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FUNCTIONAL REPRESENTATIONS AND UNIVERSALS FOR MV- AND GMV-ALGEBRAS

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Dedicated to Ján Jakubík on his 80th birthday, as a tribute to his work and influence

ABSTRACT. We represent every normal-valued GMV-algebra as a GMV-algebra of real-valued functions; we also describe the universal MV-algebras and universal normal-valued GMV-algebras with a prescribed set of components.

1. Introduction

Łukasiewicz infinite valued propositional logic is one of the important ingredients involved in the theory of fuzzy sets. It has truth values in the real interval $[0,1]$. This interval is a (linearly) ordered MV-algebra; it is obtained from the additive ordered group $\mathbb{R}$ (with 1 as strong order unit) using truncated addition (at 1). In general, every MV-algebra M can be obtained in this manner from an Abelian lattice-ordered group with strong order unit [Mu]. The fundamental significance of the MV-algebra $[0,1]$ is underlined by Chang's Completeness Theorem ([CDM], Theorem 2.5.3): an MV-equation holds in every MV-algebra iff it holds in $[0,1]$. Moreover, by Di Nola's Representation Theorem ([CDM], Theorem 9.5.1), every MV-algebra can be represented as an algebra of $[0,1]^*$-valued functions over some set, where $[0,1]^*$ is an ultrapower of $[0,1]$. GMV-algebras have been recently introduced as non-commutative generalizations of MV-algebras. They can be viewed as models of an algebraic semantics

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that is a non-commutative generalization of multi-valued reasoning. The corresponding logic then lies between Łukasiewicz infinite valued logic and bilinear propositional logic ([Ra2]). Dvurečenskij ([Dv1]) essentially extended Mundici's theorem proving that every GMV-algebra is analogously obtainable from a (not necessarily Abelian) lattice-group with a strong order unit. Because of the importance of the real interval $[0,1]$ for MV-algebras and fuzzy logics, a natural question arises:

Is it possible to represent every GMV-algebra as a GMV-algebra of real-valued functions?

In this paper we provide a positive answer for normal-valued GMV-algebras (including all MV-algebras).

In [CHH], universals were obtained for Abelian lattice-ordered groups (with a fixed set of components) using generalized valuation (real-valued function) groups. In [HMc], universals were obtained for transitive ℓ-permutation groups (with a fixed set of primitive components). In [GW], the rooted valuation product and rooted Wreath product constructions were introduced and developed. The latter provides universals (which are real-valued function groups) in the intransitive case when the ℓ-permutation group is normal-valued and extends the results in [HMc] to this setting; the former is a re-examination of the universals for Abelian lattice-ordered groups and locates them in appropriate rooted Wreath products. In this article we adapt the constructions of [GW] when strong order units are present and obtain constructions of universals for (Abelian) MV-algebras and normal-valued GMV-algebras (with fixed sets of primitive components). Both are groups of real-valued functions.

To make the paper self-contained, we reproduce the pertinent portions of [GW].

2. Universals for Abelian MV-algebras

2.1 Background

Throughout we will consider groups with partial orders defined on them which are compatible with the group operation (i.e., $xfy \leq xgy$ whenever $f \leq g$). If the partial order is total, we call the structure an *ordered group* or *o-group* for short; if the order is a lattice (the least upper bound ($\vee$) and greatest lower bound ($\wedge$) exist for any pair of elements), then we call the structure a *lattice-ordered group* or *ℓ-group* for short. In all these cases, we will write G^+ for $\{g \in G \colon g \geq 1\}$ and G_+ for $G^+ \setminus \{1\}$. If G is an ℓ-group, let $|g| = g \vee g^{-1}$. Then $|g| \in G^+$ for all $g \in G$ and $|g| = 1$ iff $g = 1$ [G2], Lemma 2.3.8. We call an element $u \in G^+$ a *strong order unit* if the convex subgroup generated by u is equal to G. Equivalently, for each $g \in G$, there is $n \in \mathbb{Z}_+$ such that $|g| \leq u^n$.

Throughout, we will use the standard abbreviation ℓ-subgroup for sublattice subgroup.

MV-algebras were introduced by Chang [Ch] (see also [CDM]) as an algebraic counterpart of the Łukasiewicz infinite valued propositional logic. Recently, the second author [R1], and independently Georgescu and Iorgulescu [GI1], [GI2], have introduced equivalent non-commutative generalizations of MV-algebras, which, following [R1], we call GMV-algebras. These have been extensively studied by Jakubík ([J1]–[J7]) and are also called pseudo MV-algebras.

A *GMV-algebra* M is a monoid $(M;\oplus,0)$ with two unary operations $\neg$ and $\sim$ and one nullary operation 1 satisfying the conditions:

(where $a\odot b$ is defined as $\sim(\neg a\oplus\neg b)$)

$a\oplus 1=1=1\oplus a$, $\neg(\sim a\oplus\sim b)=\sim(\neg a\oplus\neg b)$,

$a\oplus(b\odot\sim a)=b\oplus(a\odot\sim b)=(\neg b\odot a)\oplus b=(\neg a\odot b)\oplus a$,

$(\neg a\oplus b)\odot a=b\odot(a\oplus\sim b)$, $\sim\neg a=a$.

If M is a GMV-algebra, $n\in\mathbb{Z}_+$ and $a\in M$, let $\oplus_1\, a=a$ and $\oplus_{n+1}\, a=(\oplus_n\, a)\oplus a$.

If we put $a\le b$ iff $\neg a\oplus b=1$ then $(A;\le)$ is a bounded distributive lattice (0 is the least element and 1 is the greatest) with $a\vee b=a\oplus(b\odot\sim a)$ and $a\wedge b=a\odot(b\oplus\sim a)$.

If the monoid $(M;\oplus,0)$ is commutative, then the unary operations $\neg$ and $\sim$ coincide and exactly such GMV-algebras are MV-algebras.

MV- and GMV-algebras are closely related to Abelian ℓ-groups and ℓ-groups. If G is an Abelian ℓ-group, $0<g\in G$ and $[0,g]=\{a\in G\colon 0\le a\le g\}$, then for any $a,b\in[0,g]$, let $a\oplus b=(a+b)\wedge g$, $\neg a=g-a$ and $\sim a=-a+g$. Then the monoid $\big([0,g];\oplus,0\big)$ with unary operations $\neg$ and $\sim$ and nullary operation g is a GMV-algebra (and is an MV-algebra provided G is Abelian); we denote it $\mathcal{M}(G,g)$.

Conversely, Mundici [Mu] proved that if M is any MV-algebra then there is an Abelian ℓ-group G and a strong order unit $u\in G$ such that M is isomorphic to $\mathcal{M}(G,u)$, and that such G and u are unique up to isomorphism. So we will denote by $\mathcal{G}(M)$ any of these ℓ-groups with strong unit. Recently, Dvurečenskij [Dv1] extended these results to GMV-algebras and ℓ-groups (which need not be Abelian) with strong units. The denotations $\mathcal{G}(M)$ and $\mathcal{M}(G,u)$ will be used in the same sense as for the commutative cases.

If M is an MV- or GMV-algebra and $\emptyset\ne I\subseteq M$, then I is called an *ideal* of M if it is closed under $\oplus$ and if $b\le a$ implies $b\in I$ for any $b\in M$ and $a\in I$. An ideal I of M is called *normal* if $\neg a\odot b\in I$ iff $b\odot\sim a\in I$ for any $a,b\in M$. Let G be an ℓ-group with strong order unit u. Denote by $\mathcal{C}(G)$ the set of convex ℓ-subgroups of G and by $\mathcal{C}\big(\mathcal{M}(G,u)\big)$ the set of ideals of $\mathcal{M}(G,u)$. Both $\mathcal{C}(G)$ and $\mathcal{C}\big(\mathcal{M}(G,u)\big)$ are complete lattices with respect to set-inclusion and the mapping $\varphi\colon \mathcal{C}(\mathcal{M}(G,u)) \longrightarrow \mathcal{C}(G)$ given by $\varphi\colon I \longmapsto \{g \in G\colon |g| \wedge u \in I\}$ is a lattice isomorphism of $\mathcal{C}(\mathcal{M}(G,u))$ onto $\mathcal{C}(G)$. (For MV-algebras see [CT], for GMV-algebras see [R2].) Let $\mathcal{I}(G)$ be the set of ℓ-ideals (i.e., normal convex ℓ-subgroups) of G and $\mathcal{I}(\mathcal{M}(G,u))$ the set of normal ideals of $\mathcal{M}(G,u)$. The restriction of φ to $\mathcal{I}(\mathcal{M}(G,u))$ is an isomorphism between the complete lattices $\mathcal{I}(\mathcal{M}(G,u))$ and $\mathcal{I}(G)$ (see [Dv2]).

We provide examples at the end of each subsection which the reader is encouraged to consider while reading the definitions and proofs of the theorems; they should help clarify matters.

