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POLAR FUNCTIONS, III: ON IRREDUCIBLE MAPS VS. ESSENTIAL EXTENSIONS OF ARCHIMEDEAN ℓ-GROUPS WITH UNIT

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Dedicated to Professor Ján Jakubík on the occasion of his 80th birthday

ABSTRACT. This paper further develops the notion of a polar function, extending the scope to sublattices of polars. The category of discourse is $\mathbf{W}$, the category of archimedean ℓ-groups with designated unit and all ℓ-homomorphisms. The dual notion of a covering function of compact spaces is similarly extended.

The paper builds on the ideas of summand-inducing hulls, defining, for each invariant polar function $\mathcal{L}$ on objects in $\mathbf{W}$, an associated "$\mathcal{L}$-essential extension". With certain natural assumptions on the function $\mathcal{L}$, there is, for each $\mathbf{W}$-object G, a largest $\mathcal{L}$-essential extension $G^{\mathcal{L}}$. Given $\mathcal{L}$ and a $\mathbf{W}$-object G, $\widehat{\mathcal{L}}(G)$ is the boolean algebra of all polars in $\mathcal{L}(G)$ whose complements also lie in $\mathcal{L}(G)$. This boolean center $\widehat{\mathcal{L}}$ of $\mathcal{L}$ gives rise to the least summand-inducing hull $G[\widehat{\mathcal{L}}]$, which, under the same assumptions, is contained in $G^{\mathcal{L}}$.

In tandem with these ideas goes the development of $\mathfrak{I}$-irreducible maps of compact spaces, associated with an invariant covering function $\mathfrak{I}$.

This paper builds on the ideas of [M02a] and [M02b], and the terminology and notation of those articles is carried over, except where noted. The concept of a polar function is generalized to sublattices of polars, with the objective of applying the ideas developed here to abstractly characterize essential extensions with special properties. Dually, the original notion of a covering function on compact spaces is likewise extended to enable one to discuss particular notions of irreducibility in maps.

As in the aforementioned papers, the ambient category of lattice-ordered groups is $\mathbf{W}$, the category of all archimedean lattice-ordered groups with a designated weak order unit $u > 0$ and all ℓ-homomorphisms that preserve the designated unit. All topological spaces are assumed to be compact and Hausdorff, unless the contrary is expressly stated. We abbreviate "lattice-ordered group"

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to *ℓ-group*, as is customary. For any unexplained terminology from the theory of ℓ-groups the reader is referred to [D95]. Most of the topological references may be found in [GJ76]. Background on coverings of spaces will be provided as we make our way through the paper.

The essential background for this paper is reviewed next.

1. Splitting extensions vs. covers of spaces

If G is a $\mathbf{W}$-object, then G^e will denote its essential closure; its existence and uniqueness was first demonstrated by Conrad in [C71]. Recall that $G \leq H$ is an essential extension of G if for each $h > 0$ in H there is a $g > 0$ in G such that $g \leq nh$, for some positive integer n. G is *essentially closed* if it has no proper essential extensions (in $\mathbf{W}$).

Definition & Remarks 1.1. (a) Let G be an ℓ-group. $\mathcal{C}(G)$ denotes the lattice of all convex ℓ-subgroups of G. If $S \subseteq G$ then

$$S^{\perp} \equiv \{g \in G\colon |g| \wedge |s| = 0\,,\ \forall\, s \in S\}\,.$$

$S^{\perp}$ is the *polar of* S, and if S is a convex ℓ-subgroup then $S^{\perp}$ is the largest convex ℓ-subgroup of G intersecting S trivially. P is a *polar* of G if it is of the form $P = S^{\perp}$, for a suitable $S \subseteq G$, or, equivalently, if $P^{\perp\perp} = P$. $\mathcal{P}(G)$ stands for the set of all polars of G, which is a boolean algebra under inclusion. $\mathcal{S}(G)$ denotes the subalgebra of all summands of G.

Now, the extension $G \leq H$ in $\mathbf{W}$ is an essential extension if and only if the trace map $\mathrm{tr}_{H,G}(P) \equiv P \cap G$ is an isomorphism of boolean algebras between $\mathcal{P}(H)$ and $\mathcal{P}(G)$ ([BKW77, Theorem 11.1.15]). We take up the inverse of this map in 1.3.

(b) If G is essentially closed, then $G = K + K^{\perp}$, for each polar K of G; see [Be75] and, in a broader context, [HM96]. When a $\mathbf{W}$-object G satisfies the property that $G = K + K^{\perp}$, for each $K \in \mathcal{P}(G)$, G is called *strongly projectable.* If $G = g^{\perp\perp} + g^{\perp}$, for each $g \in G$, then G is said to be *projectable.*

No introductory discussion in this kind of context can be complete without a brief review of the Yosida Representation Theorem; see [HR77] for additional details. Note for the record that βX denotes the Stone-Čech compactification of the Tychonoff space X. $C(X)$ stands for the ring of all continuous real-valued functions defined on X, and $S(X)$ for the subalgebra of $C(X)$ consisting of all continuous functions with finite range.

Definition & Remarks 1.2. *YG* denotes the *Yosida space* of the $\mathbf{W}$-object G; that is to say, the space of values of the designated unit, with the hull-kernel topology. It is well known that YG is a compact Hausdorff space.

Now for any space X, $D(X)$ shall denote the set of all continuous functions $f\colon X \to \mathbb{R} \cup \{\pm\infty\}$, where the range is the extended reals with the usual topology, such that $f^{-1}\mathbb{R}$ is a dense subset of X. $D(X)$ is a lattice under pointwise operations, but not a group or ring under pointwise operations, unless some assumptions are made about X. We need not explore these issues here.

What is needed is an understanding of the term "ℓ-group in $D(X)$": a subset $H \subseteq D(X)$ which is an ℓ-group such that under the lattice operations it is a sublattice of $D(X)$, and, for each $h, k \in H$, $(h+k)(x) = h(x) + k(x)$ on a dense subset of X. Now we are able to state the Yosida Representation Theorem:

> *For each* $\mathbf{W}$*-object* G *with designated unit* $u > 0$, *there is an* ℓ*-isomorphism* φ *of* G *onto an* ℓ*-group* G' *in* $D(YG)$ *such that* $\varphi(u) = 1$.

The Yosida space YG has a canonical role in the above theorem when one takes into account *separation of points*; that is, φ has the feature that if $x \neq y$ in X, then there is a $g \in G$ such that $\varphi(g)(x) \neq \varphi(g)(y)$. The Yosida space is unique (up to homeomorphism) in the following sense:

> *Suppose that* Z *is a compact Hausdorff space and that there is an* ℓ*-isomorphism* θ *of* G *onto an* ℓ*-group* H *in* $D(Z)$ *which separates the points of* Z *and such that* $\theta(u) = 1$, *then there is a homeomorphism* $t\colon Z \to YG$ *such that, for each* $g \in G$ *and* $z \in Z$,
> $$\varphi(g)(tz) = \theta(g)(z).$$

Finally, if the unit u of G is strong, the image of the Yosida Representation lies in $C(YG)$.

Next, here is one of the central concepts of [M02a], for the algebraic side of the discussion. We shall defer a consideration of examples until the review of the topological aspects is completed.

Definition & Remarks 1.3. (a) First a notational convention. If $G \leq H \leq G^e$ is an intermediate $\mathbf{W}$-object then the polar of $K \in \mathcal{C}(G)$ in H is denoted by $K^{\perp_H}$. The map $\mathrm{aug}_{G,H}(K) \equiv K^{\perp_H \perp_H}$ is the isomorphism from $\mathcal{P}(G)$ onto $\mathcal{P}(H)$ which is inverse to $\mathrm{tr}_{H,G}$. If $g \in G$ and $K \in \mathcal{P}(G)$ then $g[K]$ denotes the projection of g on the summand $\mathrm{aug}_{G,G^e}(K)$ in G^e. (Note that, since G^e is strongly projectable, $K^{\perp_{G^e} \perp_{G^e}}$ is a cardinal summand, for each $K \in \mathcal{P}(G)$.)

$\mathcal{X}$ denotes a fixed subalgebra of $\mathcal{P}(G)$, and we assume that it contains $\mathcal{S}(G)$. In [M02a] the essential extensions of G are considered, for which $\mathrm{aug}_{G,H}(K)$ is a cardinal summand, for each $K \in \mathcal{X}$. Such an extension is called an $\mathcal{X}$*-splitting* extension of G. We see that G^e is a $\mathcal{X}$-splitting extension, for each subalgebra $\mathcal{X}$. In [M02a] it is shown that the set of essential $\mathcal{X}$-splitting extensions of G is closed under intersections, and so there is a least such, denoted $G[\mathcal{X}]$.

