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THE EXACT LR TEST OF THE SCALE IN THE GAMMA FAMILY

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ABSTRACT. The aim of this paper is to give overview about the statistical results on the likelihood ratio test of the scale in the Gamma family and discuss the optimality of such exact tests. The results are applied to the testing of the intensity of the Poisson and Erlang processes. An application to the time processing data set are provided to illustrate the methods discussed.

1. Introduction

Choosing the most efficient statistical test is one of the essential problems of statistics. The class of such statistics is apparently narrow, though it contains under certain conditions the likelihood ratio (LR) statistics (see [Bahadur 65], [Bahadur 67] and [Rublík 89a]).

Let $y_1, y_2, \ldots, y_N$ be univariate independent random variables distributed according to the gamma densities

$$f(y_i|\vartheta) = \begin{cases} \gamma_i(\vartheta)^{v_i} \dfrac{y_i^{v_i-1}}{\Gamma(v_i)} e^{-\gamma_i(\vartheta) y_i} & \text{for } y_i > 0\,, \\ 0 & \text{for } y_i \le 0\,. \end{cases} \tag{1}$$

Here $\vartheta := (\vartheta_1, \ldots, \vartheta_p)$ is the vector of unknown scale parameters and $v_1, \ldots, v_N$ are known shape parameters. The parameter space Θ is an open subset of $\mathbb{R}^p$, $\gamma_i \in C^2(\Theta)$ and the matrix of the first order derivatives of the mapping $\gamma := (\gamma_1, \ldots, \gamma_N)$ has full rank on Θ.

This model has been applied in many disciplines, including queueing systems, reliability models and generalized linear models. See, for example, [Gaudoin-Soler 92] for use of the model (1) in testing the software reliability. Here rather complicated expression for the distribution function of the maximum likelihood

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estimator can be found. In [Pázman 96] the probability density of the parameter estimators is derived by the use of the concept of the "I-divergence distance" between the observation and the hypothesis as presented in [Pázman 93].

The tests of the intensity of the Poisson and Erlang processes (see [Kleinrock 75]) can be constructed in the model (1). Such tests are useful in a wide variety of applications. Two generic applications are queueing models and lifetime models. Another example of the application of the model (1) is test of the proportion of the exponential distribution which can be used for constructing a statistical quality control sampling plan (see [Rublík 96]).

In this paper we discuss the exact LR tests of the scale hypothesis

$$H_0 : \gamma = \gamma_0 \quad \text{versus} \quad H_1 : \gamma \neq \gamma_0 \tag{2}$$

in the Gamma family and their properties. Details can be found in [Stehlík 00] and [Stehlík 01]. The advantages of the exact tests of the scale in the Gamma family are exact levels of significance (the asymptotical tests are oversized) and exact power. Furthermore, the exact LR tests of the scale in the Gamma family are unbiased, uniformly most powerful (UUMP) tests. The theoretical justification of the exact LR testing is its asymptotical optimality. For example, the exact LR test of the scale of the exponential distribution is asymptotically optimal in the sense of the Bahadur exact slopes.

The paper is organized as follows. In sections *Observations distributed exponentially* and *Observations having the Erlang distribution* we discuss the results of the exact LR tests of the scale in the model (1) in the cases of exponential and Erlang distribution. The table of oversizing of the asymptotical test is also given. In section *Illustrative example* we provide the case study of the time processing dataset analyzed by the use of the exact LR tests. To maintain the continuity of the explanation the properties of the Lambert W function are included into the *Appendix*.

