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THE MV-ALGEBRA OF FIRST ORDER ŁUKASIEWICZ LOGIC

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Dedicated to Professor Ján Jakubík on the occasion of his 80th birthday

ABSTRACT. In this work we begin the study of three prominent questions about Łukasiewicz predicate logic:

1) What is the algebraic significance of the "true" but unprovable well formed formula?

2) Is there a non-archimedean linearly ordered MV-algebra A for which the completeness theorem holds, that is, a formula is "true" on A iff it is provable?

3) What can be said about the structure of the Lindenbaum algebra of the Łukasiewicz predicate logic?

Question 1) is answered somewhat easily, while 2) remains open; 3) of course, is open-ended.

1. Introduction

In the years immediately following the publication by Chang [5, 6] of the first results on MV-algebras and an algebraic proof of the completeness of the Łukasiewicz propositional logic, there was an effort directed towards the investigation of the completeness properties of the monadic and first order Łukasiewicz predicate logics [1, 2, 4, 11, 12].

A result obtained from these investigations stated that the monadic predicate logic is complete with respect to the canonical set of "truth-values", namely the unit interval $[0, 1]$ of real numbers [11]. In contrast, the pure first order predicate logic in not complete with respect to the canonical set of "truth-values" [12]. However, if the "truth-values" are allowed to vary through all linearly ordered MV-algebras, then a completeness theorem results [4].

Besides, there is a model completeness, or equivalently a compactness theorem, with respect to the canonical set of truth-values; namely, every consistent set of well formed formulas can be simultaneously satisfied on $[0, 1]$ [4].

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Now, if one takes a less restricted notion of “true” formula, say calling a formula “true” if its value on all canonical models is strictly greater than some fixed $r \in [0,1]$, then there is completeness depending on the nature of r [2, 4].

Since this early flurry of activity, there has been little attention paid to Łukasiewicz predicate logic. There has been, on the other hand, a great deal of investigation into the theory of MV-algebras, see the excellent monograph [8].

It is time, then, to revisit first order Łukasiewicz logic to determine if the mass of accumulated information on MV-algebras can refine and enrich our understanding of this logic.

In this work we begin the study of three prominent questions about Łukasiewicz predicate logic:

1) What is the algebraic significance of the “true” but unprovable well formed formula?

2) Is there a non-archimedean linearly ordered MV-algebra A for which the completeness theorem holds, that is, a formula is “true” on A iff it is provable?

3) What can be said about the structure of the associated MV-algebra?

Question 1) is answered somewhat easily, while 2) remains open; 3) of course, is open-ended.

Though we do not answer definitively these questions, and many others that present themselves in this study, we do hope that we will stimulate the interest of others to pursue this investigation.

2. Preliminary results on MV-algebras

We say that a structure $A = (A, 0, 1, *, \odot, \oplus)$ is an MV-algebra iff A satisfies the following equations:

1. $(x \oplus y) \oplus z = x \oplus (y \oplus z)$;
2. $x \oplus y = y \oplus x$;
3. $x \oplus 0 = x$;
4. $x \oplus 1 = 1$;
5. $0^* = 1$;
6. $1^* = 0$;
7. $x \odot y = (x^* \oplus y^*)^*$;
8. $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$.

From (8), with $y = 0$ it follows $(x^{**}) = x$ and with $y = 1$, $x^* \oplus x = 1$. On A two new operations $\vee$ and $\wedge$ are defined as follows: $x \vee y = (x^* \oplus y)^* \oplus y$ and

$x \wedge y = (x^* \odot y)^* \odot y$. The structure $(A, \vee, \wedge, 0, 1)$ is a bounded distributive lattice. We shall write $x \leq y$ iff $x \wedge y = x$. Boolean algebras are just the MV-algebras obeying the additional equation $x \odot x = x$. Let $B(A) = \{x \in A \colon x \odot x = x\}$ be the set of all idempotents elements of A. Then, $B(A)$ is a subalgebra of A, which is also a Boolean algebra. Indeed, it is the greatest Boolean subalgebra of A.

The basic MV-algebraic operations on the unit interval of real numbers, $[0,1]$, are

$$\begin{aligned} x \oplus y &= \min(1, x+y)\,; \\ x \odot y &= \max(0, x+y-1)\,; \\ x^* &= 1-x\,. \end{aligned}$$

We refer to this MV-algebra by $[0,1]$. An *ideal* of an MV-algebra A is a non-empty subset I of A which is closed under $\oplus$ and $x \leq y$, $y \in I$ imply $x \in I$. A *prime* ideal P of A is an ideal of A such that $x \wedge y$ implies $x \in P$ or $y \in P$. An ideal M of A is called *maximal* if $M \subseteq I$ implies $I = A$, for I ideal of A. As MV-algebras form an equational class, the notions of MV-isomorphism, quotient, subalgebra, product, etc., are just the particular cases of the corresponding universal algebraic notions. Let A be an MV-algebra and B a subset of A, then the subalgebra of A generated by B will be denoted by $\langle B \rangle$. The intersection of all maximal ideals, the *radical* of A, will be denoted by $\operatorname{Rad} A$ and the set $\{x \in A \colon x^* \in \operatorname{Rad} A\}$ by *co-radical* A. We write nx instead of $x \oplus \cdots \oplus x$ (n-times) and x^n instead of $x \odot \cdots \odot x$ (n-times). An MV-algebra A having exactly one maximal ideal is called *local*.

The least integer for which $nx = 1$ is called the *order* of x. When such integer exists, we denote it by $\operatorname{ord} x$ and say that x has *finite order*; otherwise we say that x has *infinite order*.

For every MV-algebra A, we have

$$\operatorname{Rad} A = \{x \in A \colon nx \leq x^* \text{ for every positive integer } n\}$$

and

$$\langle \operatorname{Rad} A \rangle = (\operatorname{Rad} A) \cup (\operatorname{Rad} A)^*\,.$$

An MV-algebra A generated by $\operatorname{Rad} A$ is called *perfect*. An MV-algebra A such that $\operatorname{Rad} A = 0$ is called *semisimple*. If $\operatorname{ord} x$ is finite for every $x \in A$, then we say that A is *locally finite*.