(b) Theorem 2.4 of [M02a] records the existence of the least $\mathcal{X}$-splitting extension and describes it as follows:

> *For each* **W***-object* G *and each subalgebra* $\mathcal{X}$ *of* $\mathcal{P}(G)$*, the least* $\mathcal{X}$*-splitting extension of* G*,* $G[\mathcal{X}]$*, consists of the elements of* G^e *of the form*
> $$\sum_{i=1}^{n} g_i[K_i]\,,$$
> *where each* $g_i \in G$ *and each* $K_i \in \mathcal{X}$ *and* $K_i \cap K_j = \{0\}$*, whenever* $i \neq j$*.*

If A is an archimedean f-ring with identity, then $A[\mathcal{X}]$ is an ℓ-subring of A^e (Corollary 1.6).

Next, we review the companion ideas concerning covers of compact spaces. The reader is referred to [H89], where the context is more general. What we are about to describe here is presented in [HVW89] by way of an ultrafilter construction. We stress that the present discussion is for compact Hausdorff spaces.

Definition & Remarks 1.4. (a) Fix a space X. Recall that a continuous surjection $g\colon Y \to X$ is said to be *irreducible* if X is not the image under g of any proper closed subset of Y. The irreducible surjections $g\colon Y \to X$ and $h\colon Z \to X$ are *equivalent* if there is a homeomorphism $\overline{g}\colon Y \to Z$ such that $h \cdot \overline{g} = g$. Clearly, this equivalence of irreducible maps onto X is an equivalence relation. On the set of equivalence classes, $\mathrm{Cov}(X)$, we define a partial ordering. Actually, we define a quasi-ordering on the class of all irreducible coverings, which, modulo the equivalence, is the partial ordering we want. Thenceforth, we blur distinctions between a map and the equivalence class which contains it.

Again, suppose that $g\colon Y \to X$ and $h\colon Z \to X$ are irreducible surjections. We put $h \leq g$ if there is a continuous surjection $u\colon Y \to Z$, which is necessarily irreducible, such that $h \cdot u = g$.

When we speak of a *cover* of X we mean, depending on the context, an irreducible surjection g onto X or else the equivalence class containing the domain of g.

(b) Let $\mathfrak{R}(X)$ denote the set of all regular closed sets of X; recall that a closed set A of X is *regular* if $\mathrm{cl}_X \,\mathrm{intr}_X A = A$. $\mathfrak{R}(X)$ is a complete boolean algebra under inclusion. Let $\mathfrak{B}(X)$ denote the subalgebra of all clopen subsets.

It is well known that an irreducible surjection $g\colon Y \to X$ induces a isomorphism between $\mathfrak{R}(Y)$ and $\mathfrak{R}(X)$ of regular closed sets. From $\mathfrak{R}(Y) \to \mathfrak{R}(X)$ the assignment is simply $A \mapsto g(A)$, and the inverse isomorphism is $B \mapsto \mathrm{cl}_Y\, g^{-1}(\mathrm{intr}_X B)$. Let us designate the latter $g^{\leftarrow}\colon \mathfrak{R}(X) \to \mathfrak{R}(Y)$.

Suppose now that $\mathfrak{K}$ is a subalgebra of $\mathfrak{R}(X)$, containing $\mathfrak{B}(X)$. As introduced in [M02a], if $g\colon Y \to X$ is in $\mathrm{Cov}(X)$ and each $A \in g^{\leftarrow}(\mathfrak{K})$ is clopen, we say that Y is a *$\mathfrak{K}$-cover* of X.

It is shown in [M02a, Theorem 3.5] that each compact X has a minimum $\mathfrak{K}$-cover (in $\mathrm{Cov}(X)$). By way of contrast in the approach, in [HVW89], the authors construct the minimum $\mathfrak{K}$-cover, and in Theorem 2.11 of that article show that it is least. Their cover is referred to as a *Wallman cover*, and it is constructed by way of ultrafilters of member of $\mathfrak{K}$, with a "twist" in the construction when the sets in $\mathfrak{K}$ do not form a base for the closed sets of X. We refer the reader to [HVW89] for details.

Next, we recall the fundamental connection between polars of a $\mathbf{W}$-object G and the regular closed sets of its Yosida space.

Definition & Remarks 1.5. (a) Suppose that $P \in \mathcal{P}(G)$. We associate a regular closed subset $\rho(P)$ of $Y = YG$ by

$$\rho(P) \equiv \mathrm{cl}_Y\left(\bigcup_{g\in P} \mathrm{coz}(g)\right).$$

ρ defines an isomorphism of boolean algebras from $\mathcal{P}(G)$ onto $\mathfrak{R}(Y)$, the inverse of which is given by

$$\rho^{-1}(A) = \{g \in G\colon\ \mathrm{coz}(g) \subseteq A\},$$

for each $A \in \mathfrak{R}(Y)$. Observe that if $P \in \mathcal{S}(G)$ then $\rho(P)$ is clopen in YG, although the converse may fail, as explained in [M02a 2.6(c)].

(b) Now, if H is an extension of G in G^e, then there is an induced irreducible surjection $t\colon YH \to Y$. The fundamental identity tying together t and ρ is as follows; the subscripts should be self-explanatory: for each $P \in \mathcal{P}(G)$, we have that

$$\rho_H(P^{\perp_H\perp_H}) = t^{\leftarrow}(\rho_G(P)), \tag{§}$$

or more compactly,

$$\rho_H \cdot \mathrm{aug}_{G,H} = t^{\leftarrow} \cdot \rho_G. \tag{§'}$$

Using (§) one obtains [M02a, Proposition 4.2], establishing that if $H \le G^e$ is a $\mathcal{X}$-splitting extension of G, then YH is a $\rho_G(\mathcal{X})$-cover of Y. The converse is true provided the designated unit is strong.

(c) The main result of § 4 of [M02a] is this

Suppose that $\mathcal{X}$ is a subalgebra of $\mathcal{P}(G)$. Then $YG[\mathcal{X}]$ is the least $\rho(\mathcal{X})$-cover of $Y = YG$.

The following, [M02a, Proposition 4.10], is worth noting.

Proposition 1.6. *Suppose that $\mathfrak{K}$ is a subalgebra of $\mathfrak{R}(X)$. If $\mathfrak{K}$ is a base for the closed subsets of X, then $X[\mathfrak{K}]$ is zero-dimensional. This implication holds, in particular, if X itself is zero-dimensional.*

We recall the definition of a hull class and its associated hull operator.

Definition 1.7. (a) We consider a class $\mathbf{H}$ of $\mathbf{W}$-objects closed under formation of ℓ-isomorphic copies. We call the extension $G \leq hG$ in $\mathbf{W}$ an $\mathbf{H}$-*hull* of G if it is an essential extension, $hG \in \mathbf{H}$, and for each essential extension H of G, with $H \in \mathbf{H}$, there is an ℓ-embedding $\theta\colon hG \to H$ extending the identity on G.

From [HM99, Proposition 2.4] we have that each $\mathbf{W}$-object G has a $\mathbf{H}$-hull if and only if $\mathbf{H}$ is *essentially intersective*; that is, for each essentially closed $\mathbf{W}$-object E, and each collection $\mathcal{B}$ of subobjects of E, such that $\mathcal{B} \subseteq \mathbf{H}$ and $\bigcap \mathcal{B}$ is essential in E, then $\bigcap \mathcal{B} \in \mathbf{H}$.

A class $\mathbf{H}$ with these features is called a *hull class*; h as described above is a hull operator associated with $\mathbf{H}$.

(b) Dually, recall that a class $\mathbf{C}$ of compact spaces is called a *covering class* if for each compact space X, there is a smallest member of $\mathrm{Cov}(X)$ which belongs to $\mathbf{C}$.

We conclude this section with an account of the prototypical examples illustrating the above constructs. Our references to the topological counterparts of the subalgebras described below are brief and only intended to give the reader some orientation. There are subtleties to these dualities, and we point them out where appropriate, or else postpone them for a later discussion.

Examples 1.8. We fix a $\mathbf{W}$-object G; $Y = YG$.

(a) Let $\mathcal{X} = \mathcal{P}(G)$; suppose that $H \leq G^e$ is an extension of G. Since $\mathrm{aug}_{G,H}$ is an isomorphism from $\mathcal{P}(G)$ onto $\mathcal{P}(H)$, it follows that H is a $\mathcal{P}(G)$-splitting extension if and only if $H = P \oplus P^{\perp_H}$, for each $P \in \mathcal{P}(H)$; that is to say, H is strongly projectable, by definition. Thus, $G\big[\mathcal{P}(G)\big]$ is simply the strongly projectable hull of G, as discussed in [Bl74], [Ch71] and [C73].