2. Observations distributed exponentially

In this section we consider the LR test of the hypothesis (2) in the particular case of the model (1) when the observations are distributed exponentially. The LR λ of the test of the hypothesis (2) is defined as the ratio

$$\lambda(y) = \frac{\max\limits_{\gamma \in \Gamma_0} \prod\limits_{i=1}^{N} \gamma_i e^{-\gamma_i y_i}}{\max\limits_{\gamma \in \Gamma_1} \prod\limits_{i=1}^{N} \gamma_i e^{-\gamma_i y_i}}, \tag{3}$$

where $\Gamma_0 = \{\gamma_0\}$ and $\Gamma_1 = \mathbb{R}^+ \setminus \Gamma_0$. As the consequence of (3) we obtain the relation

$$-\ln\lambda = G_N\left(\gamma_0 \sum_{i=1}^{N} y_i\right) - G_N(N),$$

where

$$G_N(x) = \begin{cases} x - N\ln x & \text{for } x > 0, \\ 0 & \text{for } x \le 0. \end{cases}$$

The exact cdf F_N of the Wilks statistics $-2\ln\lambda$ of the LR test of the hypothesis (2) has under the H_0 the form (see Theorem 6 in [Stehlík 01]):

$$F_N(\tau) = \begin{cases} \mathcal{F}_N\big(-NW_{-1}(-e^{-1-\frac{\tau}{2N}})\big) - \mathcal{F}_N\big(-NW_0\big(-e^{-1-\frac{\tau}{2N}}\big)\big), & \tau > 0, \\ 0, & \tau \le 0. \end{cases} \tag{4}$$

Here W_k is for $k = \cdots -1, 0, 1, \ldots$ the kth branch of the Lambert W function (see Appendix) and $\mathcal{F}_N$ is the cdf of the Gamma distribution with the shape parameter equals to $N = 1, 2, \ldots$ and scale parameter equals to 1.

The Wilks statistics $-2\ln\lambda$ of the LR test of the hypothesis (2) is under H_0 asymptotically χ_1^2-distributed [Wilks 67] and this test based on the asymptotics is oversized. The following table gives the oversizing of the asymptotical test for small samples. Here α is the size of the test given from the the Wilks asymptotics while $\alpha_{e,N}$ is the exact size of the same test. We calculate from the formula $\alpha_{e,N} = 1 - F_N(\chi^2_{\alpha,1})$, where $\chi^2_{\alpha,1}$ denotes $(1-\alpha)$-quantile of the asymptotical χ_1^2-distribution and F_N is given by the formula (4).

TABLE 1. The exact sizes $\alpha_{e,N}$.

$\alpha \backslash N$	1	2	3	4	5
0.00001	0.229211e-4	0.178154e-4	0.155230e-4	0.142351e-4	0.134204e-4
0.00002	0.445707e-4	0.347445e-4	0.303720e-4	0.279358e-4	0.264021e-4
0.00005	0.1070863e-3	0.838629e-4	0.736765e-4	0.680620e-4	0.645496e-4
0.0001	0.2073771e-3	0.1630849e-3	0.1439040e-3	0.1334225e-3	0.1268967e-3
0.0002	0.4007327e-3	0.3167122e-3	0.2808525e-3	0.2614285e-3	0.2493896e-3
0.0005	0.9536900e-3	0.7598856e-3	0.6789730e-3	0.6356518e-3	0.6089504e-3
0.001	0.18315866e-2	0.14706397e-2	0.13227325e-2	0.12442198e-2	0.11960162e-2
0.002	0.35061415e-2	0.28417427e-2	0.25749120e-2	0.24344361e-2	0.23484946e-2
0.005	0.82247349e-2	0.67718253e-2	0.62044830e-2	0.59087828e-2	0.57286067e-2
0.01	0.015599286	0.013037809	0.012058871	0.011552053	0.011244013
0.02	0.029448482	0.025065314	0.023424550	0.022579936	0.022067611
0.05	0.067701923	0.059361294	0.056314364	0.054754992	0.053810812