2.2 MV-algebras associated with Abelian o-groups

Let Γ be a totally ordered set and $\{R_\gamma\colon \gamma \in \Gamma\}$ a family of subgroups of the additive o-group $\mathbb{R}$ of all real numbers. Let F be the additive group of all $f\colon \Gamma \to \mathbb{R}$ such that $f(\gamma) \in R_\gamma$ for all $\gamma \in \Gamma$. For each $f \in F$, let $\operatorname{supp}(f)$, *the support of* f, be the set $\{\gamma \in \Gamma\colon f(\gamma) \neq 0\}$. Let $V = V(\Gamma, \{R_\gamma\colon \gamma \in \Gamma\})$ be the set of all $f \in F$ such that every non-empty subset of $\operatorname{supp}(f)$ has a maximal element. Then V is a subgroup of F. It is an Abelian o-group where $f < g$ iff $f(\beta) < g(\beta)$ where β is the greatest element of $\operatorname{supp}(g-f)$.

If Γ has a maximal element γ_0, then any element $u \in V$ with $u(\gamma_0) > 0$ is a strong order unit. This follows immediately since $\mathbb{R}$ is Archimedean: so there is $n \in \mathbb{Z}_+$ with $g(\gamma_0) < nu(\gamma_0)$, whence $g < nu$.

If G is an Abelian o-group with a strong order unit u, then the set of convex subgroups forms a chain under inclusion [G2], Lemma 3.1.2. Thus if $g \in G \setminus \{0\}$, then there is a unique convex subgroup of G that is maximal with respect to not containing g. It is called the *value* of g and will be denoted by V_g. The intersection of all convex subgroups of G that contain g and V_g is a convex subgroup of G denoted by V_g^*; the pair (V_g, V_g^*) is called a *covering pair.* Note that $V_g = V_{-g}$ and that the convex subgroup of G generated by $|g|$ is V_g^*; i.e., $|g|$ is a strong order unit for V_g^*. Note that V_u is the maximal proper convex subgroup of G and $V_u^* = G$. Let $\Gamma(G)$ denote the set of all covering pairs totally ordered by inclusion. In 1901, Hölder [Ho] proved that V_g^*/V_g is isomorphic to an additive subgroup R_γ of $\mathbb{R}$, and that this isomorphism preserves the natural orders. In 1907, H. Hahn [Ha] obtained the crucial representation that (in modern terminology) every Abelian o-group is a group of functions; indeed, if G is an Abelian o-group, then G can be embedded in $V = V(\Gamma(G), \{\bar{R}_\gamma\colon \gamma \in \Gamma\})$ where $\bar{R}_\gamma$ is the divisible closure of R_γ in $\mathbb{R}$. Moreover, this embedding preserves order and the strong order unit, where we identify u with the function with $u(\gamma) = 0$ if $\gamma \neq (V_u, V_u^*)$, and $u(\gamma)$ is the image of $u + V_u$ in R_γ if $\gamma = (V_u, V_u^*)$.

Now let M be an MV-algebra and $g \in M \setminus \{0\}$. Assume that the associated Abelian ℓ-group $G = \mathcal{G}(M)$ is an o-group with strong order unit u. So V_g^*

is an Abelian o-group with strong order unit g. Let $M(g)$ be the MV-algebra associated with (V_g^*, g). Then $M(g)$ is isomorphic (as an MV-algebra) to the MV-algebra (M, g) which we also denote by $M(g)$. That is, we regard $M(g)$ as both $\mathcal{M}(V_g^*, g)$ and $\{m \in M\colon m \leq g\}$ with truncated addition at g. Let $\mathfrak{o}$ be the maximal ideal of $M(g)$ and $R(g)$ be the MV-algebra $M(g)/\mathfrak{o}$. Let $(\mathcal{G}(g), g)$ be the Archimedean o-group $\mathcal{G}(R(g))$ with strong order unit $g/\mathfrak{o}$. That is, if V_g is the value of g, then $V_g^*/V_g = \mathcal{G}(g)$ and $R(g) = \mathcal{M}(\mathcal{G}(g),\ g + V_g)$.

If $f, g \in M$, then define $f \sim g$ if there are $m, n \in \mathbb{Z}_+$ such that $f \leq \oplus_m g$ and $g \leq \oplus_n f$. Then $\sim$ is an equivalence relation on M that is preserved under $\oplus$. If $f \sim u$, then u is the largest element of the equivalence class of f; if $0 \neq f \not\sim u$, then the equivalence class of f has no largest element.

We therefore define $\mathcal{R}(g) = \mathcal{M}(\mathcal{G}(g), u)$ if $g \sim u$, and $\mathcal{R}(g) = \mathcal{G}(g)$ if $g \not\sim u$. So $\mathcal{R}(g)$ is an Archimedean o-group if $g \not\sim u$, and $\mathcal{R}(g)$ is an MV-algebra (with u as top element) if $g \sim u$. Also, $\mathcal{R}(f) = \mathcal{R}(g)$ if $f \sim g$. Let $\Gamma(M)$ be the set of equivalence classes. Then $\Gamma(M)$ inherits a natural total order by $[f] < [g]$ iff $f < g$ and $f \not\sim g$; it has maximal element $[u]$. We call $\{\mathcal{R}(g)\colon [g] \in \Gamma(M)\}$ the *set of components* of the MV-algebra M.

Now let M be an MV-algebra and let $G = \mathcal{G}(M)$ with strong order unit u. By Hahn's Theorem, G can be embedded in $V(\Gamma(G), \{\bar{R}_\gamma : \gamma \in \Gamma(G)\})$ where $\bar{R}_\gamma$ is the divisible closure (in $\mathbb{R}$) of R_γ. Clearly $\Gamma(M) = \Gamma(G)$. Moreover, $\mathcal{R}(g) = R_\gamma$ if $[g] = \gamma \neq [u]$, and $\mathcal{R}(u) = \mathcal{M}(\mathcal{G}(u), u)$ is the MV-algebra associated with $[u]$. In this context, we write $\bar{\mathcal{R}}(u)$ for $\mathcal{M}(\bar{\mathcal{G}}(u), u)$, where $\bar{\mathcal{G}}(u)$ is the divisible closure of $\mathcal{G}(u)$ in $\mathbb{R}$. That is,

Theorem 2.2.1. (Universals for MV-algebras associated with Abelian o-groups). *Let M be an MV-algebra with $\mathcal{G}(M)$ an Abelian o-group with strong order unit u. Let the components of M be $\{\mathcal{R}(\gamma)\colon \gamma \in \Gamma(M)\}$. Then M can be embedded* (as an MV-algebra) *in $\mathcal{M}(V, u)$ where V is the o-group $V(\Gamma(M), \{\bar{\mathcal{G}}(\gamma)\colon \gamma \in \Gamma(M)\})$ with strong order unit u. That is, $\mathcal{M}(V, u) = V(\Gamma(M), \{\bar{\mathcal{R}}(\gamma)\colon \gamma \in \Gamma(M)\})$.*

We close this subsection with some examples.

Example 1. Let $\Gamma = \{\gamma_1, \gamma_2\}$ with $\gamma_1 < \gamma_2$. Let $R_{\gamma_1} = \mathbb{R}$ and $R_{\gamma_2} = \mathbb{Z}$. Let $u(\gamma_j) = \delta_{2,j}$ ($j = 1, 2$). Then $\mathcal{M}(V, u)$ is the union of the sets $(\mathbb{R}_+, 0)$, $(\mathbb{R}^-, 1)$, and the components are the Archimedean o-group $\mathbb{R}$ and the two element MV-algebra $\{0, 1\}$. Any MV-algebra with these components can be embedded in $V(\Gamma, \{\mathbb{R}, \mathbb{Q} \cap [0,1]\}) = \mathcal{M}(V(\Gamma, \{\mathbb{R}, \mathbb{Q}\}), u)$.

Example 2. Let $\Gamma = \{\gamma_1, \gamma_2, \gamma_3\}$ with $\gamma_1 < \gamma_2 < \gamma_3$. Let $R_{\gamma_1} = \mathbb{Z}$, $R_{\gamma_2} = \mathbb{Q}$ and $R_{\gamma_3} = \mathbb{R}$. Let $u(\gamma_j) = \delta_{3,j}$ ($j = 1, 2, 3$). Then $\mathcal{M}(V, u)$ is the union of the sets $(\mathbb{Z}_+, 0, 0)$, $(\mathbb{Z}, \mathbb{Q}_+, 0)$, $(\mathbb{Z}, \mathbb{Q}, r)$ $(0 < r < 1)$, $(\mathbb{Z}, \mathbb{Q}_-, 1)$ and $(\mathbb{Z}^-, 0, 1)$, where $(\mathbb{Z}_+, 0, 0)$ is the set of all functions f such that $f(\gamma_1) \in \mathbb{Z}_+$, $f(\gamma_2) = 0 = f(\gamma_3)$, etc. The components here are $\mathbb{Z}$, $\mathbb{Q}$ and $\{r \in \mathbb{R} : 0 \le r \le 1\}$ (two o-groups and an Archimedean MV-algebra on top). The only restrictions that arise are the truncation at 0 and u. Any MV-algebra with these components can be embedded in $V\big(\Gamma, \{\mathbb{Q}, \mathbb{Q}, [0,1]\}\big) = \mathcal{M}\big(V\big(\Gamma, \{\mathbb{Q}, \mathbb{Q}, \mathbb{R}\}\big), u\big)$.

