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CHARACTERIZATION OF FIXED EFFECTS AS SPECIAL CASE OF GENERALIZED RANDOM EFFECTS

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ABSTRACT. The objective of this paper is to emphasize the common nature of fixed and random factors in a mixed linear statistical model. We show that factors can be classified into fixed and random factors by means of covariance matrices of their level effects. In particular, classical reparameterization conditions for fixed factors can be formulated in this way.

1. Introduction

Mixed linear statistical models contain both fixed and random factors. During the long history of the Analysis of Variance, various mixed model formulations have been proposed and the discussion around this topic is still continuing (Scheffè 1959, Searle 1971, Harville 1978, Seely *et al.* 1988, Khuri *et al.* 1998). Currently the most widely accepted version of mixed models is what is defined and realized in the MIXED procedure of the SAS system (SAS Institute Inc. 1996). In models of this type the fixed and random factors are clearly discriminated and the interactions between fixed and random factors are considered as independent random effects. Fixed factors are treated as constants while levels of a random factor are represented as random variables, possibly with a restricted covariance structure.

The classical distinction between random and fixed factors can be described as follows. Let us call factor levels represented in the data sampled levels. Let $U_1, \dots, U_k$ be the sets of sampled levels of factors $\gamma_1, \dots, \gamma_k$ respectively and let W_i be the population of all levels of the factor γ_i $(i = 1, \dots, k)$. The factor γ_i is called fixed if $U_i = W_i$ and random if a random sample $U_i \subset W_i$ is presented in the data and W_i is infinite. In the intermediate case where a random sample

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$U_i \subset W_i$ is presented in the data and W_i is finite, the factor γ_i may be called quasi-random.

In the present paper we consider a different approach where all factors (fixed and random) in the model are treated as generalized random factors having random effects and random interactions. In this context, the separation of factor types goes by imposing constraints on their covariance matrices. The advantages of this approach will be clear, for instance, in simulation studies where levels of fixed factors must be generated randomly but so that the classical reparameterization conditions hold.

2. Results

Let vector ξ consist of sampled levels of some factor. Let P be an orthogonal matrix of an transformation of the vector space generated by ξ.

Definition 1. Covariance matrix $D(\xi)$ is called invariant with respect to the transformation P of the factor ξ (further simply P-*invariant*) if $D(\xi) = D(P\xi)$.

Obviously, the P-invariance can be considered as a property of the covariance matrix $\Sigma = D(\xi)$ only: Σ is P-invariant if $P\Sigma P' = \Sigma$.

Definition 2. Let V be an $n \times n$ orthogonal matrix which columns are eigenvectors which correspond to the ordered eigenvalues $\lambda_1, \ldots, \lambda_k$ (with multiplicities $n_1, \ldots, n_k$ respectively) of Σ. Then the matrix C is called *associated with* V if $C = \mathrm{diag}\{C_{n_1}, \ldots, C_{n_k}\}$ where C_{n_i} is an orthogonal matrix of order n_i $\left(\sum\limits_{i=1}^{k} n_i = n\right)$.

In the following lemma we obtain a characterization of the class of orthogonal matrices P for which the matrix Σ is P-invariant.

Lemma 1. *Let Σ be an $n \times n$ covariance matrix of factor ξ, i.e., $D(\xi) = \Sigma$, let V be an orthogonal matrix which columns are eigenvectors that correspond to the ordered eigenvalues of Σ, and let C be an orthogonal matrix of order n which is associated with V. Then the matrix Σ is P-invariant if and only if $P = VCV'$.*

P r o o f. Suppose first that the $n \times n$ covariance matrix Σ is P-invariant, i.e., $\Sigma = P\Sigma P'$. Let V be an orthogonal matrix which columns are eigenvectors that correspond to the ordered eigenvalues of Σ, i.e., $\Sigma V = V\Lambda$ $(V'\Sigma V = \Lambda$, $V\Lambda V' = \Sigma)$, where Λ is a diagonal matrix which diagonal elements $\lambda_1, \ldots, \lambda_k$ are the ordered distinct eigenvalues of Σ with multiplicities $n_1, \ldots, n_k$ respectively $\left(\sum\limits_{i=1}^{k} n_i = n\right)$. We show, that then $P = VCV'$.