Under the isomorphism ρ of 1.5, we have $\mathfrak{R}(Y) = \rho\big(\mathcal{P}(G)\big)$. Dual to the strongly projectable hull stands the absolute of Y (as developed by Gleason); that is, the least extremally disconnected cover of Y. Recall that a space is *extremally disconnected* if the closure of any open set is open. We refer the reader to [H89] for details.

(b) Let $\mathcal{P}_\omega(G)$ be the subalgebra of $\mathcal{P}(G)$ generated by the lattice of principal polars, $\mathcal{P}_0(G)$. To put the matter formally, $H \leq G^e$ is an $\mathcal{P}_\omega(G)$-splitting extension of G if and only if $H = g^{\perp_H\perp_H} + g^{\perp_H}$, for each $g \in G$. $G\big[\mathcal{P}_\omega(G)\big]$ is the projectable hull. Once more, we refer to [Bl74], [Ch71], [C73] and also to [M02a].

Now $\mathfrak{R}_{\omega}(Y) \equiv \rho\big(\mathcal{P}_{\omega}(G)\big)$ is the subalgebra generated by all the closures of cozerosets of Y. And the dual to the projectable hull we have the least basically disconnected cover of Y, although the reader should heed the cautions regarding the construction, by either going to the source, [V84b], or else referring to the discussion in [M02a, 5.11]. In any event, recall that a space X is *basically disconnected* if the closure of every cozeroset is clopen.

In the next two cases let α denote a regular, uncountable cardinal, or else the symbol ∞. The infinite cardinals serve to impose bounds on the size of a generating set; the case $\alpha = \infty$ is the one in which no such restriction is made.

(c) Let $\mathcal{P}_{\alpha}^{\alpha}(G)$ stand for the α-generated polars having an α-generated complement. (Note: a polar is *α-generated* if it can be generated by fewer than α elements.) It is easy to see that $\mathcal{P}_{\alpha}^{\alpha}(G)$ is a subalgebra of $\mathcal{P}(G)$. We shall abbreviate "$\mathcal{P}_{\alpha}^{\alpha}(G)$-splitting" to *$\alpha$-splitting*. We also use $G[\alpha]$ in place of $G\big[\mathcal{P}_{\alpha}^{\alpha}(G)\big]$. $G[\alpha]$ already appears in [HM02], in the context of the ring of α-quotients of an f-ring.

Let $\mathfrak{R}_{\alpha}^{\alpha}(Y) \equiv \rho\big(\mathcal{P}_{\alpha}^{\alpha}(G)\big)$; this is the subalgebra of the closures of α-cozerosets whose complements are, likewise, closures of α-cozerosets. Recall that an *α-cozeroset* if it is the union of fewer than α cozerosets. Dual to the least α-splitting objects are the so-called α-cloz spaces; X is *α-cloz* if every each member of $\mathfrak{R}_{\alpha}^{\alpha}(Y)$ is clopen. We refer the reader to [M02a, Example 5.5(b)] and [M02b, § 4] for additional discussion.

Observe that $\mathcal{P}_{\infty}^{\infty} = \mathcal{P}$.

(d) Let $\mathcal{P}_{\alpha}(G)$ denote the sublattice of $\mathcal{P}(G)$ consisting of all α-generated polars of G. Dually, for a space X, let $\mathfrak{R}_{\alpha}(X)$ stand for the sublattice of $\mathfrak{R}(X)$ of all closures of α-cozerosets. We denote the subalgebra generated by $\mathcal{P}_{\alpha}(G)$ in $\mathcal{P}(G)$ by $\langle\mathcal{P}_{\alpha}\rangle(G)$ and the subalgebra of $\mathfrak{R}(X)$ generated by $\mathfrak{R}_{\alpha}(X)$ is denoted $\langle\mathfrak{R}_{\alpha}\rangle(X)$. Note that $\mathfrak{R}_{\alpha}(Y) = \rho\big(\mathcal{P}_{\alpha}(G)\big)$. The reader will also observe that, as in (c), $\mathcal{P}_{\infty} = \mathcal{P}$.

Let us point out that, here, the notation differs from that of [M02a].

We feel it is better to postpone a discussion of the hulls and covers associated with $\mathcal{P}_{\alpha}(G)$ until the next section.

Remark 1.9. The examples outlined in 1.8 have been amply discussed in [M02a] and [M02b]. In fact, in formulating open questions for study at the conclusion of [M02b], we seem to have put our ignorance on display, admitting that we were unaware of any additional examples of ipfs. Elsewhere we shall discuss a number of new examples. They are different in character, and, consequently, do not really fit in the present discussion.

This concludes the most basic part of our review of [M02a]. In the next section we take up the polar and covering functions, and through these reach the heart of the matter.

2. Polar and covering functions

Definition & Remarks 2.1. (a) A *polar function* is an assignment $G \mapsto \mathcal{X}(G)$, associating to each $\mathbf{W}$-object G a subalgebra $\mathcal{X}(G)$ of $\mathcal{P}(G)$ with the property that $\mathcal{S}(G) \subseteq \mathcal{X}(G)$ for each G. If, in addition the polar function $\mathcal{X}$ has the feature that, whenever H is an essential extension of G, and $K \in \mathcal{X}(G)$, then $\mathrm{aug}_{G,H}(K) \in \mathcal{X}(H)$, we say that $\mathcal{X}$ is an *invariant polar function*, abbreviated *ipf.* A word about notation and terminology: if $\mathcal{X}$ is a polar function and G is a $\mathbf{W}$-object, we write $G[\mathcal{X}]$ (in place of $G[\mathcal{X}(G)]$) for the least $\mathcal{X}(G)$-splitting extension. We shall also abbreviate to $\mathcal{X}$*-splitting* extensions when discussing $\mathcal{X}(G)$-splitting extensions.

(b) The dual notion for compact spaces is a function $X \mapsto \mathfrak{K}(X)$ which assigns to each space a subalgebra $\mathfrak{K}(X)$ of $\mathfrak{R}(X)$ containing $\mathfrak{B}(X)$; this is called a *covering function.* If $\mathfrak{K}$ also satisfies the condition: for each irreducible surjection $g\colon Y \to X$ and each $A \in \mathfrak{K}(X)$, $g^{\leftarrow}(A) \in \mathfrak{K}(Y)$, we say that $\mathfrak{K}$ is an *invariant covering function*, or, briefly, an *icf.* (This invariance is closely related to one of Vermeer's criteria defining covering classes. See [V84a] and also [H89, Theorem 8.2].)

With regard to covering functions we will observe the same economy of notation and terminology here that was outlined in (a) for polar functions.

(c) Call $\mathcal{X}$ *idempotent* if $\big(G[\mathcal{X}]\big)[\mathcal{X}] = G[\mathcal{X}]$, for each $\mathbf{W}$-object G. Given $\mathcal{X}$ there is an idempotent polar function $\mathcal{X}^\flat$ such that $\mathcal{X} \le \mathcal{X}^\flat$ and $\mathcal{X}^\flat$ is the least idempotent polar function with this property. (The partial ordering on polar functions is the obvious one: $\mathcal{X} \le \mathcal{Y}$ if and only if $\mathcal{X}(G)$ is a subalgebra of $\mathcal{Y}(G)$, for each $\mathbf{W}$-object G.) If $\mathcal{X}$ is invariant, then so is $\mathcal{X}^\flat$ ([M02a, Proposition 5.2]). $\mathcal{X}^\flat$ is the *idempotent closure of* $\mathcal{X}$. $\mathcal{P}$, $\mathcal{P}_\omega$, $\mathcal{P}_\alpha^\alpha$ and $\langle \mathcal{P}_\alpha \rangle$ in 1.8 are all ipfs; the first three are idempotent, whereas the last one is not (see [M02a, 5.5] and [HM99]).

Note as well that the polar function $\mathcal{X}$ is idempotent if and only if the map $\mathrm{aug}_{G,G[\mathcal{X}]}$ carries $\mathcal{X}(G)$ onto $\mathcal{X}\big(G[\mathcal{X}]\big)$. Keep in mind that the latter is simply $\mathcal{S}\big(G[\mathcal{X}]\big)$.

One may iterate $\mathcal{X}^\flat$ using transfinite induction. The inductive process will not concern us here; the reader is referred to [M02a, § 5].

The reader will quickly appreciate that the relationship between polar and covering functions is not in any conventional sense a duality. The reason should be obvious: two objects which polar functions are able to distinguish may have the same Yosida space. On the other hand, the following convention, from [M02a], seems adequate to capture the nature of this relationship.

Definition & Remarks 2.2. Suppose that $\mathcal{X}$ is a polar function and $\mathfrak{K}$ is a covering function. If for each compact space X we have that

$$\rho\big(\mathcal{X}\big(C(X)\big)\big) = \mathfrak{K}(X)\,, \tag{*}$$

we shall say that $\mathfrak{K}$ is *derived from* $\mathcal{X}$. We shall also refer to $\mathfrak{K}$ as the *covering derivative* of $\mathcal{X}$.