Example 3. Let $\Gamma = \mathbb{Z}_+$ with the reverse order: so $1 > 2 > \dots$. Let $R_n = \mathbb{R}$ for all $n \in \mathbb{Z}_+$. Let $u(n) = 0$ if $n > 1$ and $u(1) = 1$. Then the top component is the MV-algebra $\{r \in \mathbb{R} : 0 \le r \le 1\}$, and every other component is the Archimedean o-group $\mathbb{R}$. Thus $\mathcal{M}(V)$ is the union of the sets $(0, \dots, 0, \mathbb{R}_+, \mathbb{R}, \mathbb{R}, \dots)$, $(r, \mathbb{R}, \mathbb{R}, \dots)$ $(r \in (0,1))$, $(1, 0, 0, \dots)$, $(1, 0, \dots, \mathbb{R}_-, \mathbb{R}, \mathbb{R}, \dots)$. Moreover, any MV-algebra with these components can be embedded in $\mathcal{M}(V, u)$.

2.3 Rooted valuation groups

Now let Γ be a root system; i.e., a partially ordered set such that γ and δ have a common lower bound only if $\gamma \le \delta$ or $\delta \le \gamma$. For each $\gamma \in \Gamma$, let R_γ be a subgroup of $\mathbb{R}$. Let F be the additive group of all functions $f \colon \Gamma \to \mathbb{R}$ with $f(\gamma) \in R_\gamma$ for all $\gamma \in \Gamma$. For each $g \in F$, let $\operatorname{supp}(g)$ be defined as before. Let $V = V\big(\Gamma, \{R_\gamma : \gamma \in \Gamma\}\big)$ be the subgroup of all $g \in F$ such that every non-empty totally ordered subset of $\operatorname{supp}(g)$ has a maximal element. Then V is an Abelian ℓ-group where $g > 0$ iff $g(\delta) > 0$ for *every* maximal element δ of $\operatorname{supp}(g)$.

Let G be an Abelian ℓ-group. By Zorn's Lemma, if $g \in G \setminus \{0\}$, then there is a (not necessarily unique) convex sublattice subgroup of G that is maximal with respect to not containing g. It is called a *value* of g and will be denoted by V_g. The intersection of all convex ℓ-subgroups of G that contain g and V_g is a convex ℓ-subgroup of G denoted by V_g^*; the pair (V_g, V_g^*) is called a covering pair. For fixed $g \in G \setminus \{0\}$, let $\Gamma(g)$ be the set of all such pairs $\big(V_g, V_g^*\big)$ with V_g a value of g and V_g^* the cover of V_g. Now $\Gamma(g) = \Gamma(g \vee 0) \cup \Gamma(-g \vee 0)$ where $\Gamma(0) = \emptyset$ ([G2], Lemma 2.3.8). So $\Gamma(G) = \bigcup\big\{\Gamma(g) : g \in G_+\big\}$ is the set of *all* covering pairs. Partially order $\Gamma(G)$ by inclusion. Then $\Gamma(G)$ is a root system ([G2], Corollary 3.5.5). Hölder's proof applies and establishes that V_g^*/V_g is isomorphic to an additive subgroup of $\mathbb{R}$, and that this isomorphism preserves the natural orders.

A subset Λ of the root system $\Gamma(G)$ is called a *plenary subset* if (i) $\lambda \in \Lambda$ implies $\big\{\gamma \in \Gamma(G) : \gamma \ge \lambda\big\} \subseteq \Lambda$, and (ii) $\bigcap\big\{V_g : (V_g, V_g^*) \in \Lambda\big\} = \{0\}$.

In 1963, Conrad, Harvey and Holland [CHH] extended Hahn's Theorem and proved that every Abelian ℓ-group is a group of functions; indeed, if G is an Abelian ℓ-group and $\Gamma'(G)$ is a plenary subset of $\Gamma(G)$, then G can be

ℓ-embedded in $V = V\big(\Gamma'(G), \{\bar{R}_\gamma \colon \gamma \in \Gamma'\}\big)$ (an embedding that preserves the group and lattice operations), where $\bar{R}_\gamma$ is the divisible closure of R_γ in $\mathbb{R}$.

We can obtain the Conrad-Harvey-Holland representation using rooted valuation products.

Let Γ be a root system and $V = V\big(\Gamma, \{R_\gamma \colon \gamma \in \Gamma\}\big)$ be as above.

Let Δ be a maximal totally ordered subset of Γ. Since Γ is a root system, if $\gamma \geq \delta \in \Delta$, then $\gamma \in \Delta$. Let

$$K(\Delta) = \big\{g \in V \colon \Delta \cap \operatorname{supp}(g) = \emptyset\big\}.$$

Then $K(\Delta)$ is a convex ℓ-subgroup of V. Now $V/K(\Delta)$ is an ℓ-group under the naturally induced order: $K(\Delta) + f < K(\Delta) + g$ iff ($\Delta \cap$ $\operatorname{supp}(f-g) \neq \emptyset$ and $f(\delta) < g(\delta)$ where δ is the maximal element of $\Delta \cap$ $\operatorname{supp}(f-g)$). Let $\nu(\Delta)$ be the natural ℓ-surjection from V onto $V/K(\Delta)$. Clearly, $V/K(\Delta)$ is naturally ℓ-isomorphic to the Hahn group, $V(\Delta) = V\big(\Delta, \{R_\delta \colon \delta \in \Delta\}\big)$. Call this ℓ-isomorphism $\phi(\Delta)$. So $\psi(\Delta) := \phi(\Delta)\nu(\Delta) \colon V \to V(\Delta)$ is an ℓ-surjection.

Let $\mathfrak{M}$ be the set of all totally ordered maximal subsets of Γ. Then we can map V into $V(\mathfrak{M})(\sharp) = \prod\limits_{\Delta \in \mathfrak{M}} V(\Delta)$ using the $\psi(\Delta)$'s in the natural way: $\psi(g)_\Delta = \psi(\Delta)(g)$. Then ψ is an ℓ-homomorphism where $w \in V(\mathfrak{M})(\sharp)^+$ iff $(\forall \Delta \in \mathfrak{M})(w_\Delta \geq 0)$.

If $g \in \bigcap\big\{K(\Delta) \colon \Delta \in \mathfrak{M}\big\}$, then $\operatorname{supp}(g) = \emptyset$ (whence $g = 0$); thus ψ is an ℓ-embedding of V into $V(\mathfrak{M})(\sharp)$.

This construction is akin to writing V as a subdirect product of o-groups and then using Hahn's Theorem for each. In that sense, it is wasteful, and we tighten it by using the compatibility condition

$$\psi(g)_{\Delta_1}(\delta) = \psi(g)_{\Delta_2}(\delta) \qquad \forall \Delta_1, \Delta_2 \in \mathfrak{M} \quad \text{and} \quad \delta \in \Delta_1 \cap \Delta_2 . \tag{$*$}$$

Consider $V(\mathfrak{M})$, the set of all elements of $V(\mathfrak{M})(\sharp)$ that enjoy property $(*)$. Then $V(\mathfrak{M})$ is an ℓ-subgroup of $V(\mathfrak{M})(\sharp)$ that contains $\psi(V)$.

We call $V(\mathfrak{M})$ the *rooted valuation product of* V.

Moreover, for each element $w \in V(\mathfrak{M})$, $w_{\Delta_1}(\gamma) = w_{\Delta_2}(\gamma)$ for any $\Delta_1, \Delta_2 \in \mathfrak{M}$ to which γ belongs (by $(*)$). Thus we can obtain an element $w^\flat \colon \Gamma \to \mathbb{R}$. Since $\operatorname{supp}(w_\Delta)$ is inversely well-ordered for all $\Delta \in \mathfrak{M}$, the element $w^\flat$ corresponding to w belongs to V. Consequently, ψ maps V onto $V(\mathfrak{M})$; that is, V is ℓ-isomorphic to $V(\mathfrak{M})$. Hence we obtained in [GW]:

If Γ is a root system, then $V = \big(\Gamma, \{R_\gamma \colon \gamma \in \Gamma\}\big)$ is a rooted valuation product. Moreover:

Let G be an Abelian ℓ-group with values indexed by $\Gamma(G)$. Let Γ' be a plenary subset of $\Gamma(G)$. Then G can be regarded as an ℓ-subgroup of $V\big(\Gamma'(G), \{\bar{R}_\gamma \colon \gamma \in \Gamma'\}\big)$. We define $\mathcal{V}(G)$ to be $V(\mathfrak{M})$, where $\mathfrak{M}$ is the set of all maximal totally ordered subsets of $\Gamma'(G)$. Thus:

Every Abelian ℓ-group G is an ℓ-subgroup of a rooted valuation product $\mathcal{V}(G)$.

Now, let M be an MV-algebra and $G = \mathcal{G}(M)$ be the corresponding Abelian ℓ-group with strong order unit u. We can proceed similarly to the totally ordered case. If $g \in M$, then $u \notin V_g$ for any value $V_g \in \Gamma(g)$. Hence there is a value $V_u \in \Gamma(u)$ with $V_u \supseteq V_g$. So each maximal chain in $\Gamma(M)$ has a greatest element. We again take $R(g) = V_g^*/\mathfrak{o}$ and take the corresponding Archimedean o-group $(\mathcal{G}(g), g)$, etc. However, the rooted valuation product V does not have a strong order unit if u has infinitely many values: let w be any strictly positive element in the rooted valuation product having non-zero values at each component of u. Let $\Lambda = \{\gamma_n : n \in \mathbb{Z}_+\}$ be an infinite subset of $\Gamma(u)$. If $f(\gamma_n) = nw(\gamma_n)$, $f(\gamma) = u(\gamma)$ if $\gamma \in \Gamma(u) \setminus \Lambda$, and $f(\gamma) = 0$ if $\gamma \notin \Lambda$, then $f \nleq mw$ for any $m \in \mathbb{Z}_+$ (consider their values at γ_{m+1}). Hence w is not a strong order unit.