Because Σ is P-invariant

$$\Sigma = P\Sigma P' = PV\Lambda V'P' = (PV)\Lambda(PV)' = U\Lambda U' .$$

Thus, $U = PV$ is an orthogonal matrix which columns are eigenvectors of matrix Σ: $\Sigma U = U\Lambda$. Then there exists an orthogonal matrix C such that $U = VC$, and because the columns of U are eigenvectors of Σ, matrix C should be associated with V. As a result, $U'\Sigma U = C'V'\Sigma VC = C'\Lambda C = \Lambda$, where $\Lambda = \mathrm{diag}\{\lambda_1 I_{n_1}, \ldots, \lambda_k I_{n_k}\}$ (here I_{n_i} is the identity matrix of order n_i). Taking into account $U = PV$ and $U = VC$, we get $P = VCV'$.

Let us show now that if $P = VCV'$, where V and C are defined as above, then Σ is P-invariant. It is easy to see that

$$P\Sigma P' = VCV'\Sigma VC'V' = VC\Lambda C'V' = V\Lambda V' = \Sigma ,$$

and the covariance matrix Σ is P-invariant. This completes the proof of the lemma. □

It is intuitively justified to require that the covariance matrix of a factor must be invariant with respect to permutation of its levels. Suppose that all the levels of a factor are random variables that form a population. In a particular experiment we have a random sample from this population. It is natural to assume that all levels have the same chance to be sampled, and the order of levels must not affect the covariance matrix of sampled levels of the factor. In other words, the covariance matrix of a factor should be invariant with respect to the permutation of levels of this factor. The next lemma shows, that the covariance matrix must have specific form to be permutation invariant.

Lemma 2. *If the covariance matrix $D(\xi)$ of a factor ξ is invariant, with respect to permutations of factor levels, then*

$$D(\xi) = \begin{pmatrix} \sigma^2 & \tau & \ldots & \tau \\ \tau & \sigma^2 & \ldots & \tau \\ \vdots & \vdots & \ddots & \vdots \\ \tau & \tau & \ldots & \sigma^2 \end{pmatrix} ,$$

where $\sigma^2 = \mathrm{Cov}(\xi_i, \xi_i)$ and $\tau = \mathrm{Cov}(\xi_i, \xi_j)$, $i \neq j$.

P r o o f. Since the covariance matrix $D(\xi)$ of ξ is invariant with respect to permutations of levels of ξ, the equality $D(\xi) = D(P\xi)$ must hold for each permutation matrix P. Let ξ_i and ξ_j $(i \neq j)$ be levels of the factor ξ and let $D(\xi)$ be the covariance matrix of ξ:

$$D(\xi) = (d_{ij}) = \mathrm{Cov}(\xi_i, \xi_j) = \begin{cases} \sigma_i^2 , & i = j , \\ \sigma_{ij} , & i \neq j . \end{cases}$$

Permutation of ξ_i and ξ_j in ξ implies the interchange of the elements of matrix $D(\xi)$: d_{ik} and d_{jk} will be interchanged for each k, $k \neq i, j$, and d_{ii} and d_{jj} will be interchanged also. From here the stated form for $D(\xi)$ follows. □

The next theorem shows that classical reparameterization of a factor can be considered as a specific restriction on the covariance matrix of factor levels.

THEOREM 1. *Let $\xi = (\xi_1, \dots, \xi_n)'$ be a vector which consists of sampled levels of a factor and let $\xi_i \neq \xi_j$ a.s. Let $E(\xi) = 0$ and assume that $D(\xi)$ is P-invariant, i.e., $PD(\xi)P' = D(\xi)$, where P is an orthogonal matrix of permutations of the coordinates of ξ. Then the following conditions are equivalent:*

(**a**) $\sum\limits_{i=1}^{n} \xi_i = 0$.

(**b**) $D(\xi) = h\left(I_n - \frac{1}{n} J_n\right)$,

where h is a known constant, $J_n = \mathbf{1} \cdot \mathbf{1}'$ is $n \times n$ matrix, where $\mathbf{1}$ is $n \times 1$ vector of ones.