Let X be a non-empty set. Then the set $B = [0,1]^X$ of all $[0,1]$-valued functions over X, equipped with pointwise operations, is an MV-algebra. Up to isomorphism, subalgebras of B provide the most general possible examples of semisimple MV-algebras, and locally finite MV-algebras are subalgebras of the MV-algebra $[0,1]$.

Lemma 1. *Let A be an MV-algebra. Suppose that $x \in A$ and $x \oplus x^n = 1$ for all $n > 0$. Then $x^* \in \operatorname{Rad} A$.*

Proof. Suppose there is a maximal ideal M with $x^* \notin M$. Then for some $n > 0$, $x^n \in M$. But $x \oplus x^n = 1$, so $x^* \leq x^n$. □

Lemma 2. *Suppose in an MV-algebra A we have an element x such that for all $n \in \omega$, $\operatorname{ord}(x^*)^n < \infty$. Then $x \in \operatorname{Rad} A$.*

Proof. If $x \notin M$ for some maximal ideal M, then for some n, $(x^*)^n \in M$. □

An element a of an MV-algebra A is called *infinitesimal* if $a \neq 0$ and $na \leq a^*$ for every $n \in \omega$. An element a of an MV-algebra A is called *co-infinitesimal* if a^* is an infinitesimal.

For a non-empty subset S of an MV-algebra A, the set $S^\perp = \{a \in A \colon a \wedge x = 0$, for every $x \in S\}$ is called the *polar* or the *annihilator* of S. If $a \in A$ then the polar of $\{a\}$ will be simply denoted by $a^\perp$.

We recall that a non-zero ideal I in an MV-algebra is *essential* iff it has a non-zero intersection with every non-zero ideal (equivalently, $I^\perp = \{0\}$), and that a *weak unit* in an MV-algebra A is an element $a \in A$ such that $a^\perp = \{0\}$.

Recall that an ideal I in an MV-algebra is *local* provided that A/I is a local MV-algebra. We say that an ideal I of an MV-algebra A is *orthogonally free* if and only if every element $a \in I$ is such that $a^\perp \cap I = \{0\}$.

Lemma 3. *Let A be an MV-algebra and I an ideal of A. Then the following are equivalent:*

(j) *I is linearly ordered;*

(jj) *I is orthogonally free.*

Proof. Clear if I is linearly ordered. Conversely, assume I orthogonally free but not linearly ordered. Then, there are elements $a, b \in I$ such that $a \odot b^* \neq 0$ and $b \odot a^* \neq 0$. But $a \odot b^*, b \odot a^* \in I$ and $a \odot b^* \wedge b \odot a^* = 0$. Hence, I is not orthogonally free. □

Lemma 4. *Let A be an MV-algebra and I a linearly ordered ideal of A. Then A is linearly ordered or I does not contain any weak unit.*

Proof. Since I is linearly ordered, then $I^\perp$ is a prime ideal of A. If $I^\perp = \{0\}$, then A is linearly ordered. If $I^\perp \neq \{0\}$, then any element k of I has a non-zero annihilator. Hence, k is not a weak unit. □

Lemma 5. *Suppose that I is a local ideal of an MV-algebra A. Then $I^{\perp}$ is either contained in the radical of A or is generated by an idempotent. In the latter case I is also generated by an idempotent.*

Proof. Let I be a local ideal. Suppose $x \in I^{\perp}$, $x \neq 0$. Then in A/I, $x/I \neq 0$. If $\mathrm{ord}(x/I) < \infty$, then for some n, $nx/I = 1$, hence $(x^*)^n \in I$. But $nx \in I^{\perp}$ so $nx \wedge (x^*)^n = 0$, that is, $nx \in B(A)$.

If $\mathrm{ord}(x/I) = \infty$, then, for every n, $nx/I \in \mathrm{Rad}(A/I)$ and so $nx \odot nx \in I$. But $(nx \odot nx) \wedge nx = (nx \odot nx) = 0$. It follows that $x \in \mathrm{Rad}\,A$.

Assume that $I^{\perp} \nsubseteq \mathrm{Rad}\,A$; let $y \in I^{\perp} \setminus \mathrm{Rad}\,A$. Then $\mathrm{ord}(y/I) < \infty$. Hence for some n we get $ny \in B(A)$. Since A/I is local and then $B(A/I) = \{0,1\}$, we have that $(ny)/I = 1$, i.e., $(ny)^* \in I$. Moreover, if $z \in I$, then $0 = z \wedge ny$ and so $z \leq (ny)^*$. Also, if $x \in I^{\perp}$, $x \wedge (ny)^* = 0$ so $x \leq ny$. Thus, $I^{\perp} = \mathrm{id}(ny)$, and $I = \mathrm{id}\big((ny)^*\big)$.

Consequently, if I is not generated by an idempotent, then $I^{\perp} \subseteq \mathrm{Rad}\,A$. □

Suppose now that I is a local ideal and $\mathrm{Rad}\,A \cap I = \{0\}$. Then $\mathrm{Rad}\,A \subseteq I^{\perp}$. Hence if I is not generated by an idempotent, then $I^{\perp} = \mathrm{Rad}\,A$.

Call an ideal I of an MV-algebra A *anti-boolean* if A/I has no non-trivial idempotents. It is easy to see that an ideal I is anti-boolean iff for all x, $x \wedge x^* \in I$ implies $x \in I$ or $x^* \in I$.

Proposition 6. *Let A be an MV-algebra, I and P an ideal and a prime ideal, respectively. Let $\{0\} \neq I^{\perp} \cap P^{\perp}$. Then $I \subseteq P$.*

Proof. Let $0 \neq x \in I^{\perp} \cap P^{\perp}$. Let $y \in I$. Then, $x \wedge y = 0 \in P$. Since $x \in P^{\perp}$, we know that $x \notin P$. Thus $y \in P$ and we have that $I \subseteq P$. □

Corollary 7. *Let P, Q be prime ideals of an MV-algebra A. If $\{0\} \neq P^{\perp} \subseteq Q^{\perp}$, then $P = Q$.*

Observe that this corollary also shows that if a prime ideal P is such that $P^{\perp} \neq \{0\}$, then P is minimal prime.