We therefore take the *u-restricted rooted valuation product* $\tilde{\mathcal{V}}(G)$; i.e., the convex (necessarily ℓ-)subgroup of $\mathcal{V}(G)$ generated by u. Then as G is generated as a convex ℓ-subgroup by u, the ℓ-embedding of G into $V = \mathcal{V}(G)$ will be an ℓ-embedding of G into $\tilde{\mathcal{V}}(G)$.

If we call the maximal components of an MV-algebra the set of MV-algebras $\mathcal{R}(\gamma)$ (γ ranging over an index set for $\Gamma(u)$) and all other components the Archimedean Abelian o-groups $\mathcal{R}(\gamma)$ (γ ranging over the complement in Γ of the index set for $\Gamma(u)$), then we obtain the MV-algebra analogue of the Conrad–Harvey–Holland Theorem for Abelian ℓ-groups.

THEOREM 2.3.1. (Universals for MV-algebras) *Every MV-algebra M can be embedded (as an MV-algebra) in $\mathcal{M}(\tilde{\mathcal{V}}(M), u)$ where $\tilde{\mathcal{V}}(M) = \tilde{\mathcal{V}}(\Gamma'(M), \{\bar{\mathcal{R}}(\gamma) : \gamma \in \Gamma(M)\})$ with strong order unit u and $\{\mathcal{R}(\gamma) : \gamma \in \Gamma'(M)\}$ is any plenary set of components of M.*

EXAMPLE 1. Let $\Gamma = \{\gamma_1, \gamma_2, \gamma_3\}$ be the root system with $\gamma_j < \gamma_3$ ($j = 1, 2$) and γ_1, γ_2 incomparable. Let $R_{\gamma_1} = \mathbb{R}$, $R_{\gamma_2} = \mathbb{Z}$ and $R_{\gamma_3} = \mathbb{Q}$. Let $G = V(\Gamma, \{R_\gamma : \gamma \in \Gamma\})$ and $M = \mathcal{M}(G, u)$ where $u(\gamma_1) = u(\gamma_2) = 0$ and $u(\gamma_3) = 1$. Then the components of G are $\mathbb{R}, \mathbb{Z}, \mathbb{Q}$ and those of M are $\mathbb{R}, \mathbb{Z}$ and $[0,1] \cap \mathbb{Q}$. There are two maximal chains in Γ, namely $\mathfrak{C}_1 = \{\gamma_1, \gamma_3\}$ and $\mathfrak{C}_2 = \{\gamma_2, \gamma_3\}$. These give valuation groups $V(\mathfrak{C}_1) = V(\mathfrak{C}_1, \{\mathbb{R}, \mathbb{Q}\})$ and $V(\mathfrak{C}_2) = V(\mathfrak{C}_2, \{\mathbb{Z}, \mathbb{Q}\})$ with strong order unit $(0,1)$. Hence we get $M(\mathfrak{C}_1) = V(\mathfrak{C}_1, \{\mathbb{R}, \mathbb{Q} \cap [0,1]\})$ and $M(\mathfrak{C}_2) = V(\mathfrak{C}_2, \{\mathbb{Z}, \mathbb{Q} \cap [0,1]\})$. Here $\tilde{\mathcal{V}}(G) = \mathcal{V}(G) = G$ with strong order unit u and $M = \mathcal{M}(G, u)$.

EXAMPLE 2. More generally, let $\mathfrak{D}_n$ ($n \in \mathbb{Z}_+$) be a family of chains and γ_0 be an extra element. Let $\Gamma = \{\gamma_0\} \cup \bigcup\limits_{n \in \mathbb{Z}_+} \mathfrak{D}_n$ with $\gamma_0 \geq \gamma$ for all $\gamma \in \Gamma$; thus $\mathfrak{C}_n = \{\gamma_0\} \cup \mathfrak{D}_n$ ($n \in \mathbb{Z}_+$) are the maximal chains for Γ. Let $\{R_\gamma : \gamma \in \Gamma\}$ be a family of subgroups of $\mathbb{R}$ with $u_0 \in (R_{\gamma_0})_+$. Let u be defined by $u(\gamma) = 0$ if $\gamma \neq \gamma_0$, and $u(\gamma_0) = u_0$. Let $G = V\big(\Gamma, \{R_\gamma \colon \gamma \in \Gamma\}\big)$. Then u is a strong order unit of G, and $\tilde{\mathcal{V}}(G) = \mathcal{V}(G) = G$. The components are $\bar{R}_\gamma$ ($\gamma \in \Gamma \setminus \{\gamma_0\}$) (all of which are components of $\bar{M} = \mathcal{M}\big(\mathcal{V}(G), u\big) = \mathcal{M}\big(V(\Gamma, \{\bar{\mathcal{R}}(\gamma) \colon \gamma \in \Gamma\})\big)$), and $\bar{R}_{\gamma_0}$ (which gives an MV-algebra top component $\mathcal{M}(\bar{R}_{\gamma_0}, u_0)$ of $\bar{M}$). Moreover, $\bar{M}$ is the resulting MV-algebra guaranteed by Theorem 2.3.1; it is universal for these components.

Example 3. Let G be as in the previous example and $H = G \oplus \mathbb{Z}$ with $H^+ = G^+ \oplus \mathbb{Z}^+$. Then H is an ℓ-group and has strong order unit $v = (u, 1)$. Then $\tilde{\mathcal{V}}(H) = \mathcal{V}(H) = \mathcal{V}(G) \oplus \mathbb{Z}$ and the associated MV-algebra is $\bar{M} \oplus \{0, 1\}$ with maximal element v (where $\bar{M}$ is given by the previous example). It is universal for MV-algebras with non-maximal components $\bar{R}_\gamma$ ($\gamma \neq \gamma_0$), and maximal components $\mathcal{M}(\bar{R}_{\gamma_0}, u_0)$ and $\{0, 1\}$.

We close with an easy example to illustrate how complicated things can become with even a simple example.

Example 4. Let M be the set of all real sequences s such that $s(n) \in [0, 1]$ for all $n \in \mathbb{Z}_+$. Let G be the additive group of all bounded real sequences and let $u \in G$ be the constant sequence 1. Then G is an ℓ-group under the pointwise ordering: $(f \vee g)(n) = \max\big\{f(n), g(n)\big\}$, etc., and has strong order unit u. Note that $M = \mathcal{M}(G, u)$. Among the values of u are $G(m)$, the set of the bounded sequences $g \in G$ with $g(m) = 0$ ($m \in \mathbb{Z}_+$). Also there are values of u which properly contain $\sum\limits_{n=1}^{\infty} \mathbb{R}$. For each $g \in G_+$ and value $V_g \in \Gamma(g)$, we have $V_g^*/V_g \cong \mathbb{R}$. All components of G are isomorphic to $\mathbb{R}$ as are all non-maximal components of M; all maximal components of M are $[0, 1]$. So $\mathcal{V}(G) = \big(V\big(\Gamma(M), \{\mathbb{R} \colon \gamma \in \Gamma(M)\}\big)\big)$ and $\tilde{\mathcal{V}}(M) = \tilde{\mathcal{V}}(G)$ is the ℓ-subgroup of all elements $f \in \mathcal{V}(G)$ for which there is $n_0 = n_0(f) \in \mathbb{Z}_+$ such that $\big|f(\gamma)\big| < n_0$ for all $\gamma \in \Gamma(u)$. Moreover, M is embedded in $\mathcal{M}\big(\tilde{\mathcal{V}}(M), u\big)$ as an MV-algebra (where u is the function with value 1 on all maximal elements of $\Gamma(M)$ and 0 on all other elements of $\Gamma(M)$). In this example, $\tilde{\mathcal{V}}(G) \neq \mathcal{V}$.

3. Rooted Wreath products

3.1 Background

Let $A(\Omega)$ denote the group of all order-preserving permutations of a totally ordered set $(\Omega, \leq)$; i.e., $A(\Omega) = \mathrm{Aut}(\Omega, \leq)$. Under the pointwise ordering, this group of functions (under composition) is an ℓ-group:

$$\alpha(f \vee g) = \max\{\alpha f, \alpha g\} \qquad \text{and} \qquad \alpha(f \wedge g) = \min\{\alpha f, \alpha g\},$$

where, as is standard in permutation groups, we write αf for the image of $\alpha \in \Omega$ under $f \in A(\Omega)$.

Let $A(\Omega)^+ = \{g \in A(\Omega)\colon (\forall \alpha \in \Omega)(\alpha g \geq \alpha)\}$.

Let $(\overline{\Omega}, \leq)$ denote the Dedekind completion of $(\Omega, \leq)$; that is, the set obtained by non-empty cuts with the inherited order (as in the construction of $(\mathbb{R}, \leq)$ from $(\mathbb{Q}, \leq)$). Each element of $A(\Omega)$ extends uniquely to an element of $A(\overline{\Omega})$ and we will identify $A(\Omega)$ with this corresponding ℓ-subgroup of $A(\overline{\Omega})$.