$\alpha\backslash N$	10	20	30	40	50
0.00001	0.117232e-4	0.108587e-4	0.105711e-4	0.104276e-4	0.103418e-4
0.00002	0.232203e-4	0.216047e-4	0.210675e-4	0.207998e-4	0.206392e-4
0.00005	0.573068e-4	0.536422e-4	0.524229e-4	0.518152e-4	0.514510e-4
0.0001	0.1134937e-3	0.1067253e-3	0.1044757e-3	0.1033529e-3	0.1026806e-3
0.0002	0.2247564e-3	0.2123424e-3	0.2082143e-3	0.2061548e-3	0.2049209e-3
0.0005	0.5545638e-3	0.5272108e-3	0.5181143e-3	0.5135745e-3	0.5108540e-3
0.001	0.10981231e-2	0.10489474e-2	0.10325913e-2	0.10244260e-2	0.10195322e-2
0.002	0.21744048e-2	0.20870295e-2	0.20579594e-2	0.20434441e-2	0.20347421e-2
0.005	0.53646073e-2	0.51820403e-2	0.51212668e-2	0.50909105e-2	0.50727079e-2
0.01	0.010622626	0.010311006	0.010207222	0.010155366	0.010124268
0.02	0.021035190	0.00517344	0.020344780	0.020258532	0.020206799
0.05	0.051909321	0.050954881	0.050636560	0.050477398	0.050381907

The hypothesis testing problem (2) has in the full exponential family [Barndorff-Nielsen 78] nice statistical properties, let us mention the UUMP property. For more extensive discussion about the other properties see [Johansen 79] and [Lehmann 64]. The interesting property of the LR test of the hypothesis (2) in the exponential case is its asymptotical optimality in the sense of the Bahadur exact slopes (see [Bahadur 65]), which can be proved using the Theorem 4.1 in [Rublík 89,b].

3. Observations having the Erlang distribution

The family of density functions

$$f(y|\,\gamma) = \frac{\gamma(\gamma y)^{k-1}}{(k-1)!}e^{-\gamma y}, \qquad y>0\,, \quad k=2,3,\dots\,,$$

is referred to as the family of Erlang (k,γ) distributions. The length of time interval required in order to collect k arrivals from a Poisson process is distributed according to the Erlang distribution with the shape parameter k. The family of Erlang distributions with the known shape parameter $v_i = k > 1$ and an unknown scale parameter γ belongs to the model (1). Erlang distribution is frequently used in queueing systems (see [Kleinrock 75]). The distribution is named after Agner Krarup Erlang (see `http://www-groups.dcs.st-and.ac.uk/history/Mathematicians/Erlang.html`), a Danish mathematician.

In this section we consider the LR test of the hypothesis (2) in the particular case of the model (1) when the observations are Erlang (k,γ) distributed. The test statistics $-\ln\lambda$ of the LR test of the hypothesis (2) has the form

$$-\ln\lambda = G_{kN}\left(\gamma_0\sum_{i=1}^{N} y_i\right) - G_{kN}(kN)\,.$$

Theorem 1. *The exact cdf of the Wilks statistics $-2\ln\lambda$ of the* LR *test of the scale hypothesis* (2) *of the Erlang* (k,γ) *distribution has under the H_0 the form*

$$F_N(\rho) = \begin{cases} \mathcal{F}_{kN}\Big(-kNW_{-1}\big(-e^{-1-\frac{\rho}{2kN}}\big)\Big) - \mathcal{F}_{kN}\Big(-kNW_0\big(-e^{-1-\frac{\rho}{2kN}}\big)\Big)\,, & \rho > 0\,, \\ 0\,, & \rho \le 0\,. \end{cases}$$

Proof. Theorem 1 is a corollary of Theorem 4 in [Stehlík 01]. □

The Wilks statistics $-2\ln\lambda$ of the LR test of the hypothesis (2) of the Erlang distribution is under H_0 asymptotically χ_1^2-distributed [Wilks 67] and the test based on this asymptotics is oversized. The LR test of the hypothesis (2) is UUMP.