3. First order Łukasiewicz logic and its Lindenbaum algebra

We begin with a review of first order Łukasiewicz logic.

Start with a language $\mathcal{L}$ built from the symbols

1) a set Var of individual variables, v_0 $v_1, \dots$,

2) logical symbols, $\rightarrow$, $\neg$,

3) predicate symbols R_n, $n = 0, 1, \dots,$

4) quantifier symbol $\exists$,

5) improper symbols (,),

6) function $d(n)$ from the non-negative integers to itself.

We construct the class W of well-formed formulas in the usual manner starting from the atomic formula, $R_n(v_{i_1}, \dots, v_{i_{d(n)}})$.

From given axioms ([1, 2, 4, 9]) and two rules of inference, modus ponens and generalization, notions of *formal proof* and *formal theorem* are defined in the usual manner.

If α is a formal theorem we write $\vdash \alpha$; more generally, if α is formally provable from a set Δ of wff, we write $\Delta \vdash \alpha$.

Two formulas α, β, are defined to be equivalent iff both $\alpha \to \beta$ and $\beta \to \alpha$ are formally provable. We shall write $\alpha + \beta$ as an abbreviation of the formula $\neg\alpha \to \beta$. The set of equivalence classes, $\{[\alpha] \colon \alpha \in W\}$, forms an MV-algebra L, the Lindenbaum algebra of $\mathcal{L}$.

4. Models and interpretations

Let A be an MV-algebra, X a non-empty set. An $\{A, X\}$-model is a system $\mathcal{M} = \{A, X, F_n, n \in \omega\}$ such that for each n we have a function, $F_n \colon X^{d(n)} \to A$.

An $\{A, X\}$-model $\mathcal{M}$ will be called *linear* iff A is linearly ordered. An $\{A, X\}$-model $\mathcal{M}$ will be called *canonical* iff $A \subseteq [0, 1]$.

Given an $\{A, X\}$-model, $\mathcal{M}$, an $\mathcal{M}$-*assignment* is a function $f \colon \mathrm{Var} \to X$. If f is an $\mathcal{M}$-assignment, then f_{vx} is the $\mathcal{M}$-assignment defined by:

i) $f_{vx}(v_i) = f(v_i)$ if $v \neq v_i$.

ii) $f_{vx}(v) = x$.

For an MV-algebra A and subset $S \subseteq A$, define ΣS to be the least upper bound of S in A if it exists; otherwise set $\Sigma S = A$.

With these notions we can assign "truth values" to the elements of W.

Let $\mathcal{M} = \{A, X, F_n, n \in \omega\}$ be a model, and f an $\mathcal{M}$-assignment. We define by induction on W a function

$$\mathrm{Val}(\alpha, \mathcal{M}, f) \colon W \to A \cup \{A\}$$

as follows

1) $\mathrm{Val}\Big(R_n(v_{i_1}, \dots, v_{i_{d(n)}}), \mathcal{M}, f\Big) = F_n\Big(fv_{i_1}, \dots, fv_{i_{d(n)}}\Big)$.

2) Assuming the function to be defined for $\alpha \in W$, then

$\mathrm{Val}(\neg\alpha, \mathcal{M}, f) = \big(\mathrm{Val}(\alpha, \mathcal{M}, f)\big)^*$ if the latter is in A, and equal to A otherwise.

3) Assuming the function to be defined for $\alpha, \beta \in W$, then

$\mathrm{Val}(\alpha \to \beta, \mathcal{M}, f) = \big(\mathrm{Val}(\alpha, \mathcal{M}, f)\big)^* \oplus \mathrm{Val}(\beta, \mathcal{M}, f)$ provided both values are in A, and A otherwise.

4) $\mathrm{Val}\big((\exists v)\alpha, \mathcal{M},\ f\big) = \Sigma_{x\in X}\,\mathrm{Val}(\alpha, \mathcal{M}, f_{\varepsilon vx})$

Call an $\mathcal{M}$-assignment f an *interpretation* if $\mathrm{Val}(\alpha, \mathcal{M}, f) \in A$ for *all* $\alpha \in W$.

An interpretation f on a model $\mathcal{M} = \{A, X, F_n, n \in \omega\}$ will be called *entire* provided $f(\mathrm{Var}) = X$. We point out that all the models used in [4] to obtain the completeness results were entire. (This requires, of course, that the set X of the model be countable.)

We note that for a closed wff α, and $\mathcal{M}$-interpretations, f, f', there holds that $\mathrm{Val}(\alpha, \mathcal{M}, f) = \mathrm{Val}(\alpha, \mathcal{M}, f')$ [4].

We do not require that every assignment to a model determines an interpretation (cf. [8]).

We note that there always exist models and interpretations. Moreover linear models are related to certain prime ideals as follows: Let $\beta_0, \beta_1, \ldots, \beta_n, \ldots$ be an enumeration of all wff β of the form $(\exists v_j)\alpha$. For each $n, m \geq 0$, let $\beta(n,m)$ be the formula obtained from $\beta_n = (\exists v_j)\alpha_n$ in the following manner; let v be the first element in Var not occurring in α_n, and let $\beta(n,m)$ be the result of first replacing all bound occurrences of v_m in α_n by v, then replacing all free occurrences of v_j by v_m.

Proposition 8. [4] *In the MV-algebra L, $\Sigma_{m\in\omega}[\beta(n,m)]$ exists for each $n \in \omega$ and equals $[\beta_n]$.*

An ideal I of the Lindenbaum algebra L is said to *preserve* the sums $[\beta_n]$ if for any n, $[\beta(n,m)] \in I$ for all m implies $[\beta_n] \in I$. We call β-ideal any ideal P of L which preserves the sums $[\beta_n]$. In addition, if P is prime we say that P is β-prime, if P is maximal we say that P is β-maximal. A principal result of [4] is the following proposition.