For $g \in A(\Omega)$, let $\operatorname{supp}(g) = \{\alpha \in \Omega\colon \alpha g \neq \alpha\}$, the *support* of g, and $\operatorname{Fix}(g) = \{\alpha \in \Omega\colon \alpha g = \alpha\} = \Omega \setminus \operatorname{supp}(g)$. If $\alpha \in \Omega$, let $\Delta(g, \alpha)$ be the interval in Ω that is the convexification of the orbit of α under g; so

$$\Delta(g, \alpha) = \{\beta \in \Omega\colon (\exists m, n \in \mathbb{Z})(\alpha g^m \leq \beta \leq \alpha g^n)\}.$$

If $\alpha g \neq \alpha$, then $\Delta(g, \alpha)$ is an open interval in Ω; otherwise it is a singleton.

Throughout, let G be an ℓ-subgroup of $A(\Omega)$; so $G^+ = G \cap A(\Omega)^+$. A non-empty convex subset X of Ω is called a *convex G-block* if $(\forall g \in G)(Xg = X$ or $Xg \cap X = \emptyset)$.

The convex G-block X is called an *extensive block* if for each $x, y, z \in X$, there are $f, g \in G$ such that $y \leq xf \in X$ and $z \geq xg \in X$.

The convex G-block X is called a *fat block* if $\{Xg\colon g \in G$ and $Xg > X\}$ has no least element (under $<$) and $\inf\big(\bigcup\{Xg\colon g \in G$ and $Xg > X\}\big) = \sup(X)$, and similarly with $<$ in place of $>$, where we write $X < Y$ iff $x < y$ for all $x \in X$, $y \in Y$ and take the supremum and infimum in $\overline{\Omega}$.

If X is a convex G-block, then $X^\sharp = \bigcup\{Xg\colon g \in G\}$ is a G-invariant set, and $\{Xg\colon g \in G\}$ partitions $X^\sharp$ into convex (in Ω) blocks which are fat (or extensive) if X is.

More generally, let $Y \subseteq \Omega$ be a G-invariant set and $\mathcal{C}$ be an equivalence relation on Y. If each $\mathcal{C}$-class is a fat or extensive block, then $\mathcal{C}$ is a congruence and we call it a *natural G-congruence on Y*, or a *natural partial G-congruence* (on Ω). So equivalence classes of natural partial G-congruences are convex.

We write $\operatorname{dom}(\mathcal{C})$ for the *domain* of the natural partial G-congruence $\mathcal{C}$; that is, all $\alpha \in \Omega$ such that $\alpha\ \mathcal{C}\ \beta$ for some $\beta \in \Omega$ (and so, all $\alpha \in \Omega$ such that $\alpha\ \mathcal{C}\ \alpha$). Note that $\alpha \in \operatorname{dom}(\mathcal{C})$ implies that $\alpha g \in \operatorname{dom}(\mathcal{C})$ for all $g \in G$. As is standard, we write $\alpha\mathcal{C}$ for $\{\beta \in \Omega\colon \alpha\ \mathcal{C}\ \beta\}$.

We will write $\mathcal{C} \subseteq \mathcal{D}$ if $\alpha\ \mathcal{D}\ \beta$ whenever $\alpha\mathcal{C}\beta$. This is clearly equivalent to $\alpha\mathcal{C} \subseteq \alpha\mathcal{D}$ for all $\alpha \in \operatorname{dom}(\mathcal{C})$.

Under this partial order ($\subseteq$), the set of natural partial G-congruences forms a root system ([Mc] or [G1], Theorem 3B[†]); so if $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ are natural partial G-congruences and $\mathcal{C}_1 \subseteq \mathcal{C}_2 \cap \mathcal{C}_3$, then $\mathcal{C}_2 \subseteq \mathcal{C}_3$ or $\mathcal{C}_3 \subseteq \mathcal{C}_2$. Moreover, the union and non-empty intersections of natural partial G-congruences are natural partial G-congruences.

As shown in [Mc] (or see [G1], Theorem 3C†), for any distinct $\alpha, \beta \in \Omega$ there are natural partial G-congruences $\mathcal{C} \subseteq \mathcal{C}^*$, such that $\alpha\, \mathcal{C}^*\, \beta \,\&\, \neg(\alpha\, \mathcal{C}\, \beta)$, where $\mathcal{C}$ and $\mathcal{C}^*$ are natural G-congruences on the same set $(\alpha\mathcal{C}^*)^{\sharp}$—whence G is transitive on the set of $\mathcal{C}^*$-classes—and no natural G-congruence on $(\alpha\mathcal{C}^*)^{\sharp}$ lies strictly between $\mathcal{C}$ and $\mathcal{C}^*$:

The intersection of all natural partial G-congruences in which α, β belong to the same class provides $\mathcal{C}^*$. $\mathcal{C}$ is obtained by using Zorn's Lemma: let $\Lambda = \mathrm{dom}(\mathcal{C}^*)$ and consider the set of all natural G-congruences on Λ (contained in $\mathcal{C}^*$) in which α and β belong to separate classes. This set is non-empty (it includes the natural G-congruence on Λ all of whose classes are singletons) and is closed under unions of chains. $\mathcal{C}$ is any maximal element thereof.

We write $\mathrm{val}(\alpha, \beta)$ for such a pair $(\mathcal{C}, \mathcal{C}^*)$ of natural partial G-congruences. It is further shown that if X is any $\mathcal{C}^*$-class, then G induces a permutation action on X as follows:

Let $G_{\{X\}} = \{g \in G \colon Xg = X\}$, an ℓ-subgroup of G (convex in G under the pointwise ordering). Let

$$L(X, G) = \big\{g \in G_{\{X\}} \colon (\forall x \in X)\big((x\mathcal{C})g = x\mathcal{C}\big)\big\}\,.$$

Then $L(X, G)$ is a normal ℓ-subgroup of $G_{\{X\}}$ called the *lazy subgroup associated with* $(\mathcal{C}, \mathcal{C}^*)$.

Let $\hat{G}(X) = G_{\{X\}}/L(X, G)$. Then $\hat{G}(X)$ acts faithfully on $X/\mathcal{C} := \{x\mathcal{C} \colon x \in X\}$. The resulting permutation group $\big(\hat{G}(X), X/\mathcal{C}\big)$ is called a *primitive component of* G.

As shown in [Mc] (or see [G1], Theorems 4C and 4A), if $\hat{G}(X) \neq \{1\}$, then $\big(\hat{G}(X), X/\mathcal{C}\big)$ satisfies a trichotomy: it is either integral, or transitively derived from a subgroup of $\mathbb{R}$, or transitively derived from an order-two transitive faithful action on X. That is, either there is a subset of $X/\mathcal{C}$ that is isomorphic to $\mathbb{Z}$ and $\hat{G}(X)$ acts as $\mathbb{Z}$ on this set (and on all of $X/\mathcal{C}$), or there is a dense subset $Y/\mathcal{C}$ of $X/\mathcal{C}$ on which: either $\hat{G}(X)$ acts as a right regular subgroup of $\mathbb{R}$ or for all $w, x, y, z \in Y$ with $w < x$ and $y < z$, there is $f \in \hat{G}(X)$ such that $w\,\mathcal{C}\, f = y\mathcal{C}$ and $x\mathcal{C} f = z\mathcal{C}$.

Now let M be a GMV-algebra. Let $(G, u) = \mathcal{G}(M)$. By the Cayley-Holland Theorem [H1] (or [G1] or [G2]), there is a totally ordered set Ω such that (G, Ω) is an ℓ-permutation group. For each $\alpha \in \Omega$, if $\alpha u \neq \alpha$, then $\mathrm{val}(\alpha u, \alpha)$ gives rise to a maximal primitive component (G_K, Ω_K) of (G, Ω) which has strong order unit u_K corresponding to u. Furthermore, every element of the index set of primitive components (an induced root system) is less than or equal to one of these. We call $\big(\mathcal{M}(G_K, u_K), \Omega_K\big)$ a *maximal primitive component of the GMV-algebra* M *associated with* Ω; or, briefly, a *maximal primitive component of the permutation GMV-algebra* (M, Ω). The *non-maximal primitive components of the* *permutation GMV-algebra* (M,Ω) are the non-maximal primitive components of $\mathcal{G}(M)$ (c.f., Sections 2.2 and 2.3).

As before, if H is an ℓ-group and $g \in H \setminus \{1\}$, then there is a convex ℓ-subgroup V_g of H maximal with respect to not containing g. (V_g is called a *value* of g). Let V_g^* be the intersection of all convex ℓ-subgroups of H that contain V_g and g. If $V_g \lhd V_g^*$ for all $g \in H \setminus \{1\}$ and values V_g of g, then we call H *normal-valued.*

Analogously, if M is a GMV-algebra and $a \in M \setminus \{0\}$, then there is an ideal W_a of M maximal with respect to not containing a, called a *value* of a. Let W_a^* be the intersection of all ideals of M that contain W_a and a. Then $W_a \subset W_a^*$. If $x \oplus W_a = W_a \oplus x$ for all $a \in M \setminus \{0\}$, values W_a of a and elements x in W_A^*, then M is called *normal-valued.*

If G is an ℓ-group with strong order unit u, then G is a normal-valued ℓ-group iff $\mathcal{M}(G,u)$ is a normal-valued GMV-algebra by [Dv2].

