerable and C_2 has the power of the continuum.

A *core of an object-system* S is any subsystem S^* of S such that $S \to S^*$ while S fails to be homomorphic to any proper subsystem of S^*. Up to isomorphism every object-system S has a unique core, denoted by $C(S)$. A system S is called a *core* if S is a core for itself, i.e., S is isomorphic to $C(S)$, which is equivalent to the fact, that every homomorphism $S \to S$ is an automorphism of S. We can show that the hom-properties form a sublattice of $\mathbb{L}^a(\mathbf{C})$ isomorphic to the lattice of the cores partially ordered by the existence of homomorphisms (it is also called as "the colouring poset"). The proof, the same as for graphs, is omitted here (see [16, 19]).

4. Remarks and open problems

Several results on additive and hereditary properties of graphs were generalized and extended during the last five years. A comprehensive overview is given with many deep new results in the thesis of Alastair Farrugia [9] and the forthcoming paper [10]. Substantial part of the results on the structure of additive hereditary properties of graphs can be generalized for digraphs and hypergraphs, moreover one can see that they are valid for object-systems over concrete categories as well. The reason is, that the proofs are not using concrete graphs and they have "slightly categorial" character. Let us mention here only two topics with references to papers, whose results can be generalized for object-systems: Unique Factorization Theorem ([8, 9, 14, 16, 18, 23]) and the structure of the lattice of additive hereditary properties ([1, 2, 3]).

Let us remark that this paper is not devoted to the category theory. We are using the notions of category theory only as an appropriate language. In the language of category theory we would like to formulate now some general questions.

Let us consider the concrete small category **SC** of all finite simple object-systems over **C** with homomorphisms defined as in Section 2. A property in **SC** is a class of object-systems of **SC** closed under isomorphisms. We say, that a class of object-systems in **SC** is *homomorphism-closed property* (hc-property), if it is closed under all homomorphic preimages, i.e., $S_2 \in p$ and $S_1 \to S_2$ implies $S_1 \in p$. Obviously, the set of hc-properties is a complete lattice (cf. [4]). The classes closed under all monomorphism in **SC** are hereditary properties (they need not to be additive!). What can we say about classes closed under all epimorphisms in **SC**? A property p of **SC** has *the amalgamation property*, if for any object-systems A, B, C, for any isomorphism f from A onto an induced subsystem of B and any isomorphism g from A onto an induced subsystem of C, there is an object-system D in p, an isomorphism f' from B onto an induced subsystem of D and an isomorphism g' from C onto an induced subsystem of D, so that for all $v \in V(A)$ we have $(f'f)(v) = (g'g)(v)$ (cf. [5, 12, 13]). What can we say about the structure of the properties in **SC** which have amalgamation property? Many similar questions about properties closed under other closure operators in **SC** can be posed. Let us finish with a basic problem:

PROBLEM 1. A class of object-systems of **SC** is said to be a variety in **SC**, if it is closed under all homomorphic images, subsystems and (categorial) products. Characterize the lattice of all varieties in **SC**.

For example the class of all bipartite graphs is a variety in *Graph*. As far as we know, the structure of the lattice of varieties of graphs is not characterized. Let us remark, that hc-properties of object-systems are closed under subsystems and products, but they need not to be closed under homomorphic images.

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Mathematical Institute
Slovak Academy of Sciences
Grešákova 6
SK–040 01 Košice
SLOVAKIA
E-mail: mihok@saske.sk

Department of Applied Mathematics
Faculty of Economics
Technical University
B. Němcovej
SK–040 01 Košice
SLOVAKIA
E-mail: Peter.Mihok@tuke.sk