By [M02a, Proposition 5.7], if $\mathcal{X}$ is invariant, then so is its covering derivative. However, $\mathfrak{B}$—which "selects" the clopen sets—is derived from the polar function $\mathcal{S}$ of summands; $\mathfrak{B}$ is invariant, whereas $\mathcal{S}$ is not.

Next, briefly, a word concerning idempotent covering functions.

Definition & Remarks 2.3. Suppose that $\mathfrak{K}$ is a covering function. We say that $\mathfrak{K}$ is *idempotent* if $\big(X[\mathfrak{K}]\big)[\mathfrak{K}] = X[\mathfrak{K}]$, for each compact space X. For each $\mathfrak{K}$ there is a least idempotent covering function $\mathfrak{K}^\flat$ exceeding $\mathfrak{K}$. The construction is by transfinite induction. We refer the reader to [M02a, 5.8].

A word of caution: if $\mathcal{X}$ is idempotent its covering derivative $\mathfrak{K}$ need not have this property. Example 5.11 of [M02a] witnesses this.

Finally, observe that the covering function $\mathfrak{K}$ is idempotent if and only if the map $A \mapsto t^{\leftarrow}(A)$, from $\mathfrak{K}(X) \to \mathfrak{K}\big(X[\mathfrak{K}]\big)$ is surjective, where $t\colon X[\mathfrak{K}] \to X$ is the canonical irreducible surjection.

To conclude this section we paraphrase [M02a, Theorem 5.13].

Theorem 2.4.

(a) *For any ipf $\mathcal{X}$, the class*

$$\mathbb{H}(\mathcal{X}) = \big\{G\colon G[\mathcal{X}] = G\big\}$$

is a hull class, for which the extension $G \le G\big[\mathcal{X}^\flat\big]$ is the hull operator.

(b) *For any icf $\mathfrak{K}$,*

$$\mathbb{T}(\mathfrak{K}) = \big\{X : X[\mathfrak{K}] = X\big\}$$

is a covering class. The minimum cover of X in $\mathbb{T}(\mathfrak{K})$ is $X\big[\mathfrak{K}^\flat\big]$.

3. Distinguished essential extensions and irreducible maps

Having recalled the background in the first two sections, we now proceed to generalize the notions of polar functions and covering functions to encompass, respectively, sublattices of polars and regular closed sets. This generalization will afford a distinction between classes of essential extensions, on the one hand, and irreducible surjections, on the other.

Definition 3.1. Suppose $\mathcal{L}$ is a function assigning to each $\mathbf{W}$-object G a sublattice $\mathcal{L}(G)$ of $\mathcal{P}(G)$ containing G and $\{0\}$, containing all the summands of G. As before, we say that $\mathcal{L}$ is *invariant* if, for each essential extension, $G \leq H$, the map $\mathrm{aug}_{G,H}$ carries $\mathcal{L}(G)$ into $\mathcal{L}(H)$. In this context we shall also use the abbreviation *ipf* for an invariant polar function.

Likewise, a function $\mathfrak{I}$ which associates to each compact space X, a sublattice $\mathfrak{I}(X)$ of $\mathfrak{R}(X)$ containing X and $\emptyset$, as well as all the clopen sets of X, a *covering function.* $\mathfrak{I}$ is *invariant* if for each irreducible surjection $g\colon Y \to X$, the induced $g^{\leftarrow}$ maps $\mathfrak{I}(X)$ into $\mathfrak{I}(Y)$. The abbreviation *icf* has the same meaning as before.

We shall also say, in this generalized context, that $\mathfrak{I}$ is *derived from* $\mathcal{L}$, when $\mathfrak{I}(X) = \rho\big(\mathcal{L}(C(X))\big)$, for each compact space X. Alternatively, and as before, we say that $\mathfrak{I}$ is the *covering derivative* of $\mathcal{L}$.

Now we come to the central definition of the article.

Definition & Remarks 3.2. Suppose that $\mathcal{L}$ is an ipf. An essential extension $G \leq H$ is said to be *$\mathcal{L}$-essential* if $\mathrm{aug}_{G,H}$ induces a lattice isomorphism from $\mathcal{L}(G)$ onto $\mathcal{L}(H)$. Dually, if $\mathfrak{I}$ is an icf, then the irreducible surjection $g\colon Y \to X$ is *$\mathfrak{I}$-irreducible* if $g^{\leftarrow}$ induces a lattice isomorphism of $\mathfrak{I}(X)$ onto $\mathfrak{I}(Y)$.

Note the following facts: if $G \leq H \leq H'$ are essential extensions, then $G \leq H'$ is $\mathcal{L}$-essential if and only if both $G \leq H$ and $H \leq H'$ are $\mathcal{L}$-essential. The corresponding property also holds for icfs.

Assume that $\mathfrak{I} = \rho(\mathcal{L})$; it follows from (§) in 1.5(b) that the irreducible surjection $g\colon Y \to X$ is $\mathfrak{I}$-irreducible if and only if the induced extension $C(X) \leq C(Y)$ is $\mathcal{L}$-essential.

Here is an example, to give an idea of how the generalization of this section, the departure from subalgebras of $\mathcal{P}(G)$ and $\mathfrak{R}(X)$, might be useful. The discussion is spread out over the next item and the commentary in 3.5, where our discussion is put in perspective of the pertinent literature. This example is typical of the more general discussion to come.

Example 3.3. As in Examples 1.8, (c) and (d), α denotes a regular, uncountable cardinal, or else the symbol ∞.

(a) Suppose that $g\colon Y \to X$ is an irreducible surjection of compact spaces. Recall that g is *α-irreducible* if for each α-cozeroset U of Y there is an α-cozeroset V in X such that $g^{-1}(V)$ is dense in U. α-irreducibility is "dual to" the following type of essential extension: $G \leq H$ is *α-essential* if each α-generated polar of H traces to an α-generated polar of G. Note that "α-essential" and "$\mathcal{P}_\alpha$-essential" are synonymous, as are the labels "α-irreducible" and "$\mathcal{R}_\alpha$-irreducible".

The notion of an α-irreducible map first appears in [BHN90], and again in [Ma92].

(b) Consider the polar function $\mathcal{P}^\alpha_\alpha$ and its covering derivative $\mathfrak{R}^\alpha_\alpha$. We say that g is *weakly α-irreducible* if g is $\mathfrak{R}^\alpha_\alpha$-irreducible. The reader will quickly be able to verify that an α-irreducible map is, indeed, weakly α-irreducible. As to the algebraic counterpart of the foregoing, $G \leq H$ is *weakly α-essential* if it is $\mathcal{P}^\alpha_\alpha$-essential. Note that $G \leq H$ is weakly α-essential if and only if the induced $YH \to YG$ is weakly α-irreducible.

For zero-dimensional spaces the two strains of α-irreducibility coincide.

Proposition 3.4. *Suppose that $g\colon Y \to X$ is an irreducible surjection. If Y is zero-dimensional, then the following are equivalent:*

(a) *g is weakly α-irreducible.*

(b) *For each clopen set $B \subseteq Y$, there is an α-cozeroset V in X such that $g^{-1}(V) \subseteq B$ is dense.*

(c) *g is α-irreducible.*

Proof. It should be clear that (a) implies (b), while we have already remarked that (c) implies (a). Finally, assuming (b), note that every α-cozeroset of Y is the union of fewer than α clopen sets, owing to zero-dimensionality. Since α is a regular cardinal, the union of fewer than α α-cozerosets is an α-cozeroset, which proves (c). $\square$

Next, a brief review of some important material from the theory of minimum covers. References will appear as they are needed.

Definition & Remarks 3.5. (a) Recall that a space X is *quasi F* if every dense cozeroset is C^*-embedded in X. It is well known that X is quasi F precisely when the pointwise addition in $D(X)$ defines a group, which then implies that $D(X)$ is an ℓ-group and a $\mathbf{W}$-object under this operation.

The work of [DHH80] and [HVW87] establishes that every compact space X has a minimum quasi F-cover, denoted qFX, which is, in fact, the largest ω_1-irreducible pre-image of X. Now, it is established in [BH87, Theorem 2.7(b)] that $D(qFYG)$ is the largest ω_1-essential extension of G. The reader is advised, however, that the terminology of [BH87] is different from ours.

(b) More generally, in [BHN90, Theorem 4.9], the authors establish that there is, for each compact X, a maximum α-irreducible pre-image; denote that $qF_\alpha X$. Next, recall that a compact space X is *quasi F_α* if every dense α-Lindelöf subspace of X is C^*-embedded. (A space Y—not necessarily compact—is *α-Lindelöf* if every open cover of Y has a subcover consisting of fewer than α members.) For any compact X, $qF_\alpha X$ is simultaneously the minimum quasi F_α-cover of X.