It is well-known that an ℓ-group H is normal-valued iff it satisfies the identity $|f||g| \leq |g|^2|f|^2$ where $|h| = h \vee h^{-1}$ (see [G2], Section 4.2 and [G1], Chapter 11). For other equivalent conditions, see *op. cit.* Moreover, G is normal-valued iff ($\Delta(g,\alpha) \subseteq \Delta(f,\alpha)$ or $\Delta(f,\alpha) \subseteq \Delta(g,\alpha)$ for all $f,g \in G$, $\alpha \in \Omega$), [G1], Theorem 11A. That is, (writing $P(G)$ for $\{\Delta(g,\alpha)\colon g \in G,\ \alpha \in \Omega\}$),

LEMMA 3.1.1. *G is normal valued iff $P(G)$ is a root system under inclusion: if $I,J,K \in P(G)$ then $K \subseteq I \cap J$ implies $I \subseteq J$ or $J \subseteq I$.*

Equivalently, G is normal valued iff each non-trivial primitive component of G is integral or transitively derived from a right regular representation of a subgroup of $(\mathbb{R},+)$ (and so every primitive component is Abelian).

Note that the root system of covering pairs of natural G-congruences is a subset of the root system $\Gamma(G)$: if $g \in G_+$, let $\alpha \in \Omega$ be such that $\alpha g \neq \alpha$. Let $\operatorname{val}(\alpha g,\alpha) = (\mathcal{C},\mathcal{C}^*)$, and $G(\alpha,g) = \{f \in G\colon (\forall \beta \in \alpha\mathcal{C}^*)(\beta f\ \mathcal{C}\ \beta)\}$, a convex ℓ-subgroup of G. Then $G(\alpha,g)$ is a value of g, and $\bigcap\{G(\alpha,g)\colon g \in G_+,\ \alpha \in \Omega,\ \alpha g \neq \alpha\} = \{1\}$. Thus $\Gamma'(G) = \{G(\alpha,g)\colon g \in G_+,\ \alpha \in \Omega,\ \alpha g \neq \alpha\}$ is a plenary subset of $\Gamma(G)$.

Note that if (G,Ω) is normal valued, then $G(\alpha,g) = \{f \in G\colon \alpha f\ \mathcal{C}\ \alpha\}$ by Lemma 3.1.1. If G is an o-group or a transitive Abelian ℓ-permutation group, then $\Gamma'(G) = \Gamma(G)$.

3.2 Transitive Wreath products

We recall the main theorem in [H] and [HMc] (or [G1], p. 122 ff. or [G2], p. 158 ff). Let Ω be a chain and (G,Ω) be a transitive group of order-preserving permutations of Ω with G an ℓ-subgroup of $A(\Omega)$. Let $\mathfrak{K}_0$ be the set of all natural (in this case, extensive) G-congruences on Ω. Then $\mathfrak{K}_0$ is a chain under inclusion. Let $\mathfrak{K}$ be the set of all covering pairs of natural (extensive) G-congruences in $\mathfrak{K}_0$. So if $K \in \mathfrak{K}$ (say, $K = \mathrm{val}(x,y)$), then (G,Ω) can be embedded in

$$(W,\hat{\Omega}) = \mathrm{Wr}\{(\hat{G}(x\mathcal{C}^K), x\mathcal{C}^K/\mathcal{C}_K) : K \in \mathfrak{K}\},$$

the Wreath product of its primitive actions (where we write $(\mathcal{C}_K, \mathcal{C}^K)$ for the covering pair associated with K).

Specifically, if $\mathfrak{K}$ is the set of covering pairs of natural G-congruences and the primitive components of (G,Ω) are (G_K, Ω_K) $(K \in \mathfrak{K})$, then $\mathfrak{K}$ is totally ordered by $(\mathcal{C},\mathcal{C}^*) < (\mathcal{D},\mathcal{D}^*)$ iff $\mathcal{C}^* \subseteq \mathcal{D}$. Let $\Omega^\dagger = \prod\{\Omega_K : K \in \mathfrak{K}\}$. Choose an arbitrary fixed reference point in $\Omega^\dagger$ denoted by $\underline{0}$. For each $\alpha \in \Omega^\dagger$, let $\mathrm{supp}(\alpha) = \{K \in \mathfrak{K} : \alpha_K \neq \underline{0}_K\}$. Let

$$\hat{\Omega} = \{\alpha \in \Omega^\dagger : \mathrm{supp}(\alpha) \text{ is an inversely well-ordered subset of } \mathfrak{K}\}.$$

Note that if $\alpha, \beta \in \hat{\Omega}$ are distinct, then $\emptyset \neq \mathfrak{K}(\alpha,\beta) = \{K \in \mathfrak{K} : \alpha_K \neq \beta_K\} \subseteq \mathrm{supp}(\alpha) \cup \mathrm{supp}(\beta)$, and so $\mathfrak{K}(\alpha,\beta)$ is also inversely well-ordered. It therefore has a greatest element, say K_0. We make $\hat{\Omega}$ a chain via: $\alpha < \beta$ if and only if $\alpha_{K_0} < \beta_{K_0}$.

We next define natural equivalence relations on $\hat{\Omega}$. For each $K \in \mathfrak{K}$, define $\equiv^K$ and $\equiv_K$ by:

$$\begin{aligned} \alpha \equiv^K \beta \quad &\text{if} \quad \alpha_{K'} = \beta_{K'} \qquad \text{for all } K' > K \\ \alpha \equiv_K \beta \quad &\text{if} \quad \alpha_{K'} = \beta_{K'} \qquad \text{for all } K' \geq K. \end{aligned}$$

Hence if $\alpha \neq \beta$ and K_0 is the largest element of $\mathfrak{K}(\alpha,\beta)$, then $\alpha \equiv^K \beta$ if $K \geq K_0$ and $\alpha \equiv_K \beta$ if $K > K_0$. Clearly, $\equiv^K$ and $\equiv_K$ have convex classes (for all $K \in \mathfrak{K}$). We wish them to be convex *congruences*; so let $W_1 = \{g \in A(\hat{\Omega}) : (\forall K \in \mathfrak{K})(\forall \alpha,\beta \in \hat{\Omega})[(\alpha \equiv^K \beta \Leftrightarrow \alpha g \equiv^K \beta g) \;\&\; (\alpha \equiv_K \beta \Leftrightarrow \alpha g \equiv_K \beta g)]\}$. Then $\equiv^K$ and $\equiv_K$ are convex W_1-congruences. Observe that $(\alpha(\equiv^K))/(\equiv_K)$ is just Ω_K for each $\alpha \in \hat{\Omega}$ and $K \in \mathfrak{K}$.

For each $K \in \mathfrak{K}$ and $\alpha \in \hat{\Omega}$, let $\alpha^K \in \prod\{\Omega_{K'} : K' > K\}$ with $(\alpha^K)_{K'} = \alpha_{K'}$; i.e., α^K is α above K. Note that $\alpha^K = \beta^K$ precisely when $\alpha \equiv^K \beta$.

For each $g \in W_1$, $\alpha \in \hat{\Omega}$ and $K \in \mathfrak{K}$, g induces an element of $A(\Omega_K)$: Let $\sigma \in \Omega_K$ and define g_{K,α^K} by:

$$\sigma g_{K,\alpha^K} = (\alpha' g)_K \in \Omega_K,$$

where $\alpha' \equiv^K \alpha$ and $\alpha'_K = \sigma$.

Lemma 3.2.1. *With the above notation, $g_{K,\alpha^K} \in A(\Omega_K)$ for each $\alpha \in \hat{\Omega}$, $g \in W_1$ and $K \in \mathfrak{K}$.*

Let $W = \{g \in W_1 : (\forall K \in \mathfrak{K})(\forall \alpha \in \hat{\Omega})(g_{K,\alpha^K} \in G_K)\}$, an ℓ-subgroup of $A(\hat{\Omega})$; $(W,\hat{\Omega})$ is called the *Wreath Product* of $\{(G_K,\Omega_K) : K \in \mathfrak{K}\}$ and is written $\mathrm{Wr}\{(G_K, \Omega_K) : K \in \mathfrak{K}\}$. The elements of W may be thought of as $\mathfrak{K} \times \hat{\Omega}$ matrices $(g_{K,\alpha})$ with $g_{K,\alpha} = g_{K,\beta}$ if $\alpha^K = \beta^K$.

Lemma 3.2.2. *Assume that each (G_K, Ω_K) is transitive. Then so is $(W, \hat{\Omega}) = \mathrm{Wr}\{(G_K, \Omega_K) : K \in \mathfrak{K}\}$. Moreover, if $\underline{0}' \in \Omega^\dagger$ is chosen as reference point and the resulting Wreath product is (W', Ω'), then (W, Ω) and (W', Ω') are ℓ-isomorphic.*

The culmination of these considerations is:

Theorem 3.2.3. [Holland & McCleary 1969] *Let Ω be a totally ordered set and (G, Ω) be a transitive ℓ-permutation group. Let $\mathfrak{K} = \mathfrak{K}(G, \Omega)$, an index set for the set of all covering pairs of convex congruences of (G, Ω) ordered in the natural way by the induced inclusions. Let $\{(G_K, \Omega_K) : K \in \mathfrak{K}\}$ be the set of all primitive components of (G, Ω) and $(W, \hat{\Omega}) = \mathrm{Wr}\{(G_K, \Omega_K) : K \in \mathfrak{K}\}$. Then there are order-preserving injections $\phi : \Omega \to \hat{\Omega}$ and $\psi : G \to W$ such that $f\psi \vee g\psi = (f \vee g)\psi$ for all $f, g \in G$. Moreover, $(\alpha g)\phi = (\alpha\phi)(g\psi)$ for all $\alpha \in \Omega$, $g \in G$.*

3.3 An example.

Let Ω be the totally ordered set obtained from $\mathbb{R}$ by replacing each rational number by a copy of $\mathbb{Z}$ and each irrational number by a copy of $\mathbb{R}$. So

$$\Omega = \{(n, q) : n \in \mathbb{Z},\ q \in \mathbb{Q}\} \cup \{(r, s) : r \in \mathbb{R},\ s \in \mathbb{R} \setminus \mathbb{Q}\},$$

ordered by $(a, x) < (b, y)$ iff ($x < y$ in $\mathbb{R}$ or, $x = y$ & $a < b$ (in $\mathbb{Z}$ or $\mathbb{R}$)).