Proposition 9. *Let $[\alpha], [\beta] \in L$ be such that $[\alpha] \notin \mathrm{id}\big([\beta]\big)$. Then there exists an ideal P of L such that*

1) *$[\beta] \in P$;*
2) *P is maximal with respect to not containing $[\alpha]$, therefore is prime;*
3) *P preserves the sums $[\beta_n]$ for all $n \in \omega$.*

As a corollary to this we obtain

Corollary 10. *Let $[\beta] \in L$ be such that $\operatorname{ord}[\beta] = \infty$. Then there is a maximal ideal M of L such that:*

a) $[\beta] \in M$;

b) *M preserves the sums $[\beta_n]$ for all $n \in \omega$.*

Proof. We know that $1 \notin \operatorname{id}([\beta])$. Hence there is a prime ideal M that preserves the sums $[\beta_n]$ with $[\beta] \in M$. Suppose that for some $[\alpha]$, $[\alpha] \notin M$. Then $1 \leq p \oplus n[\alpha]$ for some $p \in M$ and some $n \in \omega$. But then $1 = p \oplus n[\alpha]$ from which it follows that $[\alpha]^* \leq p$. Thus $[\alpha]^* \in M$ and we see M is a maximal ideal. □

Proposition 11. *For each β-prime ideal P of L there exists a linear model $\mathcal{M} = \{A, X, F_n\}$ and an entire interpretation f such that $A = L/P$ and $\operatorname{Val}(\alpha, \mathcal{M}, f) = [\alpha]/P$.*

Conversely, given a linear model $\mathcal{M} = \{A, X, F_n\}$ and an entire interpretation f, there is a β-prime ideal P of L such that $\operatorname{Val}(\alpha, \mathcal{M}, f)$ is $[\alpha]/P$ modulo an isomorphism.

Proof. Let P be a β-prime ideal of L. Then we get a linear model $\mathcal{M} = \{A, X, F_n\}$ and an interpretation f such that:

1) $A = L/P$,

2) $X = \mathbb{N} = \{0, 1, 2, \dots\}$,

3) $F_n(n_1, \cdots, n_{d(n)}) = \big[R_n(v_{n_1}, \cdots, v_{n_{d(n)}})\big]/P$,

4) $f(v_n) = n$.

Conversely, given a linear model $\mathcal{M} = \{A, X, F_n, n \in \omega\}$ and an entire interpretation f, we can find a β-prime ideal P of L such that the model $\{L/P, \mathbb{N}, F'_n, n \in \omega\}$, built as above, will satisfy the following condition:

for each wff $\alpha, \operatorname{Val}(\alpha, \mathcal{M}, f) = \varphi\big([\alpha]/P\big)$ where $\varphi\colon L/P \to A$ is an injection.

To see this, let $P = \big\{[\alpha] | \operatorname{Val}(\alpha, \mathcal{M}, f) = 0\big\}$. Since A is linearly ordered, P is a prime ideal. Now let $[\beta_{n,m}] \in P$ for all n. Then $\operatorname{Val}\big(\beta(n,m), \mathcal{M}, f\big) = 0$ for all m. Suppose that $[\beta_n] \notin P$. Then for some $x \in X$ we have $\operatorname{Val}(\alpha, \mathcal{M}, f_{nx}) > 0$. Since f is entire, there is an m with $f(v_m) = x$. It is easy to see that $\operatorname{Val}(\alpha, \mathcal{M}, f_{nx}) = \operatorname{Val}\big(\beta(n,m), \mathcal{M}, f\big) = 0$. It follows that $[\beta_n] \in P$. The mapping $f_*\colon L \to A$ given by $f_*([\alpha]) = \operatorname{Val}(\alpha, \mathcal{M}, f)$ is clearly a homomorphism with kernel P, and φ is the induced natural injection from L/P to A. □

Consequently, we have a countable set of countable linearly ordered models $\mathcal{M}_n = \{A_n, X, F_k\}$ and interpretations f_n such that a wff α is provable iff $\operatorname{Val}(\alpha, \mathcal{M}_n, f_n) = 1$ for all n.

It is also easy to see that every MV-algebra A determines $\{A, X\}$-models and $\{A, X\}$interpretations. (For take X to be finite, and $F_n \colon X^{d(n)} \to A$ any function, $n \in \omega$.)

A formula α is said to be *A-valid* provided $\mathrm{Val}(\alpha, \{A, X\}, f) = 1$ for all $\{A, X\}$-models and all $\{A, X\}$-interpretations f.

We call α *strongly valid* provided $\mathrm{Val}\big(\alpha, \{A, X\}, f\big) = 1$ for all linear models $\{A, X\}$ and all $\{A, X\}$-interpretations f.

Call α *valid* if $\mathrm{Val}(\alpha, \{A,\ X\}, f) = 1$ for all assignments into any canonical model $A \subseteq [0, 1]$.

We observe that from the work in [4], it suffices to consider only countable models and onto interpretations with regard to the completeness theorems. We know that for $\alpha \in W$ that α is provable iff $[\alpha] = 1$ in L. We also know

1) α is strongly valid iff α is provable;

2) there exists valid α such that α is not provable.

Our aim, now, is to show that the algebra L is not semisimple.

Proposition 12. *L is not semisimple.*

P r o o f . Let α be a valid wff that is not provable. Then, in L, $[\alpha] \neq 1$. But by [1, 2, 4], $[\alpha] \oplus [\alpha]^n = 1$ for all $n > 0$. By Lemma 1, $[\alpha]^* \in \mathrm{Rad}\, A$. Since $[\alpha] \neq 1$ we see that $\mathrm{Rad}\, L \neq 0$. □

Call a wff α a *Scarpellini* formula provided α is valid but not provable.

Let $\mathcal{U}$ be the set of Scarpellini formulas, and let $\mathcal{S}$ be the subset of $\mathcal{U}$ consisting of *closed* Scarpellini formulas. Thus if $\alpha \in \mathcal{U}$, then $\forall\alpha \in \mathcal{S}$ where $\forall\alpha$ is the universal closure of α. We also note that $\mathcal{U}$ is closed under $\odot$ and $\wedge$. We have

Corollary 13. $\mathrm{Rad}\, L = \big\{[\alpha] \colon \neg\alpha \in \mathcal{U}\big\} \cup \{0\}$.