As was remarked in [M02a, 4.9(a)], $X[\mathfrak{R}_\alpha^\alpha]$ is the minimum α-cloz cover of X, there denoted $\mathrm{Cz}_\alpha X$. See also Example 1.8(c). Next, thanks to Proposition 1.6, if X is zero-dimensional, then $\mathrm{Cz}_\alpha X$ is zero-dimensional as well. This is known; it appears explicitly as [HVW89, Theorem 3.15], for the ordinal ω_1.

It is difficult to say more. It is well known that $\mathrm{Cz}_\alpha X \leq qF_\alpha X$, and they agree if X is zero-dimensional and $\alpha = \omega_1$. They also agree for $\alpha = \infty$, and equal the absolute of X. That $\mathrm{Cz}_\alpha X \leq qF_\alpha X$, is a consequence of Theorem 4.3. There is additional commentary in § 6 of [HM02]. It is fair to say, however, that the situation is shrouded in mystery.

We now proceed to generalize the illustrations of 3.3 and 3.5.

Definition & Remarks 3.6. (a) In this discussion $\mathcal{L}$ stands for an ipf (as generalized in this section). $\widehat{\mathcal{L}}$ will denote the ipf defined by

$$\widehat{\mathcal{L}}(G) \equiv \{K \in \mathcal{L}(G)\colon K^\perp \in \mathcal{L}(G)\},$$

for each $\mathbf{W}$-object G. It is easy to see that $\widehat{\mathcal{L}}(G)$ is a subalgebra, and, indeed, it is the largest subalgebra of $\mathcal{P}(G)$ contained in $\mathcal{L}(G)$. Moreover, as the reader will easily verify, $\widehat{\mathcal{L}}$ is an ipf. We shall refer to $\widehat{\mathcal{L}}$ as the *boolean center* of $\mathcal{L}$.

Dually, suppose that $\mathfrak{I}$ is an icf. $\widehat{\mathfrak{I}}$ here too denotes the icf for which $\widehat{\mathfrak{I}}(X)$ is the largest subalgebra of $\mathfrak{R}(X)$ contained in $\mathfrak{I}(X)$. In this context too, we call $\widehat{\mathfrak{I}}$ the *boolean center* of $\mathfrak{I}$. It is easy to verify that if $\mathfrak{I}$ is the covering derivative of $\mathcal{L}$, then $\widehat{\mathfrak{I}}$ is the covering derivative of $\widehat{\mathcal{L}}$.

Note that, for each uncountable, regular cardinal α, $\widehat{\mathcal{P}_\alpha} = \mathcal{P}_\alpha^\alpha$, and $\widehat{\mathfrak{R}_\alpha} = \mathfrak{R}_\alpha^\alpha$.

(b) We now abstract a feature of $\mathcal{P}_\alpha$: let us say that the ipf $\mathcal{L}$ is *local* if, for each $\mathbf{W}$-object G,

(i) $\mathcal{P}_0(G) \subseteq \mathcal{L}(G)$, and
(ii) the essential extension $G \leq H$ is $\mathcal{L}$-essential if $\mathrm{tr}_{H,G}(h^{\perp\perp}) \in \mathcal{L}(G)$, for each $h \in H$.

If each $\mathcal{L}(G)$ is closed under forming countable suprema we say that $\mathcal{L}$ is a *σ-ipf.*

(c) Dually, the icf $\mathfrak{I}$ is *local* if, for each compact space X, each $\mathrm{cl}_X \operatorname{coz}(f) \in \mathfrak{I}(X)$, and for each irreducible surjection $g\colon Y \to X$, g is $\mathfrak{I}$-irreducible if each $\mathrm{cl}_Y \operatorname{coz}(f)$ lies in $g^{\leftarrow}(\mathfrak{I}(X))$. Likewise, we call an icf $\mathfrak{I}$ a *σ-icf* if $\mathfrak{I}(X)$ is closed under forming countable suprema, for each compact X.

Evidently, for each uncountable cardinal α and $\alpha = \infty$, $\mathcal{P}_\alpha$ is a local σ-ipf and $\mathfrak{R}_\alpha$ a σ-icf. $\mathcal{P}_0$ itself is local, but not a σ-ipf.

We begin with what is doubtless already anticipated. It is also immediate from (§) of 1.5(b).

Proposition 3.7. *The covering derivative of a local ipf (resp. σ-ipf) is local (resp. a σ-ipf).*

The next result generalizes the discussion of [M02a, 5.5(a)], which showed that $\mathcal{P}^{\alpha}_{\alpha}$ is an idempotent ipf.

Proposition 3.8. *Suppose that $\mathcal{L}$ is a local ipf. Then*

(a) *$G\big[\widehat{\mathcal{L}}\,\big]$ is an $\mathcal{L}$-essential extension of G, for each $\mathbf{W}$-object G.*

(b) *$\widehat{\mathcal{L}}$ is idempotent.*

Proof. Suppose that G is a $\mathbf{W}$-object. We will show that for each $a \in G\big[\widehat{\mathcal{L}}\,\big]$, $a^{\perp\perp}$ is generated by a polar in $\mathcal{L}(G)$. Since $\mathcal{L}$ is local, this suffices to prove the first claim. Then also each such $a^{\perp\perp}$ traces to a polar in $\mathcal{L}(G)$, and hence each $K \in \mathcal{L}\big(G\big[\widehat{\mathcal{L}}\,\big]\big)$ does as well. In particular then, each $K \in \widehat{\mathcal{L}}\big(G\big[\widehat{\mathcal{L}}\,\big]\big)$ traces to a polar in $\widehat{\mathcal{L}}(G)$, and is therefore a summand of $G\big[\widehat{\mathcal{L}}\,\big]$, which proves the second claim.

Now, by 1.3(b), each $a \in G\big[\widehat{\mathcal{L}}\,\big]$ can be written as

$$a = \sum_{i=1}^{n} g_i\,[K_i]\,,$$

where the K_i are pairwise disjoint polars in $\widehat{\mathcal{L}}(G)$, each $g_i \in G$, and $g[K]$ denotes the projection of g on $\mathrm{aug}_{G,G^e}(K)$. Now, it suffices to consider the polars $g[K]^{\perp\perp}$, with $0 \le g \in G$, and $K \in \widehat{\mathcal{L}}(G)$. But observe that

$$\mathrm{tr}_{G[\widehat{\mathcal{L}}],G}\big(g[K]^{\perp\perp}\big) = g^{\perp_G\perp_G} \cap K\,,$$

and the polar on the right side of the equation lies in $\mathcal{L}(G)$ since $\mathcal{L}$ is local and each $\mathcal{L}(G)$ is a sublattice of $\mathcal{P}(G)$. This completes the proof. $\square$

We record the dual of Proposition 3.8, leaving the proof as an exercise for the reader.

Corollary 3.9. *Suppose that $\mathcal{L}$ is a local ipf and $\mathfrak{I} = \rho(\mathcal{L})$. Then the canonical irreducible map $t\colon X\big[\widehat{\mathfrak{I}}\big] \to X$ is $\mathfrak{I}$-irreducible, and $\widehat{\mathfrak{I}}$ is idempotent.*

Next, the result which, in a sense, generalizes the discussion of Example 3.3 and Proposition 3.4. We first make a definition, motivated by the aforementioned discussion.

Definition 3.10. Let $\mathfrak{I}$ be an icf, and V be an open set of the space X. V is called an *$\mathfrak{I}$-cozeroset* if its closure lies in $\mathfrak{I}(X)$.

Recall that a space X is *cozero-complemented* if for each cozeroset U there is a cozeroset V such that $U \cap V = \emptyset$ and $U \cup V$ is dense in X.

Proposition 3.11. *Suppose that $\mathfrak{I}$ is a local icf. Let $g\colon Y \to X$ be an irreducible surjection. Consider the following four statements:*

(a) *For each cozeroset V of Y there is an $\mathfrak{I}$-cozeroset W such that $g^{\leftarrow}(W) \subseteq V$ is dense.*

(b) *g is $\mathfrak{I}$-irreducible.*

(c) *g is $\widehat{\mathfrak{I}}$-irreducible.*

(d) *For each clopen set V of Y there is an $\mathfrak{I}$-cozeroset W such that $g^{\leftarrow}(W) \subseteq V$ is dense.*

Then (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d). *Moreover, we have the following:*

(i) *If Y is cozero-complemented then* (c) $\Rightarrow$ (a).

(ii) *If Y is zero-dimensional and $\mathfrak{I}$ is a σ-icf, then* (d) $\Rightarrow$ (a).