Let G be the group of all "generalized translations" of Ω; so if $g \in A(\Omega)$, then $g \in G$ iff there are $q \in \mathbb{Q}$, $f_1 : \mathbb{Q} \to \mathbb{Z}$ and $f_2 : \mathbb{R} \setminus \mathbb{Q} \to \mathbb{R}$ such that

$$(a, x)g = \begin{cases} (a + f_1(x), x + q) & \text{if } x \in \mathbb{Q}, \\ (a + f_2(x), x + q) & \text{if } x \notin \mathbb{Q}. \end{cases}$$

There are two non-trivial natural partial G-congruences whose domains are not all of Ω: $\mathcal{C}_1$ has classes $C(q) = \{(n, q) : n \in \mathbb{Z}\}$ ($q \in \mathbb{Q}$); $\mathcal{C}_2$ has classes $C(s) = \{(r, s) : r \in \mathbb{R}\}$ ($s \in \mathbb{R} \setminus \mathbb{Q}$). In this case, $\mathfrak{K}$ the associated root system of all covering pairs of partial natural G-congruences, is a three element root system with a single maximal element K and two (unrelated) elements $C_1, C_2 < K$. The maximal totally ordered subsets of $\mathfrak{K}$ are $\mathfrak{C}_1 = \{C_1, K\}$ and $\mathfrak{C}_2 = \{C_2, K\}$. Note that the points (a, x) with $x \in \mathbb{Q}$ have no bearing on C_2, and the points (a, x) with $x \in \mathbb{R} \setminus \mathbb{Q}$ have no bearing on C_1. We therefore do not wish to consider $W_1 = (\mathbb{Z}, \mathbb{Z}) \mathrm{Wr}(\mathbb{Q}, \mathbb{R})$ and $W_2 = (\mathbb{R}, \mathbb{R}) \mathrm{Wr}(\mathbb{Q}, \mathbb{R})$ but instead $W(\mathfrak{C}_1) = (\mathbb{Z}, \mathbb{Z}) \mathrm{Wr}(\mathbb{Q}, \mathbb{Q})$ and $W(\mathfrak{C}_2) = (\mathbb{R}, \mathbb{R}) \mathrm{Wr}(\mathbb{Q}, \mathbb{R} \setminus \mathbb{Q})$, and then sew these together.

So, in considering $W(\mathfrak{C}_1)$, we delete from $\Omega/\mathcal{C}_K$ those elements $x\mathcal{C}_K$ for which $x \in \mathbb{R} \setminus \mathbb{Q}$; that is, we remove all classes whose points do not belong to $\mathrm{dom}(C_1)$. Similarly, for $W(\mathfrak{C}_2)$. So instead of taking $\Omega_K = \mathbb{R}$, we take $\Omega_K(\mathcal{C}_1) := \mathbb{Q}$ and $\Omega_K(\mathcal{C}_2) := \mathbb{R} \setminus \mathbb{Q}$.

Then $W(\mathfrak{C}_j) = \big(\hat{G}(C_j), \Omega_{(C_j)}\big)\,\mathrm{Wr}\big(\hat{G}(K), \Omega_K(\mathcal{C}_j)\big)$ for $j = 1, 2$.

In both cases we have a translation by a rational number in the "upstairs" part. Analogously to the rooted valuation product, we form the rooted Wreath product:

$$\mathcal{W}(G) = \big\{(w_1, w_2) \in W(\mathfrak{C}_1) \times W(\mathfrak{C}_2) \colon (x\mathcal{C}_1)w_1 = (x\mathcal{C}_2)w_2\big\}\,,$$

where we take the natural extensions of w_j from $\Omega_K(\mathcal{C}_j)$ to Ω_K $(j = 1, 2)$. This is possible since both are translations of subgroups of $\mathbb{R}$.

Let $u \in G$ be given by $(a, x)u = (a, x + 1)$ for all $(a, x) \in \Omega$. Then u is a strong order unit in G. Let $M = \mathcal{M}(G, u)$. Then the non-maximal primitive components of (M, Ω) are the ℓ-permutation groups $(\mathbb{Z}, \mathbb{Z})$ and $(\mathbb{R}, \mathbb{R})$, whereas the maximal component is $(\mathbb{Q} \cap [0, 1], \mathbb{R})$. The rooted Wreath product for G is that obtained from $(\mathbb{Z}, \mathbb{Z})\,\mathrm{Wr}(\mathbb{Q}, \mathbb{Q})$ & $(\mathbb{R}, \mathbb{R})\,\mathrm{Wr}(\mathbb{Q}, \mathbb{R} \setminus \mathbb{Q})$ by identifying the "upstairs" part, whereas the rooted Wreath product for M is that obtained from $(\mathbb{Z}, \mathbb{Z})\,\mathrm{Wr}(\mathbb{Q} \cap [0, 1], \mathbb{Q})$ & $(\mathbb{R}, \mathbb{R})\,\mathrm{Wr}\big(\mathbb{Q} \cap [0, 1], \mathbb{R} \setminus \mathbb{Q}\big)$ by identifying the "upstairs" part.

3.4 Rooted Wreath products

We wish to generalize the Wreath product construction ([H2], [HMc]) to give a universal representation for normal-valued permutation groups (G, Ω) which are not necessarily transitive.

Consider the normal-valued permutation group (G, Ω). As in the transitive case, let $\mathfrak{K}_0$ be the root system of all *partial* natural G-congruences on Ω and $\mathfrak{K}$ the associated root system of all covering pairs of partial natural G-congruences.

If $K \in \mathfrak{K}$, then $\mathrm{dom}(\mathcal{C}^K) = \mathrm{dom}(\mathcal{C}_K)$, and we will write $\mathrm{dom}(K)$ as an abbreviation for this common domain.

For each G-orbit $\mathcal{O}$ of $\mathrm{dom}(K)$, choose exactly one point $x(K, \mathcal{O})$, and let $T(K)$ be the resulting set of points (a subset of $\mathrm{dom}(K)$). We do this in such a way that $K < K'$ implies that $T(K') \subseteq T(K)$ (existence, *op. cit.*).

For each $K \in \mathfrak{K}$ and orbit $\mathcal{O}$, let $X(K, \mathcal{O}) = x(K, \mathcal{O})\mathcal{C}^K/\mathcal{C}_K$. Let $\mathfrak{M}$ be the set of all maximal chains in $\mathfrak{K}$ and $\mathfrak{C} \in \mathfrak{M}$. For each $K \in \mathfrak{C}$, let

$$T(K, \mathfrak{C}) = \big\{x(K, \mathcal{O}) \in T(K) \colon x(K, \mathcal{O}) \notin \bigcup\{T(K') \colon K' \notin \mathfrak{C}\}\big\}\,,$$

and

$$\Omega_K(\mathfrak{C}) = \big\{X(K, \mathcal{O}) \colon x(K, \mathcal{O}) \in T(K, \mathfrak{C})\big\}\,,$$

the "$\mathfrak{C}$ restricted" domain of $K \in \mathfrak{C}$.

Remark. Since (G,Ω) is normal-valued, if the induced restriction of $g \in \hat{G}(K)$ to $\Omega_K(\mathfrak{C})$ is the identity, then it is the identity on all of Ω_K.

Let

$$\Omega(\mathfrak{C}) := \bigcup_{C\in\mathfrak{C}} \Omega_C(\mathfrak{C}),$$

the union of the "restricted" domains of the members of $\mathfrak{C}$.

Note that $\Omega(\mathfrak{C})g = \Omega(\mathfrak{C})$ for all $g \in G$, $\mathfrak{C} \in \mathfrak{M}$. Then, as in Section 3.2 (or *op. cit.*), we can form the Wreath product

$$\big(W(\mathfrak{C}),\Omega(\mathfrak{C})\big) = \mathrm{Wr}\Big\{\Big(\hat{G}\big(X(C,\mathcal{O})\big), X(C,\mathcal{O})\Big) \colon x(C,\mathcal{O}) \in T(C,\mathfrak{C}),\ C \in \mathfrak{C}\Big\}.$$

Since (G,Ω) is a normal-valued permutation group, each $\hat{G}\big(X(C,\mathcal{O})\big)$ is (ℓ)-isomorphic to a subgroup $R\big(x(C,\mathcal{O})\big)$ of $\mathbb{R}$, and each $X(C,\mathcal{O})$ is a collection of orbits of $R\big(x(C,\mathcal{O})\big)$ on each of which its action is induced by the right regular action.