P r o o f . From the proof of Proposition 12 we see that the right side is contained in the left side. Now let $[\alpha] \in \mathrm{Rad}\, L$, $[\alpha] \neq 0$. Then for all $n > 0$, we also have $n[\alpha] \in \mathrm{Rad}\, L$. Thus, $[\alpha] \odot n[\alpha] = 0$. Therefore, $[\alpha]^* \oplus \big([\alpha]^*\big)^n = 1$ for all $n > 0$. It follows that $\neg\alpha$ is valid. But $[\alpha] \neq 0$ therefore $[\alpha]^* \neq 1$, that is $\neg\alpha$ is not provable, and so is in $\mathcal{U}$. □

Corollary 14. *Let α and β be valid formulas. Then $\neg\alpha \to \beta$ is provable.*

P r o o f . We know from above that $[\neg\alpha], [\neg\beta] \in \mathrm{Rad}\, L$. Thus $[\neg\alpha] \odot [\neg\beta] = 0$. Therefore, $[\alpha] \oplus [\beta] = 1$. It now follows that $\neg\alpha \to \beta$ is provable. □

Thus, the valid but unprovable formulas are precisely the formulas whose negations determine the radical of L, that is the *co-infinitesimals* of L.

We observe that the valid but unprovable formulas generate a *perfect* subalgebra of L. Therefore, the following corollary holds:

Corollary 15. *$L_0 = \operatorname{Rad} L \cup (\operatorname{Rad} L)^*$ is a perfect subalgebra generated by $\mathcal{U}$.*

It follows from the incompleteness of first order logic that $\operatorname{Rad} L$ is not a principal ideal. In fact $\operatorname{Rad} L$ cannot be generated by a recursive set of generators.

Recall that a *weak unit* in an MV-algebra A is an element $a \in A$ such that $a^{\perp} = 0$. Does $\operatorname{Rad} L$ have a weak unit? We shall see later that this question has some relevance.

Proposition 16. *Let I be a β-ideal of L. Then the following statements hold:*

1) $\bigcap\{P \colon P$ *is a β-prime ideal*$\} = 0$;
2) $\bigcap\{M \colon M$ *is a β-maximal ideal*$\} = \operatorname{Rad} L$.

P r o o f . 1) follows easily from Proposition 9.

2) follows from the Corollary 10 which implies that if $[\alpha] \in \bigcap\{M \colon M$ is a β-maximal ideal$\}$, so is $n[\alpha]$ for all $n \in \omega$, and, consequently, $\operatorname{ord}\big([\alpha]^*\big)^n < \infty$. The Lemma 2 completes the argument. □

Now let $\mathcal{M} = \{A, X, F_n\}$ be a given linear model, and let f be an $\mathcal{M}$-interpretation and A be not locally finite. Consider the canonical model $\{\hat{A}, X, \hat{F}_n\}$ where $\hat{A} = A/M$, $\hat{F}_n(x_1, \dots, x_{d(n)}) = F_n(x_1, \dots, x_{d(n)})/M$ and M is the maximal ideal of A.

Proposition 17.

(1) *f is an $\langle \hat{A}, X\rangle$-interpretation;*
(2) $\operatorname{Val}\big(\alpha, \langle \hat{A}, X\rangle, f\big) = \operatorname{Val}\big(\alpha, \langle A, X\rangle, f\big)/M$.

P r o o f . We need only show that M preserves sums. Thus let $x_j \in M$ with $\Sigma_j x_j = x$. From [11] we know that $nx = \Sigma_j nx_j$. Thus, $(nx)^2 = \Sigma_j \Sigma_k nx_j \odot mx_k = 0$, since nx_j and mx_k are in the radical M. It follows that $\operatorname{id}(x)$ is nilpotent, and so $x \in M$. □

We note that the above proof actually shows that the radical in any MV-algebra is a closed ideal with respect to arbitrary supremum. Thus,

Corollary 18. *Let P be β-prime in L, and let M be the unique maximal ideal over P. Then M is a β-maximal ideal.*

Question: Are all maximal ideals β-ideals?

5. Model satisfiability

Similar to the case of classical logic, first order Łukasiewicz logic enjoys a compactness theorem. However, unlike the classical case, the compactness theorem does not imply completeness. This is so for two principal differences between the theories: 1) the deduction theorem in the Łukasiewicz case takes the form, if Δ is a set of sentences and $\Delta \cup \{\delta\} \vdash \alpha$, then for some integer $n > 0$, $\Delta \vdash \delta^n \to \alpha$, 2) from $\vdash \alpha \to \beta$ and $\vdash \neg\alpha \to \beta$, one cannot infer that β.

Thus, call a set of sentences Δ *satisfiable* if for some canonical model $\mathcal{M}$, we have $\mathrm{Val}(\delta, \mathcal{M}) = 1$ for all $\delta \in \Delta$. We set $\mathrm{Val}(\Delta, \mathcal{M}) = \{\mathrm{Val}(\delta, \mathcal{M})\}$.

A set Δ of wff of $\mathcal{L}$ is *consistent* provided there is no wff α such that $\Delta \vdash \alpha$ and $\Delta \vdash \neg\alpha$. Let $\triangle$ be a set of wff of $\mathcal{L}$. The $I_\triangle$ denotes the ideal, in the Lindenbaum algebra, generated by $\{[\delta]^* \colon \delta \in \triangle\}$. From [4] we know

Proposition 19. (Model Completeness Theorem) *Δ is consistent if and only if Δ is satisfiable.*

We note the following easy proposition.

Proposition 20. *With the above notations the following statements hold:*

1) *If Δ is consistent, then $L/I_\triangle \cong L_\triangle$;*
2) *If Δ is not consistent, then $I_\triangle$ is not proper.*

We further make the observation that for the compactness theorem we need not restrict ourselves to canonical models. That is, if Δ is satisfiable on some linear model, then it is satisfiable on a canonical model.