Proof. (Sketch.) (b) follows from (a), using the local feature and the invariance of $\mathfrak{I}$. The implications (b) $\Rightarrow$ (c) $\Rightarrow$ (d) are clear. With the assumption that Y is cozero-complemented, (a) is a special case of (c). That settles (i), and we leave (ii) to the reader. □

4. Maximum $\mathcal{L}$-essential extensions

In this section we establish (reasonable) sufficient conditions which ensure that, for an ipf $\mathcal{L}$, each **W**-object has a maximum $\mathcal{L}$-essential extension. We shall closely follow in the spirit of [BHN90], which involves $\mathcal{P}_\alpha$ and its covering derivative.

Throughout the subsequent discussion, Y will stand for YG^e. We identify G with the image under the Yosida representation; in particular, the designated unit is taken to be the constant function 1.

Definition & Remarks 4.1. The containment $G \le G^d$ of G in its divisible hull satisfies $\mathrm{tr}_{G^d,G}(a^{\perp\perp}) = (na)^{\perp_G\perp_G}$, for a suitable positive multiple n of a. Thus, if $\mathcal{L}$ is local, it is clear that $G \le G^d$ is $\mathcal{L}$-essential. For this reason we assume, through much of the development in this section, that G is divisible.

Assume G is divisible, and define

$$G^{\mathcal{L}} \equiv \Big\{a \in G^e\colon\ \mathrm{tr}_{G^e,G}\Big(\big((a-r)^+\big)^{\perp\perp}\Big),\ \mathrm{tr}_{G^e,G}\Big(\big((r-a)^+\big)^{\perp\perp}\Big) \in \mathcal{L}(G),\ \forall r \in \mathbb{Q}\Big\}.$$

The first objective is the next lemma. The reader should note the parallels between this and the development in [BHN90, § 4].

LEMMA 4.2. *Let $\mathcal{L}$ be a local σ-ipf. For each divisible $\mathbf{W}$-object G, $G^{\mathcal{L}}$ is an ℓ-subgroup of G^e and an extension of G.*

Proof. All polars occurring in this proof are calculated in G^e.

Now, it is clear that $G^{\mathcal{L}}$ is closed under taking additive inverses. Also, since $\mathcal{L}$ is local we also have that $G \subseteq G^{\mathcal{L}}$. Consider now $a_1, a_2 \in G^{\mathcal{L}}$ and a rational number r. From the identities

$$\big(((a_1 \vee a_2) - r)^+\big)^{\perp\perp} = \big((a_1 - r)^+\big)^{\perp\perp} \vee \big((a_2 - r)^+\big)^{\perp\perp}$$

and

$$\big(((a_1 \wedge a_2) - r)^+\big)^{\perp\perp} = \big((a_1 - r)^+\big)^{\perp\perp} \cap \big((a_2 - r)^+\big)^{\perp\perp}$$

and their duals it follows immediately that $G^{\mathcal{L}}$ is a sublattice of G^e. Finally, it is a routine matter to verify that, in Y,

$$\mathrm{cl}_Y \operatorname{coz}\big((a_1 + a_2 - r)^+\big) = \mathrm{cl}_Y \left(\bigcup_{r=r_1+r_2} \operatorname{coz}\big((a_1 - r_1)^+\big) \cap \operatorname{coz}\big((a_2 - r_2)^+\big) \right).$$

(One mimics the proof of [BHN90, Lemma 4.2].) The above identity means that

$$\big((a_1 + a_2 - r)^+\big)^{\perp\perp} = \bigvee_{r=r_1+r_2} \big((a_1 - r_1)^+\big)^{\perp\perp} \cap \big((a_2 - r_2)^+\big)^{\perp\perp}.$$

Since $\mathcal{L}$ is a σ-ipf, the preceding identity, together with its dual, implies that $G^{\mathcal{L}}$ is an ℓ-subgroup of G^e. □

The next result generalizes [BHN90, Proposition 4.4].

THEOREM 4.3. *Suppose that $\mathcal{L}$ is a local σ-ipf, and $\mathfrak{I} = \rho(\mathcal{L})$. Then,*

(a) *for each $\mathbf{W}$-object G, $G^{\mathcal{L}}$ is the largest $\mathcal{L}$-essential extension of G;*

(b) *each compact space X has a maximum $\mathfrak{I}$-irreducible pre-image, $X^{\mathfrak{I}}$. In fact, $X^{\mathfrak{I}} = YC(X)^{\mathcal{L}}$;*

(c) *$G\big[\widehat{\mathcal{L}}\big] \leq G^{\mathcal{L}}$, for each $\mathbf{W}$-object G, and, for each compact space X, $X\big[\widehat{\mathfrak{I}}\big] \leq X^{\mathfrak{I}}$ in $\mathrm{Cov}(X)$.*

Proof. Without loss of generality, as explained before, G is assumed to be divisible. From the very definition it should be clear that $G \leq G^{\mathcal{L}}$ is a $\mathcal{L}$-essential extension. On the other hand, if $G \leq H$ is $\mathcal{L}$-essential then, for each rational number r and each $a \in H$, we have $\mathrm{tr}_{H,G}\big(\big((a-r)^+\big)^{\perp_H\perp_H}\big)$ and $\mathrm{tr}_{H,G}\big(\big((r-a)^+\big)^{\perp_H\perp_H}\big)$ in $\mathcal{L}(G)$. But these traces are the same as the traces of the principal polars of $(a-r)^+$ and $(r-a)^+$ computed in G^e; that is, $a \in G^{\mathcal{L}}$ and so $H \leq G^{\mathcal{L}}$. This proves (a).

As to (b), suppose that $g\colon Y \to X$ is $\mathfrak{I}$-irreducible. Then the induced embedding $C(g)\colon C(X) \to C(X)$ is $\mathcal{L}$-essential. Thus, $C(Y) \le C(X)^{\mathcal{L}}$, and dualizing again, we see that the induced irreducible surjection $YC(X)^{\mathcal{L}} \to X$ is $\mathfrak{I}$-irreducible and factors through g.

(c) follows by applying Propositions 3.7 and 3.8. This completes the proof. $\square$

The next step after Theorem 4.3 is to show that the extension $G \le G^{\mathcal{L}}$ is a hull, for each local σ-ipf. We need a definition first.

Definition & Remarks 4.4. Suppose that $\mathcal{L}$ is an ipf. If G has no proper $\mathcal{L}$-essential extensions we will say that G is *$\mathcal{L}$-essentially closed.* Evidently, per Theorem 4.3, $G^{\mathcal{L}}$ is $\mathcal{L}$-essentially closed.

Dually, if $\mathfrak{I}$ is an icf, and the space X has no proper $\mathfrak{I}$-irreducible pre-images, we will say that *$\mathfrak{I}$-irreducibly closed.* The maximum $\mathfrak{I}$-irreducible pre-image $X^{\mathfrak{I}}$ of X is, obviously, $\mathfrak{I}$-irreducibly closed.

Let $\mathcal{L}^{\mathrm{cl}}$ denote the class of all $\mathcal{L}$-essentially closed $\mathbf{W}$-objects, and $\mathfrak{I}^{\mathrm{cl}}$ stand for the class of all $\mathfrak{I}$-irreducibly closed compact spaces.

Here is the main result of the section.

Theorem 4.5. *Suppose that $\mathcal{L}$ is a local σ-ipf and $\mathfrak{I} = \rho(\mathcal{L})$. Then we have the following:*

(a) *$\mathcal{L}^{\mathrm{cl}}$ is a hull class of $\mathbf{W}$-objects.*

(b) *For each $\mathbf{W}$-object G, $G^{\mathcal{L}}$ is the least extension of G in $\mathcal{L}^{\mathrm{cl}}$.*

(c) *$\mathfrak{I}^{\mathrm{cl}}$ is a covering class of compact spaces.*

(d) *For each compact space X, $X^{\mathfrak{I}}$ is the minimum cover of X in $\mathfrak{I}^{\mathrm{cl}}$.*

Proof. Suppose that $G \le H$ is an essential extension and that H is $\mathcal{L}$-essentially closed. For each $a \in G^{\mathcal{L}}$, we have that $\mathrm{tr}_{G^e,G}\big(((a-r)^+)^{\perp\perp}\big)$ and $\mathrm{tr}_{G^e,G}\big(((r-a)^+)^{\perp\perp}\big)$ lie in $\mathcal{L}(G)$, for each $r \in \mathbb{Q}$, and hence $\mathrm{tr}_{G^e,H}\big(((a-r)^+)^{\perp\perp}\big)$ and $\mathrm{tr}_{G^e,H}\big(((r-a)^+)^{\perp\perp}\big)$ lie in $\mathcal{L}(H)$, by the invariance of $\mathcal{L}$. Thus, since H is $\mathcal{L}$-essentially closed, $H = H^{\mathcal{L}}$ and it follows that $a \in H$. This proves that $G^{\mathcal{L}}$ is the maximum $\mathcal{L}$-essential extension of G, and establishes both (a) and (b).