4. Illustrative example

In this section we illustrate the discussed methods of the exact testing and analyze the data from the case study provided by [Gautam 99]. Gautam studied a typical small scale company which manufactures a single item on demand and according to the specifications of the user. The products are made one after the other. Assume that the users can demand only a single unit of the product. As soon as a demand for a customized item comes in, the company registers the requirements into a database. The processing begins by scanning the database according to a first come first served manner and selecting an item to be processed. Once the processing is complete, the next item's processing begins. Due to the customization of very similar items, the processing times are random variables with very small variation. Table 2 lists the sample processing times in hours.

Table 2. The sample processing times.

1.9174	1.0866	0.9324	1.7576	0.5757	1.3961	1.3726	1.1121	1.3469	1.7477
1.2796	1.5442	1.3416	1.8200	1.0853	1.4793	0.9549	0.8739	1.1211	1.9341
1.6733	1.5880	1.1860	1.3422	1.6634	1.6364	2.7820	1.3701	1.6089	1.5860
1.1534	1.3417	1.6009	1.6476	1.3069	1.4546	1.7063	1.4398	1.1611	1.1278
2.0458	1.3681	0.9886	1.9371	1.6347	1.3754	2.0093	1.2315	1.4088	1.0767
1.0636	1.8241	1.3653	1.3242	1.5804	1.3875	1.2286	1.2866	1.7617	1.4828
1.6378	0.8287	1.1559	1.1054	1.4677	1.1568	1.6271	1.3273	1.5050	1.4858
1.1695	2.3610	1.3828	2.0455	1.1538	1.0897	1.7877	1.2850	1.3557	2.1030
1.0352	1.6838	1.6230	0.8023	1.6138	1.7363	0.8154	1.1386	2.1088	1.0676
1.5356	1.7195	1.9940	1.7607	1.2447	1.4794	1.1470	1.4041	1.1489	1.7280

The following question is of prime concern for the company: Given the number of orders waiting to be processed, by what time should the company promise to deliver the product so that company meets the deadline with the probability p?

One of the probability distributions representing time random variable with relatively small variation is the Erlang distribution. To fit the processing time data as Erlang (k, γ) random variable we will estimate the parameters from the data and test whether the estimated distribution fits the data well. The maximum likelihood estimators (MLEs) $\hat{k}$ and $\hat{\gamma}$ of the parameters k and γ are given by

$$\hat{k} = \left\lfloor \frac{m^2}{s^2} \right\rfloor \quad \text{or} \quad \left\lfloor \frac{m^2}{s^2} \right\rfloor + 1 \qquad \text{and} \qquad \hat{\gamma} = \frac{m}{s^2}\,, \tag{5}$$

where m and s^2 denote the sample mean and sample variance and $\lfloor x \rfloor$ denotes the greatest integer less then or equal to x. The estimation (5) from the data in Table 2 gives MLEs $\hat{\gamma} = 11.1992$ and $\hat{k} = 16$ or 17.

[Gautam 99] used the Kolmogorov-Smirnov (KS) test to test the hypothesis that the random variable for the processing time has an Erlang $(\hat{k}, \hat{\gamma})$ distribution. The KS critical value $C = D\big(\sqrt{N} + 0.12 + \frac{0.11}{\sqrt{N}}\big)$ is $C = 0.6002$ in the case of the Erlang (16, 11.1992) and $C = 1.1958$ in the case of the Erlang (17, 11.1992) distribution. Here D denotes the maximum $\max\big|F(x) - F_N(x)\big|$ where F is the cdf of an Erlang $(\hat{k}, \hat{\gamma})$ and F_N is the empirical distribution function based on the data in Table 2. The KS test states that the null hypothesis cannot be rejected at the level 0.05 if $C < 1.358$. This implies that both Erlang (16, 11.1992) and Erlang (17, 11.1992) null hypotheses cannot be rejected at the level 0.05. For example in the case of the Normal distribution is $C = 2.7029$ and the null hypothesis of the normality is rejected at the level 0.05. We prefer to use Erlang $(16, 11.1992)$ as the distribution of the processing time random variable (the corresponding critical constant is smaller!).