Let

$$L(\mathfrak{C}) = \bigcap\big\{L\big(x(C,\mathcal{O})\mathcal{C}^C, G\big) \colon x(C,\mathcal{O}) \in T(C,\mathfrak{C}),\ C \in \mathfrak{C}\big\},$$

and $G(\mathfrak{C}) = G/L(\mathfrak{C})$. By the remark,

$$L(\mathfrak{C}) = \bigcap\big\{L\big(x(C,\mathcal{O})\mathcal{C}^C, G\big) \colon x(C,\mathcal{O}) \in T(C),\ C \in \mathfrak{C}\big\}. \qquad (**)$$

As above, we get a pair of embeddings $(\phi_{\mathfrak{C}}, \psi_{\mathfrak{C}})$ of $\big(G(\mathfrak{C}), \hat{\Omega}(\mathfrak{C})\big)$ as in the transitive case (by the remark).

Now $\bigcap\big\{L(\mathfrak{C}) \colon \mathfrak{C} \in \mathfrak{M}\big\} = \{1\}$ (since if $g \neq 1$, then $xg \neq x$ for some $x \in \Omega$; then $g \notin \mathrm{val}(xg,x)$ and so $g \notin L(\mathfrak{C})$ for any chain $\mathfrak{C}$ containing $\mathrm{val}(xg,x)$ by $(**)$). Thus we obtain an ℓ-embedding $\theta \colon G \to \prod\limits_{\mathfrak{C}\in\mathfrak{M}} G/L(\mathfrak{C})$ induced by the natural maps $\nu_{\mathfrak{C}} \colon g \mapsto L(\mathfrak{C})g$ $(\mathfrak{C} \in \mathfrak{M})$. Thus we have an ℓ-embedding of G into $\prod\limits_{\mathfrak{C}\in\mathfrak{M}} W(\mathfrak{C})$ induced by $\{\nu_{\mathfrak{C}}\psi_{\mathfrak{C}} \colon \mathfrak{C} \in \mathfrak{M}\}$.

To complete the analysis, we need two further observations:

(1) Since (G,Ω) is normal-valued, we have that for each $K \in \mathfrak{K}$ and $x_K \in T(K)$, either

$$\begin{aligned} (y\mathcal{C}_K)g &= y\mathcal{C}_K \qquad \text{for all } y\mathcal{C}^K x_K, \\ \text{or}\quad (y\mathcal{C}_K)g &\neq y\mathcal{C}_K \qquad \text{for all } y\mathcal{C}^K x_K. \end{aligned}$$

So, as in standard group theory, the right regular actions provide an index set and we do not need to resort to a permutation representation approach as in [HMc].

(2) Suppose that $\mathfrak{C}_1, \mathfrak{C}_2 \in \mathfrak{M}$. Then for all $g \in G$, $y \in \mathrm{dom}(C)$ and $C \in \mathfrak{C}_1 \cap \mathfrak{C}_2$,

$$(y\mathcal{C}_C)(g\psi_{\mathfrak{C}_1}) = (y\mathcal{C}_C)(g\psi_{\mathfrak{C}_2}).$$

Since $\mathfrak{K}$ is a root system, it follows that if $C \in \mathfrak{C}_1 \cap \mathfrak{C}_2$, and $C < K \in \mathfrak{K}$, then $K \in \mathfrak{C}_1 \cap \mathfrak{C}_2$.

In analogy with the rooted valuation product, we need to consider compatibility conditions to get a tighter embedding.

First note that we may uniquely extend each element of $\big(G_K, \Omega_K(\mathfrak{C})\big)$, and that if $K \in \mathfrak{C}_1 \cap \mathfrak{C}_2$, then $g \in \hat{G}(K)$ is the same translation of Ω_K as that given by the extensions of the corresponding elements of each of $\big(G_K, \Omega_K(\mathfrak{C}_1)\big)$ and $\big(G_K, \Omega_K(\mathfrak{C}_2)\big)$.

We define the *rooted Wreath product* $\mathcal{W}(G)$ to comprise all $w \in \prod_{\mathfrak{C} \in \mathfrak{M}} W(\mathfrak{C})$ that satisfy

$$(\forall \mathfrak{C}_1, \mathfrak{C}_2 \in \mathfrak{M})(\forall C \in \mathfrak{C}_1 \cap \mathfrak{C}_2)\big(\forall y \in \mathrm{dom}(C)\big)\big((y\mathcal{C}_C)w_{\mathfrak{C}_1} = (y\mathcal{C}_C)w_{\mathfrak{C}_2}\big). \qquad (*)$$

Thus if $w \in \mathcal{W}(G)$, then $w_{\mathfrak{C}_1}$ agrees with $w_{\mathfrak{C}_2}$ on the (possibly empty) upper segment of $\mathfrak{C}_1 \cap \mathfrak{C}_2$.

By (1) and (2) we have an embedding χ of G into $\mathcal{W}(G)$ that preserves the (pointwise) ordering on G and any finite suprema and infima that exist in G. Consequently, in [GW] we obtained the desired universal:

Theorem 3.4.1. *If (G, Ω) is any normal-valued (coherent) permutation group with natural primitive components (G_K, Ω_K) $(K \in \mathfrak{K})$, then (G, Ω) can be ℓ-embedded in the rooted Wreath product of $\big\{(G_K, \Omega_K) : K \in \mathfrak{K}\big\}$.*

Now if Ω is a totally ordered set and M is a GMV-algebra contained in $A(\Omega)$ (i.e., $(\mathcal{G}(M), \Omega)$ is an ℓ-permutation group and u is the strong unit of $G = \mathcal{G}(M)$), then the non-maximal primitive components of (M, Ω) are those of (G, Ω), and the maximal primitive components of (M, Ω) are those of the form $\mathcal{M}\big((G_K, \Omega_K), u_K\big)$ where K corresponds to $\mathrm{val}(\alpha u, \alpha)$ for some $\alpha \in \Omega$. We ℓ-embed (G, Ω) in the rooted Wreath product $\mathcal{W}(G)$ and continue to denote the image of u by u. As in the case of rooted valuation products, if $\big\{\mathrm{val}(\alpha u, \alpha) : \alpha \in \Omega \ \&\ \alpha u \neq \alpha\big\}$ is an infinite set, then u is no longer a unit in $\mathcal{W}(G)$; indeed, $\mathcal{W}(G)$ has no unit in this case. We therefore form $\tilde{\mathcal{W}}(G)$, the convex ℓ-subgroup of $\mathcal{W}(G)$ generated by u. Note that the ℓ-embedding of G into $\mathcal{W}(G)$ is actually an ℓ-embedding of G into $\tilde{\mathcal{W}}(G)$, and $\tilde{\mathcal{W}}(G)$ has strong order unit u. Let $\mathcal{W}(M) = \mathcal{M}(\tilde{\mathcal{W}}(G), u)$ acting on $\hat{\Omega}$. Then $\mathcal{W}(M)$ is a GMV-algebra and $\big(\mathcal{W}(M), \hat{\Omega}\big)$ has the same primitive components as those of M. We therefore obtain the GMV-algebra analogue of the normal-valued ℓ-permutation result of [GW] (Theorem 3.4.1 above):

THEOREM 3.4.2. *If M is any normal-valued GMV-algebra acting as a subgroup of $A(\Omega)$ with natural primitive components (M_K, Ω_K) $(K \in \mathfrak{K})$, then (M, Ω) can be embedded (as a GMV-algebra) in the rooted Wreath product of $\{(M_K, \Omega_K) \colon K \in \mathfrak{K}\}$.*

The location of rooted valuation products inside rooted Wreath product products for GMV-algebras follows *mutatis mutandis* the work in [GW], Section 3.5 and we omit it here.

EXAMPLE. For each $a \in \mathbb{R}_+$, let $xf_a = ax$ if $x \geq 0$ and $xf_a = x$ if $x \leq 0$; and $xg_a = ax$ if $x \leq 0$ and $xg_a = x$ if $x \geq 0$. Let $F = \{f_a \colon a \in \mathbb{R}_+\}$ and $G = \{g_a \colon a \in \mathbb{R}_+\}$. Then F, G are Abelian subgroups of $A(\mathbb{R})$ and generate $H = F \times G$; so $0h = 0$ for all $h \in H$. Now F and G are totally ordered (under the pointwise ordering) so H is an ℓ-subgroup of $A(\mathbb{R})$. Note that if $x \in \mathbb{R}$, then the orbit of x under H is either $(-\infty, 0)$, $\{0\}$ or $(0, \infty)$. Let $u = f_1 g_1 = f_1 \vee g_1$ and $M = \mathcal{M}(H, u)$. Let $M(F) = \mathcal{M}(F, f_1)$ and $M(G) = \mathcal{M}(G, g_1)$. So $M(F) = \{f_a \colon 0 \leq a \leq 1\}$ and $M(G) = \{g_a \colon 0 \leq a \leq 1\}$. The set of natural covering congruences is the four point root system with a single maximal element and three incomparable elements below it. The corresponding permutation groups are $(\{1\}, \{-, 0, +\})$, $(M(G), \mathbb{R}_-)$, $(\{1\}, \{0\})$ and $(M(F), \mathbb{R}_+)$. If we ignore the trivial actions we get $M(G) \times M(F)$ (with unit u) for both $\mathcal{W}(M)$ and $\mathcal{V}(M)$.

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