Regarding $\mathfrak{I}$ and conditions (c) and (d), suppose that X is a compact space, and that $g\colon Y \to X$ is an irreducible surjection, with $Y \in \mathfrak{I}^{\mathrm{cl}}$. Since $(\cdot)^{\mathcal{L}}$ is a hull operator, we have that $C(X)^{\mathcal{L}} \le C(Y)^{\mathcal{L}}$, and, by Theorem 4.3, we also have that

$$X \le X^{\mathfrak{I}} = YC(X)^{\mathcal{L}} \le YC(Y)^{\mathcal{L}} = Y^{\mathfrak{I}} = Y,$$

in $\mathrm{Cov}(X)$. This proves both (c) and (d). $\square$

Remark 4.6. We would like to be able to assert that if X is zero-dimensional, then $X\big[(\widehat{\mathfrak{I}})^\flat\big]$ is the maximum $\mathfrak{I}$-irreducible pre-image of X, for an icf $\mathfrak{I}$, with some reasonable hypotheses placed on $\mathfrak{I}$. Unfortunately, we cannot prove any such thing at this time; and placing assumptions on $\mathfrak{I}$ seems not to be relevant.

What we have in mind is the generalization of the well known fact ([HVW89, Corollary 3.5]) that the minimum cloz cover of a zero-dimensional space is, in fact, quasi F. The key point in establishing that corollary seems to be Theorem 3.4 of that paper, and, specifically, proving the implication (a) $\Rightarrow$ (b) of that theorem. For this facts about extendibility of functions are required ([BIH74, 1.1]), which afford no ready generalization to this abstract setting.

The next proposition records what we are able to prove.

Proposition 4.7. *Suppose that X is zero-dimensional, and that $\mathfrak{I}$ is an icf. Then $X\big[(\widehat{\mathfrak{I}})^\flat\big]$ is the maximum $\mathfrak{I}$-irreducible zero-dimensional pre-image of X.*

P r o o f. According to Proposition 1.6, $X\big[(\widehat{\mathfrak{I}})^\flat\big]$ is zero-dimensional.

We show that $X\big[(\widehat{\mathfrak{I}})^\flat\big]$ has no proper $\mathfrak{I}$-irreducible zero-dimensional pre-images. To that end, suppose that $g\colon Y \to X\big[(\widehat{\mathfrak{I}})^\flat\big]$ is $\mathfrak{I}$-irreducible with Y zero-dimensional. Now $g^\leftarrow$ induces an isomorphism of $\mathfrak{I}\big(X\big[(\widehat{\mathfrak{I}})^\flat\big]\big)$ onto $\mathfrak{I}(Y)$, and hence maps $\widehat{\mathfrak{I}}\big(X\big[(\widehat{\mathfrak{I}})^\flat\big]\big)$ isomorphically onto $\widehat{\mathfrak{I}}(Y)$. Now, by definition,

$$\widehat{\mathfrak{I}}\big(X\big[(\widehat{\mathfrak{I}})^\flat\big]\big) = \mathfrak{B}\big(X\big[(\widehat{\mathfrak{I}})^\flat\big]\big)\,,$$

which shows that $\widehat{\mathfrak{I}}(Y) = \mathfrak{B}(Y)$ as well. The point of all this is that g induces an isomorphism of the algebras of clopen sets of these spaces, by taking the direct image $A \mapsto g(A)$. Now if p, q are distinct points of Y, there is a clopen set B separating them. By the preceding arguments, $g(B)$ and $g(Y \setminus B)$ are disjoint clopen sets, and therefore $g(p) \neq g(q)$, proving that g is one-to-one, and, thereby, our claim. $\square$

What can be said, in general, about the structure of $\mathcal{L}$-essentially closed objects and $\mathfrak{I}$-irreducibly closed spaces? One consequence, of $\mathcal{L}$ being a local σ-ipf, is this:

Proposition 4.8. *Suppose that $\mathcal{L}$ is a local σ-ipf, and let $\mathfrak{I}$ be its covering derivative. Then each $\mathfrak{I}$-irreducibly closed space is quasi F, and each $\mathcal{L}$-essentially closed object is of the form $D(Y)$, for a suitable $\mathfrak{I}$-irreducibly closed space Y.*

P r o o f. Suppose that X is $\mathfrak{I}$-irreducibly closed. If $g\colon Y \to X$ is ω_1-irreducible, then for each cozeroset V of Y, there is a cozeroset U of X such that $\mathrm{cl}_Y V = g^\leftarrow(\mathrm{cl}_X U)$. Now, as $\mathfrak{I}$ is local, it follows that $\mathrm{cl}_X U \in \mathfrak{I}(X)$ and the invariance of $\mathfrak{I}$ implies that $\mathrm{cl}_Y V \in \mathfrak{I}(Y)$. Thus, g is $\mathfrak{I}$-irreducible, and hence it must be a homeomorphism, proving that X has no proper ω_1-irreducible pre-images. This means that X is a quasi F-space.

Now, if G is $\mathcal{L}$-essentially closed, and $Y = YG$, then Y is $\mathfrak{I}$-irreducibly closed. Otherwise, lift Y by $h\colon Y^{\mathfrak{I}} \to Y$ to its maximum $\mathfrak{I}$-irreducible cover (Theorem 4.3); as we have seen, $Y^{\mathfrak{I}}$ is quasi F. On the other hand, $G \le D(Y^{\mathfrak{I}})$ is an $\mathcal{L}$-essential embedding, whence $G = D(Y^{\mathfrak{I}})$, and also $Y = Y^{\mathfrak{I}}$. This completes the proof of the proposition. □

As an illustration we highlight again the situation with $\mathcal{P}_\alpha$ and its covering derivative $\mathfrak{R}_\alpha$. We refer the reader to [BHN90, Theorem 4.6] for additional details.

Example 4.9. A compact space X has no α-irreducible pre-images if and only if X is quasi F_α. Thus, Proposition 4.8 reprises some of [BHN90, Theorem 5.3], by insuring that (for the ipf $\mathcal{P}_\alpha$) if G is any $\mathbf{W}$-object, then $D(YG^{\mathfrak{R}_\alpha})$ is the maximum α-essential extension of G. We remind the reader that $YG^{\mathfrak{R}_\alpha}$ is the minimum quasi F_α-cover of YG.

5. Distinguished projectability vs. disconnectivity

As before, α denotes an infinite cardinal, or else the symbol ∞.

Definition & Remarks 5.1. Suppose that G is an ℓ-group. Recall that G is said to be *projectable* if every principal polar $g^{\perp\perp}$ is a summand. More generally, G is *α-projectable* if every α-generated polar is a summand. "∞-projectable" has the same meaning as "strongly projectable".

It is shown in [HM96, Corollary 2.4] that if G is α-projectable, with $\alpha \ge \omega_1$, then YG is *α-disconnected*; that is, every α-cozeroset of YG has clopen closure. Note that "ω_1-disconnected" is commonly called "basically disconnected" in the literature; "∞-disconnected" *is* "extremally disconnected".

We proceed to use the machinery of polar functions to generalize these ideas.

Definition & Remarks 5.2. Suppose that $\mathcal{L}$ is an ipf, and let $\langle\mathcal{L}\rangle$ denote the function which, for each $\mathbf{W}$-object G computes the subalgebra $\langle\mathcal{L}\rangle(G)$ of $\mathcal{P}(G)$ generated by $\mathcal{L}(G)$. The reader will have no trouble verifying that $\langle\mathcal{L}\rangle$ is an ipf.

Dually, if $\mathfrak{I}$ is an icf, $\langle\mathfrak{I}\rangle$ denotes the icf which assigns to each compact space X the subalgebra $\langle\mathfrak{I}\rangle(X)$ of $\mathfrak{R}(X)$ generated by $\mathfrak{I}(X)$. Note that if $\mathfrak{I} = \rho(\mathcal{L})$ then $\langle\mathfrak{I}\rangle = \rho(\langle\mathcal{L}\rangle)$; this is an immediate consequence of (§) in 1.5(b).

Our first result is easy, and sets the tone for the rest of the section.

Lemma 5.3.

(a) *If $\mathcal{L}$ is a local ipf, then each* $\mathbf{W}$*-object in* $\mathbb{H}(\langle\mathcal{L}\rangle)$ *is projectable.*

(b) *If $\mathfrak{J}$ is a local icf, then each space in* $\mathbb{T}(\langle\mathfrak{J}\rangle)$ *is basically disconnected.*

Proof. (a) For each $\mathbf{W}$-object G, $\mathcal{L}(G)$ contains all principal polars, because $\mathcal{L}$ is local. If $G \in \mathbb{H}(\langle\mathcal{L}\rangle)$ then each polar in $\mathcal{L}(G)$ is a summand, whence G is projectable.