Now we will discuss the promising a due date, e.g., by what time should the company promise to deliver the product so that company meets the deadline with the probability p? Consider that there are m orders to be processed when a customer request comes in. Denote by $X_1, \dots, X_{m+1}$ the processing times all according to an Erlang (k, γ). Denote by Z the time for the $m+1$ jobs to be completed. We have $Z = X_1 + X_2 + \cdots + X_{m+1}$ and Z has Erlang $\big(k(m+1), \gamma\big)$ distribution. Our problem is, what time should the company promise to deliver the product so that it meets the deadline with the given probability p (let us say 0.99)? We have to determine time T such that $P\{Z \le T\} = 0.99$, i.e., we can find T from the equation

$$1 - e^{-\gamma T} \sum_{i=0}^{k(m+1)-1} \frac{(\gamma T)^i}{i!} = 0.99\,. \tag{6}$$

In the case of Erlang (16, 11.1992) distributed processing times (the situation corresponding to the dataset in the Table 2), $m = 4$ and $p = 0.99$ we obtain $T = 9.1315$ hours.

Assume that the processing time in the company has Erlang distribution with the known shape parameter $k = 16$ and an unknown scale parameter γ. The customer wants to know the time to deliver and in the case of delay larger than one hour the company will be penalized. To know the uncertainty of the penalization is of prime concern for the company and such information can be useful for construction of a portfolio of company. To determine the uncertainty of the penalization we want to test $H_0 \colon T = T_0$ versus $H_1 \colon T \neq T_0$ on the appropriate level of significance α and to determine the power of this test at the point of penalization $T_0 + 1$ of the alternative. Since for γ and T the equality (6) holds, we have the correspondence $\gamma = \gamma(T)$ and our testing problem is reduced to the test of the scale hypothesis

$$H_0 \colon \gamma = \gamma(T_0) \quad \text{versus} \quad H_1 \colon \gamma \neq \gamma(T_0). \tag{7}$$

The determination of the power of the KS test of the hypothesis (7) is very laborious and since the KS test is asymptotical, the obtained power is not accurate in small samples. Therefore we recommend to compute the exact power of the LR test of the hypothesis (7) by the use of the following theorem:

THEOREM 2. *The exact power $p(\gamma, \alpha)$ of the* LR *test based on the Wilks statistics of the hypothesis* (2) *of the Erlang (k, γ) distribution on the level α at the point γ of the alternative has the form*

$$p(\gamma, \alpha) = 1 - \mathcal{F}_{kN}\Big(-kN\frac{\gamma}{\gamma_0}W_{-1}\big(-e^{-1-\frac{c_{\alpha,N}}{2kN}}\big)\Big) + \mathcal{F}_{kN}\Big(-kN\frac{\gamma}{\gamma_0}W_0\big(-e^{-1-\frac{c_{\alpha,N}}{2kN}}\big)\Big), \tag{8}$$

where $c_{\alpha,N}$ denotes the critical value of the exact test of the hypothesis (2) *on the level α.*

P r o o f. The critical region based on the Wilks statistics of the LR test of the hypothesis (2) on the level of significance α has the form

$$W_c = \{y \in Y \colon -2\ln\lambda(y) > c\} \quad \text{such that} \quad P\{W_c \mid \gamma = \gamma_0\} = \alpha, \tag{9}$$

where Y denotes the sample space. The power $p(\gamma_1, \alpha)$ of the test of the hypothesis (2) at the point γ_1 of the alternative is equal to $P\{W_c \mid \gamma = \gamma_1\}$, where W_c is defined by (9). Applying Theorem 4 in [Stehlík 01] we obtain in the case of the Erlang (k, γ) distributed observations the equality

$$\begin{aligned} 1 - P\{W_c \mid \gamma = \gamma_1\} = \mathcal{F}_{kN}\Big(-kN\frac{\gamma_1}{\gamma_0}W_{-1}\big(-e^{-1-\frac{c_{\alpha,N}}{2kN}}\big)\Big) \\ - \mathcal{F}_{kN}\Big(-kN\frac{\gamma_1}{\gamma_0}W_0\big(-e^{-1-\frac{c_{\alpha,N}}{2kN}}\big)\Big). \end{aligned}$$