Given the nature of the set of "truth values" $[0, 1]$ of Łukasiewicz logic, we may consider weaker forms of satisfiability.

Thus, call Δ *weakly satisfiable* if for some canonical model $\mathcal{M}$, and $\mathcal{M}$ interpretation h, we have $\mathrm{Val}(\Delta, \mathcal{M}, h) > 0$.

Since Δ is a set of sentences, it doesn't matter which interpretation is used.

Clearly weak satisfiability does not imply consistency, since for $0 < r < 1$ we also have $0 < 1 - r < 1$. Nevertheless, no weakly satisfiable set of sentences contains the negation of a formal theorem. Note further that if $\Delta \subseteq \mathrm{Rad}\, L$, then Δ is not weakly satisfiable.

Going a bit further we may call a set of sentences Δ *weakly linearly satisfiable* if for some linear model $\mathcal{M}$, and $\mathcal{M}$ interpretation h, we have that $\mathrm{Val}(\Delta, \mathcal{M}, h) > 0$. Again we note that no weakly linearly satisfiable set Δ can contain the negation of a formal theorem.

We can ask the converse, that is, if Δ does not contain the negation of a formal theorem, is Δ weakly linearly satisfiable? Of particular interest is the question, if $\Delta_{\mathcal{S}}$ is the set of negations of the Scarpellini sentences, is $\Delta_{\mathcal{S}}$ weakly linearly satisfiable?

We shall derive necessary and sufficient conditions for this to be true.

PROPOSITION 21. *The following are equivalent:*

1) $\Delta_{\mathcal{S}}$ *is weakly linearly satisfiable;*
2) $[\Delta_{\mathcal{S}}]$ *is linearly ordered;*
3) $\mathcal{S}$ *is closed under* $\vee$.

P r o o f. 1) implies 2).

Assume that $\Delta_{\mathcal{S}}$ is weakly linearly satisfiable. Then there is a linearly ordered MV-algebra A and an A-model $\mathcal{M} = \{A, X, F_n, n > 0\}$, such that $\mathrm{Val}(\delta, \mathcal{M}) > 0$. As the members of $\Delta_{\mathcal{S}}$ are closed wff, the choice of an $\mathcal{M}$-interpretation is not relevant. Thus, let h be any $\mathcal{M}$-interpretation. Let $P = \{[\alpha] \in L\colon \mathrm{Val}(\alpha, \mathcal{M}, h) = 0\}$. P is well-defined and easily seen to be a prime ideal of L. Moreover $[\Delta_{\mathcal{S}}] \cap P = \emptyset$ where $[\Delta_{\mathcal{S}}] = \{[\delta]\colon \delta \in \Delta_{\mathcal{S}}\}$.

Let L_c be the subalgebra of L determined by the closed wff. Let I be the ideal of L_c generated by $[\Delta_{\mathcal{S}}]$. We claim that $I = [\Delta_{\mathcal{S}}] \cup \{0\}$. For let $0 \neq [\delta] \in I$; we can assume δ is a closed wff. Then $[\delta] \leq \Sigma_{i=1}^{n} [\delta]_i$ where each $[\delta]_i \in [\Delta_{\mathcal{S}}]$. But then $[\delta] \in \mathrm{Rad}\, L$. Thus $\neg\delta$ is a Scarpellini sentence. Thus $[\neg\neg\delta] = [\delta] \in [\Delta_{\mathcal{S}}]$. Hence $I \cap P = 0$. Let $P_c = P \cap L_c$. Then P_c is prime in L_c and $I \cap P_c = 0$. Therefore $I \subseteq P_c{}^{\perp}$. Since P_c is prime, we see that I must be linearly ordered, hence $[\Delta_{\mathcal{S}}]$ is linearly ordered.

2) implies 3).

Let $\sigma_1, \sigma_2 \in \mathcal{S}$. Then $[\neg\sigma_1]$ and $[\neg\sigma_2]$ are non-zero elements of $\mathrm{Rad}\, L$. Since $[\Delta_{\mathcal{S}}]$ is linearly ordered, we can safely assume that $[\neg\sigma_1] \wedge [\neg\sigma_2] = [\neg\sigma_1]$. Then we have $[\neg(\sigma_1 \vee \sigma_2)] = [\neg\sigma_1]$. So, $[\neg(\sigma_1 \vee \sigma_2)] = [\neg\sigma_1]$ is a non-zero element of $\mathrm{Rad}\, L$, and then $(\sigma_1 \vee \sigma_2) \in \mathcal{S}$.

3) implies 1).

Let $F = \{\sigma_1, \dots, \sigma_n\} \subseteq \mathcal{C}$ and set $\sigma = \sigma_1 \vee \cdots \vee \sigma_n$. Then $\sigma \in \mathcal{C}$. Thus $[\sigma] \neq 1$, so $[\neg\sigma] \neq 0$. Thus there is a model $\mathcal{M}_F = \{A_F, X_F, \{f_{F,n}\}\}$ and an $\mathcal{M}_F$-interpretation $h_F\colon V \to X_F$ with $\mathrm{Val}(\sigma, \mathcal{M}_F, h_F) < 1$. Note that by Proposition 9 and Proposition 11, A_F can be taken linearly ordered, X_F as $\{0, 1, \dots\}$ and $h_F(v_j) = j$. Let I be the set of non-empty subsets $F \subseteq \mathcal{C}$. Thus we have a family $\{\mathcal{M}_F\}$ of models and corresponding interpretations h_F as above. For $F \in I$, let $D(F) = \{G \in I\colon \mathrm{Val}(\sigma, \mathcal{M}_G, h_G) < 1 \text{ for all } \sigma \in F\}$. Then $D(F) \neq \emptyset$ and moreover, $D(F_1 \cup F_2) \subseteq D(F_1) \cap D(F_2)$. Therefore the set of $D(F)$, $F \in I$, forms a filter base in the boolean algebra 2^I. Let $\mathcal{F}$ be a maximal filter on this base. Let $\mathcal{M}$ be the resulting ultraproduct of the family $\{\mathcal{M}_F\}_F$ and define $h = \langle h_F \rangle / \mathcal{F}$. Then h is an $\mathcal{M}$-interpretation and for $\sigma \in \mathcal{C}$ we have that $\mathrm{Val}(\sigma, \mathcal{M}, h) < 1$. $\square$