The obvious dual argument works for (b). □

Definition 5.4. The objects in $\mathbb{H}(\langle\mathcal{L}\rangle)$ will be called *$\mathcal{L}$-projectable.* The spaces in $\mathbb{T}(\langle\mathfrak{J}\rangle)$ shall be referred to as *$\mathfrak{J}$-disconnected* spaces.

Suppose that $\mathcal{L}$ is a local σ-ipf, so that the existence of the maximum $\mathcal{L}$-essential extension exists. There seems to be no satisfying correlation between $G\big[\langle\mathcal{L}\rangle^\flat\big]$ and $G^{\mathcal{L}}$. In the next result, however, we do get the expected relationship between the associated Yosida spaces. It generalizes the well known fact that basically disconnected spaces are quasi F. Notice that, for the first assertion, we do not assume that the covering function is derived from any polar function.

Proposition 5.5. *Suppose that $\mathfrak{J}$ is a local icf. Every $\mathfrak{J}$-disconnected space is $\mathfrak{J}$-irreducibly closed. Consequently, if $\mathcal{L}$ is a local σ-ipf and $\mathfrak{J} = \rho(\mathcal{L})$, then* $X^{\mathfrak{J}} \le X\big[\langle\mathfrak{J}\rangle^\flat\big]$.

Proof. We note at the outset that the second assertion about $\mathfrak{J}$ follows immediately from the first and from Theorem 4.3.

As to the first, suppose that X is $\mathfrak{J}$-disconnected. Then $\langle\mathfrak{J}\rangle(X) = \mathfrak{B}(X)$. Therefore, if $g\colon Y \to X$ is any $\mathfrak{J}$-irreducible surjection, we also have that $\langle\mathfrak{J}\rangle(Y) = \mathfrak{B}(Y)$, and, in particular, every cozeroset of Y has clopen closure. This means that Y too is basically disconnected, and hence zero-dimensional. The upshot of this is that g must be a homeomorphism; we refer the reader to the argument in the proof of Proposition 4.7. This proves that X is $\mathfrak{J}$-irreducibly closed. □

Proposition 5.7 generalizes the well known result—see [HVW87]—that a cozero-complemented quasi F-space is basically disconnected. The definition which precedes it is modeled upon the familiar notion of a cozero-complemented space.

Definition & Remarks 5.6. Let $\mathcal{L}$ be an ipf. The archimedean ℓ-group G is said to be *$\mathcal{L}$-complemented* if for each $K \in \mathcal{L}(G)$, $K^\perp \in \mathcal{L}(G)$. Evidently, this condition is equivalent to requiring that

$$\mathcal{L}(G) = \widehat{\mathcal{L}}(G) = \langle\mathcal{L}\rangle(G)\,.$$

Dually, if $\mathfrak{I}$ is an icf, we say that a compact space X is *$\mathfrak{I}$-complemented* if for each $A \in \mathfrak{I}(X)$, the complement A' in $\mathfrak{R}(X)$ lies in $\mathfrak{I}(X)$. Note that if $\mathfrak{I} = \rho(\mathcal{L})$, then X is $\mathfrak{I}$-complemented if and only if $C(X)$ is $\mathcal{L}$-complemented.

The following proposition is then self-evident. We leave the proof to the reader.

Proposition 5.7. *Suppose that $\mathcal{L}$ is a local ipf and $\mathfrak{I} = \rho(\mathcal{L})$. If X is an $\mathfrak{I}$-complemented space, then $X^{\mathfrak{I}}$ exists, and*

$$X\left[\widehat{\mathfrak{I}}\right] = X^{\mathfrak{I}} = X\left[\langle\mathfrak{I}\rangle^{\flat}\right].$$

We conclude with an observation on relative uniform limits applied to $G\left[\langle\mathcal{L}\rangle^{\flat}\right]$, when $\mathcal{L}$ is a local σ-ipf. The goal is to generalize [HM99, Theorem 5.5(a)].

First, we review the essentials concerning relative uniform convergence; for additional information the reader is referred to [LZ71] and [Ve69].

Definition & Remarks 5.8. Suppose that G is an archimedean ℓ-group. A sequence $\{a_n\}_{n\in\mathbb{N}}$ in G is said to *converge to a relatively uniformly, along the regulator u*—written $a_n \to a\,(u)$—if for each $k \in \mathbb{N}$ there is an integer $n(k)$ such that $n \geq n(k)$ implies that $k|a_n - a| \leq u$. The concept "$\{a_n\}_{n\in\mathbb{N}}$ is relatively uniformly Cauchy" is defined in a similar manner. Now G is *relatively uniformly complete* if every sequence in G which is relatively uniformly Cauchy converges relatively uniformly (along the same regulator).

There is a "relative uniform completion" hull operator ru; it is, in fact a reflection, although that is not relevant here. It is characterized by the conditions that

1. $G \leq (\mathrm{ru})G$ is essential and $(\mathrm{ru})G$ is relatively uniformly complete, and
2. if $G \leq H \leq (\mathrm{ru})G$ and H is relatively uniformly complete, then $H = (\mathrm{ru})G$.

$(\mathrm{ru})G$ is constructed by transfinitely adding limits—see the aforementioned references—through ω_1 steps. In our context we are only interested in the first step of this process. We describe this now.

Suppose $G \leq H$ with H relatively uniformly complete.

$$(\mathrm{ru})^1 G \equiv \left\{h \in H\colon \exists\,\{a_n\}_{n\in\mathbb{N}} \subseteq G \text{ and } u \in G \text{ such that } a_n \to h\,(u)\right\}.$$

Note that G is relatively uniformly complete if and only if $(\mathrm{ru})^1 G = G$. The choice of H does not matter.

In the sequel dG denotes the divisible hull of G. The reader will soon appreciate that Theorem 5.9(a) is not simply a restatement of Lemma 5.3(b).

THEOREM 5.9. *Suppose that $\mathcal{L}$ is a local σ-ipf. Then*

(a) $YG[\langle\mathcal{L}\rangle^\flat]$ *is basically disconnected—and, thus,* $D(YG[\langle\mathcal{L}\rangle^\flat])$ *is a group.*

(b) $(\mathrm{ru})^1 d(G[\langle\mathcal{L}\rangle^\flat])$ *is the ℓ-ideal generated by G in* $D(YG[\langle\mathcal{L}\rangle^\flat])$.

Proof. (a) Let V be a cozeroset of $Y = YG[\langle\mathcal{L}\rangle^\flat]$. Since the cozerosets of elements of $G[\langle\mathcal{L}\rangle^\flat]$ generate the topology, and V is Lindelöf, it follows that $\mathrm{cl}_Y V$ may be expressed as a countable supremum in $\mathfrak{R}(Y)$, $A_1 \vee A_2 \vee \cdots$, where each $A_i = \mathrm{cl}_Y \operatorname{coz}(a_i)$ for a suitable $a_i \in G[\langle\mathcal{L}\rangle^\flat]$. Since $\mathcal{L}$ is a local σ-ipf, we have that $\mathrm{cl}_Y V \in \rho(\mathcal{L}(G[\langle\mathcal{L}\rangle^\flat]))$, and therefore a clopen set. Thus, (a) is proved, since the second assertion is well known, as Y is also a quasi F-space. (See the remark following this proof.)

(b) By [M02a, Theorem 2.4] each element of $G[\langle\mathcal{L}\rangle^\flat]$ is of the form

$$h = \sum_{i=1}^{n} g_i[K_i], \tag{$\dagger$}$$

where each $g_i \in G$ and each $K_i \in \langle\mathcal{L}\rangle(G)$, with $K_i \cap K_j = \{0\}$, whenever $i \neq j$. Note that $g[K]$ refers to the projection of g on the polar $\mathrm{aug}_{G,G^e}(K)$.

What we derive from this is that if $h \in G[\langle\mathcal{L}\rangle^\flat]$ and it is expressed as in ($\dagger$), then $|h| \le |g_1| + \cdots + |g_n|$, whence G and its hull $G[\langle\mathcal{L}\rangle^\flat]$ in $\langle\mathcal{L}\rangle^\flat$ generate the same ℓ-ideal. On the other hand, according to [HM96, Theorem 4.2(a)], the ℓ-ideal generated by $\langle\mathcal{L}\rangle^\flat$ in $D(Y)$ is precisely what we are claiming. This completes the proof. □

To be prudent we should make the following observation.

Remark 5.10. In the notation of Theorem 5.9 and with the assumptions of that theorem, suppose further that $\mathfrak{I}$ denotes the covering derivative of $\mathcal{L}$. Then $\langle\mathfrak{I}\rangle = \rho(\langle\mathcal{L}\rangle)$, as has already been observed. What is not claimed in Theorem 5.9 is that $YG[\langle\mathcal{L}\rangle^\flat]$ is the $\langle\mathfrak{I}\rangle^\flat$-cover of YG. For further discussion about this the reader ought to refer to [M02b].

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