This completes the proof. □

Let us illustrate this exact procedure in the case of the dataset listed in Table 2. The company wants to known the uncertainty about the penalization mentioned above. Let us start with the exact test of the hypothesis $H_0: T = 9.1315$ versus $H_1: T \neq 9.1315$ on the level of significance $\alpha = 0.05$. Since we have correspondence $\gamma = \gamma(T)$ and $11.1992 = \gamma(9.1315)$ and $10.0938 = \gamma(10.1315)$ holds, the corresponding scale testing problem is

$$H_0: \gamma = 11.1992 \quad \text{versus} \quad H_1: \gamma \neq 11.1992\,. \tag{10}$$

The value of the Wilks statistics of the LR test of the hypothesis (10) is $-2\ln\lambda = 0.14981$ and the corresponding p value is 0.69873. Therefore the null hypothesis cannot be rejected at the 0.05 significance level. To determine the uncertain of penalization we evaluate the exact power $p(10.0938, 0.05)$ (see (8)) at the point $\gamma = 10.0938$ of the alternative, corresponding to the time of penalization $T_0 + 1 = 10.1315$. We obtain $p(10.0938, 0.05) = 0.9855$. Since the power of our test is high at the point corresponding to the penalization time, the probability of penalization is small. The power function $p(\gamma, 0.05)$ of the LR test of the hypothesis (10) is for $\gamma \in (7, 15)$ displayed in the Figure 1, the critical value $c_{0.05,100}$ of the exact LR test of the hypothesis (10) is 3.8421 (see (8)).

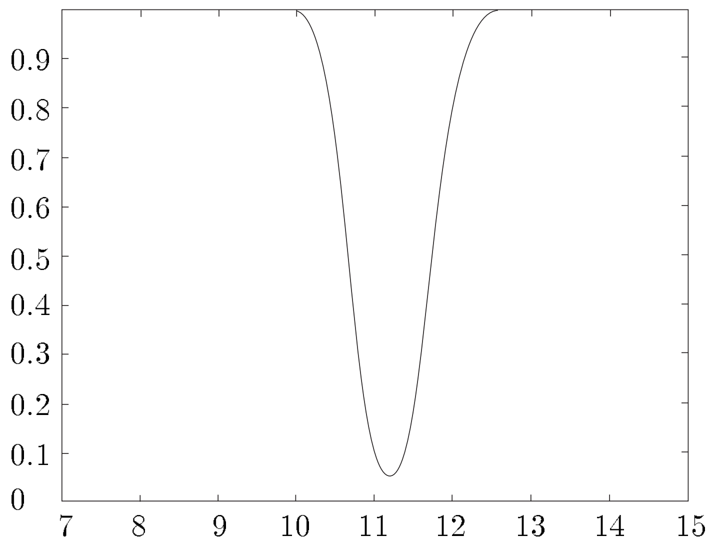

FIGURE 1

5. Appendix

The Lambert W function is defined to be the multivalued inverse of the complex function $f(y) = ye^y$. As the equation $ye^y = z$ has an infinite number of solutions for each (non-zero) value of $z \in \mathbb{C}$, the Lambert W has an infinite number of branches. Exactly one of these branches is analytic at 0. Usually this branch is referred to as the principal branch of the Lambert W and is denoted by W or W_0. The other branches all have a branch point at 0. These branches are denoted by W_k where $k \in \mathbb{Z} \setminus \{0\}$. The principal branch and the pair of branches W_{-1} and W_1 share an order 2 branch point at $z = -e^{-1}$. The branch cut dividing these branches is the subset of the real line from $-\infty$ to $-e^{-1}$ and the values of the branches of the Lambert W on this branch cut are assigned using the rule of counter-clockwise continuity around the branch point. This means that W is real-valued for $z \in (-e^{-1}, \infty)$ and the branch W_{-1} is real-valued on the interval $(-e^{-1}, 0)$. For all the branches other than the principal branch the branch cut dividing them is the negative real axis. The branches are numbered up and down from the real axis. A detailed discussion of the branches of the Lambert W can be found in [Corless96].