Proposition 22. *With the above notations we have that if* $\mathrm{id}([\Delta_{\mathcal{S}}])$ *is orthogonally free, then* $[\Delta_{\mathcal{S}}]$ *is linearly ordered.*

Proof. Assume that $\mathrm{id}([\Delta_{\mathcal{S}}])$ is orthogonally free. Then by Lemma 3, $\mathrm{id}([\Delta_{\mathcal{S}}])$ is totally ordered. Hence $[\Delta_{\mathcal{S}}]$ is linearly ordered. □

What relevance does the weak linear satisfiability of $\Delta_{\mathcal{S}}$ have? For one thing, we obtain a model and interpretation that separate all the Scarpellini formulas from the formal theorems. Thus we would obtain a stronger completeness theorem, namely

Proposition 23. *Suppose that* $\Delta_{\mathcal{S}}$ *is weakly linearly satisfiable on a model* $\mathcal{M}$. *If* α *is a closed wff that is valid and is also* $\mathcal{M}$*-valid, then* $\vdash \alpha$.

Proof. If α is valid and $\mathcal{M}$-valid, then $\neg\alpha \notin \Delta_{\mathcal{S}}$. Thus $\alpha \notin \mathcal{S}$. Therefore α is valid and not a Scarpellini formula, hence must be provable. □

It is open whether or not $[\Delta_{\mathcal{S}}]$ is linearly ordered; however, if it is linearly ordered it forces conditions on L_c. We describe some of these now.

For a wff α, let $\mathcal{I}(\alpha) = \{\mathrm{Val}(\alpha, \mathcal{M}, h) \colon \mathcal{M} \text{ is a canonical model}\}$. Then we have that $\mathcal{I}(\alpha)$ is a closed subinterval of $[0,1]$, see [2, Corollary 2 of Theorem 3, p. 74].

Proposition 24. *The MV-algebra* L *has no non-trivial idempotents.*

Proof. Suppose that $[\alpha]$ is an idempotent. Then $\vdash \alpha + \alpha \to \alpha$. Hence for any model $\mathcal{M}$ and any interpretation h, $\mathrm{Val}(\alpha, \mathcal{M}, h) \in \{0,1\}$. From the above mentioned Corollary 2 in [2], it follows that $\mathcal{I}(\alpha) = \{0\}$ or $\mathcal{I}(\alpha) = \{1\}$. Thus α is valid or $\neg\alpha$ is valid. If α is valid, then $\vdash \alpha + \alpha$ and so $[\alpha] + [\alpha] = 1$, and so $[\alpha] = 1$. If $\neg\alpha$ is valid, we have similarly that $[\alpha] = 0$. □

Proposition 25. $\mathrm{Rad}\, L$ *is an anti-boolean ideal.*

Proof. $[\alpha]/\mathrm{Rad}\, L$ is an idempotent iff $([\alpha]/\mathrm{Rad}\, L) \wedge ([\alpha]/\mathrm{Rad}\, L)^* = 0$ iff $[\alpha] \wedge [\alpha]^* \in \mathrm{Rad}\, L$. Hence $\alpha \vee \neg\alpha$ is a Scarpellini formula. Thus for all $[0,1]$-models $\mathcal{M}$ and all $\mathcal{M}$-interpretations h, we have $\mathrm{Val}(\alpha \vee \neg\alpha, \mathcal{M}, h) = 1$. Thus $\mathcal{I}(\alpha) \subseteq \{0,\ 1\}$. Now $\mathcal{I}(\alpha) = \{1\}$ implies $\vdash \alpha$ or α is a Scarpellini formula. In either case, $[\alpha]/\mathrm{Rad}\, L = 1$. Similarly, if $\mathcal{I}(\alpha) = \{0\}$, then $[\alpha]/\mathrm{Rad}\, L = 0$. Thus $L/\mathrm{Rad}\, L$ has no non-trivial idempotents. □

Corollary 26. *Suppose that* P_c *is a prime ideal such that* $\mathrm{Rad}\, L_c \cap P_c = \{0\}$. *Then* P_c *is unique and* $\mathrm{Rad}\, L_c = P_c^{\perp}$.

Proof. If $\mathrm{Rad}\, L_c \cap P_c = \{0\}$, then $\mathrm{Rad}\, L_c \subseteq P_c^{\perp}$. Now P_c is prime, therefore local, so by Lemma 5, $P_c^{\perp} \subseteq \mathrm{Rad}\, L_c$ or $P_c^{\perp}$ is generated by an idempotent. As $L_c \subseteq L$, the latter is impossible by Proposition 23, hence we may infer $\mathrm{Rad}\, L_c = P_c{}^{\perp}$. If Q were another prime in L_c with $\mathrm{Rad}\, L_c \cap Q = \{0\}$, then $Q^{\perp} \subseteq P_c{}^{\perp}$, hence, by the Corollary 7, $Q = P_c$. □

COROLLARY 27. *If for some prime $P_c \subseteq L_c$, $\mathrm{Rad}\, L_c \cap P_c = \{0\}$, then all primes in L_c other than P_c are essential.*

PROPOSITION 28. *Let A be an MV-algebra with no non-trivial idempotents. If there exist two distinct non-essential prime ideals P, Q, then the radical is not linearly ordered.*

Proof. Since P and Q are prime ideals of A, then they are local ideals. Hence, by Lemma 5, we have that $P^{\perp}, Q^{\perp} \subseteq \mathrm{Rad}(A) \neq \{0\}$. Let $x, y \neq 0$, where $x \in P^{\perp}$ and $y \in Q^{\perp}$. We can assume that $x \leq y$. Thus $x \in P^{\perp} \cap Q^{\perp}$. From Proposition 6 it follows that $P = Q$ which is absurd. □

One may also consider other forms of satisfiability and validity. For example, call a wff α *quasi-valid* (*qvalid* for short) on a model $\mathcal{M} = \{A, X, F_n\}$ if for all $\mathcal{M}$-interpretations h, we have $\mathrm{Val}(\alpha, \mathcal{M}, h) \in \text{co-Rad}\, A$.