Since the Lambert W function has many applications in pure and applied mathematics, the branches of the LW function are implemented in many mathematical computational softwares, e.g., the Maple, Matlab, Mathematica and Matcad. For more information about the implementation and some computational aspects see [Corless 93].

REFERENCES

[Barndorff-Nielsen 78] BARNDORF-NIELSEN, O. E.: *Information and Exponential Families In Statistical Theory*, John Wiley & Sons, Ltd., New York, 1978, pp. 111–115.

[Bahadur 65] BAHADUR, R. R.: *An optimal property of the likelihood ratio statistic*, in: Proc. 5th Berkeley Symp.Math. Stat. Probab., Vol. 1 (L. LeCam and J. Neyman, eds.), Berkeley and Los Angels, Univ. of California Press, Vol. 1, 1965, pp. 13–26.

[Bahadur 67] BAHADUR, R. R.: *Rates of convergence of estimates and test statistics*, Ann. Math. Statist. **38** (1967), 303–324.

[Corless 96] CORLESS, R. M.—GONNET, G. H.—HARE, D. E. G.—JEFFREY, D. J.—KNUTH, D. E.: *On the Lambert W function*, Adv. Comput. Math. **5** (1996), 329–359.

[Corless 93] CORLESS, R. M.—GONNET, G. H.—HARE, D. E. G.—JEFFREY, D. J.—KNUTH, D. E.: *Lambert's W function in maple*, MTN **9** (1993), 12–22.

[Gaudoin-Soler 92] GAUDOIN, O.—SOLER, J. L.: *Statistical analysis of the geometric de-eutrophication software reliability model*, IEEE Trans. Reliab. **41** (1992), 518–524.

[Gautam 99] GAUTAM, N.: *Erlang Distribution: Description and Case Study*, Industrial Engineering Applications and Practice: Users Encyclopedia, 1999.

[Johansen 79] JOHANSEN, S: *Introduction to the Theory of Regular Exponential Families*, Lecture Notes **3**, 1979, pp. 94.

[Kleinrock 75] KLEINROCK, L.: *Queueing Systems*, John Wiley & Sons, Toronto, Vol 1, 1975, pp. 71–72, pp. 119–134.

[Lehmann 64] LEHMANN, E. L.: *Testing Statistical Hypotheses*, John Wiley & Sons, New York, 1964.

[Pázman 93] PÁZMAN, A.: *Nonlinear statistical Models*, Math. Appl., Kluwer Acad. Publ., Dordrecht, 1993, chapters 9.1 and 9.2.

[Pázman 96] PÁZMAN. A.: *The Density of the Parameter Estimators when the Observations are Distributed Exponentially*, Metrika **44** (1996), 9–26.

[Rublík 89, a] RUBLÍK, F.: *On optimality of the* LR *tests in the sense of exact slopes, Part 1, General case*, Kybernetika **25** (1989), 13–25.

[Rublík 89, b] RUBLÍK, F.: *Appl. Individual Distributions*, Kybernetika **25** (1989), 117–135.

[Rublík 96] RUBLÍK, F.: *Some tests on exponential populations*, Tatra Mt. Math. Publ. **7** (1996), 229–235.

[Stehlík 00] STEHLÍK, M.: *On some exact densities in an exponential family*, in: ROBUST 2000 (J. Antoch and G. Dohnal, eds.), JČMF, Praha, 2001, pp. 298–307.

[Stehlík 03] STEHLÍK, M.: *Distributions of exact tests in the exponential family*, Metrika **57** (2003), 145–164.

[Wilks 67] WILKS, S. S.: *Mathematical Statistics*, Nauka, Moscow, 1967, 410–419. (In Russian)

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