It is interesting to see whether one could characterize those sentences that are qvalid on all "local" models. Indeed, we have:

PROPOSITION 29. *Let α be a sentence. Then the following are equivalent:*

(i) *α is qvalid on all local models;*

(ii) *α is valid.*

Proof. Assume (i) true. Then, in particular, α is qvalid on all canonical models. Since, for any canonical model A, we have $\text{co-Rad}\, A = \{1\}$, then we get (ii).

Viceversa, suppose α is valid. Let us also assume that there is a local model $\mathcal{M} = \{A, X, F_n\}$, such that $\mathrm{Val}(\alpha, \mathcal{M}) \notin \text{co-Rad}\, A$. As A is local, this means that $\mathrm{Val}(\alpha, \mathcal{M}) < r$, for every $r \in \text{co-Rad}\, A$. Applying Proposition 11 to $\mathcal{M}$ we obtain a model $\mathcal{M}/\mathrm{Rad}\, A = \{A/\mathrm{Rad}\, A, X, F_n/\mathrm{Rad}\, A\}$ such that $\mathrm{Val}(\alpha, \mathcal{M}/\mathrm{Rad}\, A) = \mathrm{Val}(\alpha, \mathcal{M})/\mathrm{Rad}\, A$. Thus $\mathrm{Val}(\alpha, \mathcal{M}/\mathrm{Rad}\, A) < (r/\mathrm{Rad}\, A) = 1$. This is impossible because $A/\mathrm{Rad}\, A$ is a canonical model, and then $(\mathrm{Val}(\alpha, \mathcal{M})/\mathrm{Rad}\, A)$ has to be 1. □

Calling a model $\{A, X, F_n\}$ *perfect* if A is a perfect MV-algebra, we can specialize the above result obtaining the following:

PROPOSITION 30. *Let α be a sentence. Then the following are equivalent:*

(i) *α is qvalid on all perfect models;*

(ii) *α is $\{0, 1\}$-valid.*

P r o o f . Assume (i) true. Since the MV-algebra $B = \{0,1\}$ is perfect, then we have that α is qvalid on B. Furthermore, since $\text{co-Rad}\, A = \{1\}$ we get α is $\{0,1\}$-valid.

Viceversa, suppose that (ii) is true and that, for some perfect model $\mathcal{M} = \{A, X, F_n\}$, we have $\mathrm{Val}(\alpha, \mathcal{M}) \notin \text{co-Rad}\, A$. As A is perfect this means that $\mathrm{Val}(\alpha, \mathcal{M}) \in \mathrm{Rad}\, A$. Applying Proposition 11 to $\mathcal{M}$ we obtain a model $\mathcal{M}/\mathrm{Rad}\, A = \{A/\mathrm{Rad}\, A, X, F_n/\mathrm{Rad}\, A\}$ such that $\mathrm{Val}(\alpha, \mathcal{M}/\mathrm{Rad}\, A) = \big(\mathrm{Val}(\alpha, \mathcal{M})\big)/\mathrm{Rad}\, A = 0$. This is impossible since $A/\mathrm{Rad}\, A \cong \{0,1\}$. $\square$

It would be interesting to know whether $\{\alpha\colon [\alpha] \in \mathcal{P}\}$ is recursively enumerable.

The Scarpellini sentences are, of course, those sentences qvalid on all linearly ordered models.

It might be interesting to see if one could characterize those sentences that are qvalid on all "local" models.

The question remains open as to whether or not there is a general completeness theorem relative to a non-canonical linearly ordered MV-algebra A. That is, is there a linearly ordered (necessarily non-archimedean) MV-algebra A such that a wff α is A-valid iff $\vdash \alpha$?

REFERENCES

[1] BELLUCE, L. P.: *Theory of infinite valued predicate logic*, thesis (1963).

[2] BELLUCE, L. P.: *Further remarks on infinite valued predicate logic*, J. Symbolic Logic **29** (1964), 69–78.

[3] BELLUCE, L. P.: *α-complete MV-algebras*, in: Non-Classical Logics and their Applications to Fuzzy Subsets (U. Hoehle, E. P. Klement, eds.), Kluwer Academic Publishers, Dordrecht, 1995, pp. 7–21.

[4] BELLUCE, L. P.—CHANG, C. C.: *A weak completeness theorem for infinite valued predicate logic*, J. Symbolic Logic **28** (1963), 43–50.

[5] CHANG, C. C.: *Algebraic analysis of infinite valued logic*, Trans. Amer. Math. Soc. **88** (1958), 467–490.

[6] CHANG, C. C.: *A new proof of the completeness of the Lukasiewicz axioms*, Trans. Amer. Math. Soc. **93** (1959), 74–90.

[7] CIGNOLI, R.—D'OTTAVIANO, I. M. L.—MUNDICI, D.: *Algebraic Foundations of Many-Valued Reasoning*, Trends in Logic, Vol. 7, Kluwer, Dordrecht, 2000.

[8] HÁJEK, P.: *Rational Pavelka predicate logic is a conservative extension of Lukasiewicz predicate logic*, J. Symbolic Logic **65** (2000), 669–682.

[9] HAY, L. S.: *Axiomatization of the infinite valued predicate calculus*, J. Symbolic Logic **28** (1963), 77–86.

[10] RUTLEDGE, J. R.: *A preliminary investigation of the infinite many-valued predicate calculus*, Ph.D. Thesis at Cornell University, 1959.

[11] SCARPELLINI, B.: *Die Nichtaxiomatisierbarkeit des unendlichwertigen Pradikatenkalkuls von Lukasiewicz*, J. Symbolic Logic **27** (1962), 159–170.

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