(**c**) *$D(\xi)$ has $(n-1)$ nonzero eigenvalues.*

P r o o f. Let us show that (**a**) implies (**b**). Invariance of the covariance matrix $D(\xi)$ with respect to permutations of the coordinates of ξ implies, according to Lemma 2, that $D(\xi)$ must have the following structure:

$$D(\xi) = \tilde{\sigma}^2 I_n + \tau J_n \,,$$

where $\tilde{\sigma}^2 + \tau = \sigma^2$, $\sigma^2 = \operatorname{Cov}(\xi_i, \xi_i)$, $\tau = \operatorname{Cov}(\xi_i, \xi_j)$, $i \neq j$. Further, because of the condition (**a**),

$$0 = D(\mathbf{1}'\xi) = \mathbf{1}' D(\xi) \mathbf{1} = \mathbf{1}'(\tilde{\sigma}^2 I_n + \tau J_n)\mathbf{1} = \tilde{\sigma}^2 n + \tau n^2 .$$

From here it follows that $\tau = -\frac{\tilde{\sigma}^2}{n}$. Using relationship $\sigma^2 = \tilde{\sigma}^2 + \tau$, we get

$$\tilde{\sigma}^2 = \frac{n\sigma^2}{n-1} \,, \qquad \tau = -\frac{\sigma^2}{n-1} .$$

Thereby,

$$D(\xi) = \frac{n\sigma^2}{n-1}\left(I_n - \frac{1}{n} J_n\right) = h\left(I_n - \frac{1}{n} J_n\right),$$

where $h = \frac{n\sigma^2}{n-1}$.

The condition (**c**) follows from (**b**) immediately.

Let us show that (**c**) implies (**a**). As it was noted above, the invariance of $D(\xi)$ with respect to the permutations of factor ξ levels guarantees $D(\xi) = c_1 I_n + c_2 J_n$, where c_1 and c_2 are known constants. Because $D(\xi)$ is singular, we have $c_2 \neq 0$ and $D(\xi)$ has at least one zero eigenvalue. The condition $\xi_i \neq \xi_j$ a.s. implies that $c_1 \neq 0$.

Let us define matrix

$$A = \frac{1}{nc_2} D(\xi) = cI_n + \frac{1}{n} J_n \,,$$

where $c \neq 0$. The eigenvalues $\lambda_1, \ldots, \lambda_n$ of A are the roots of the equation

$$|A - \lambda I_n| = \left| cI_n + \frac{1}{n} J_n - \lambda I_n \right| = 0 \,.$$

Rewriting the equation as

$$|A - \lambda I_n| = \left| \frac{1}{n} J_n - (\lambda - c) I_n \right| = 0$$

we find that $\lambda_i - c$ are the eigenvalues of matrix $\frac{1}{n} J_n$: one eigenvalue equals to 1 and the remaining $(n-1)$ are zeros. Therefore,

$$\begin{aligned} \lambda_1 &= 1 + c \,, \\ \lambda_2 &= \cdots = \lambda_n = c \,. \end{aligned}$$

By definition $c \neq 0$, thus $(n-1)$ eigenvalues $\lambda_2 = \cdots = \lambda_n$ are different from zero. At the same time, because $D(\xi)$ is singular, A is also singular and at least one of the eigenvalues of A should equal to zero. Consequently, $\lambda_1 = 1 + c = 0$.

Let v_1 be a nonzero eigenvector which corresponds to λ_1. Because $E(\xi) = 0$,

$$E(v_1' \xi) = 0 \,, \quad D(v_1' \xi) = v_1' D(\xi) v_1 = 0 \,.$$

Thus $v_1' \xi = 0$ a.s. The $n \times 1$ vector $\mathbf{1} = (1, \ldots, 1)'$ is the eigenvector corresponding to the eigenvalue $\lambda_1 = 0$, consequently $\mathbf{1}' \xi = 0$ what is equivalent to (**a**). □

Discussion

Theorem 1 can be used for characterizing factors on the basis of properties of their covariance matrices. Let the covariance matrix $D(\xi)$ of factor ξ be invariant with respect to permutations of factor levels. The factor ξ is then called random if its covariance matrix $D(\xi)$ is $\sigma^2 I$, where $\sigma^2 = \mathrm{Cov}(\xi_i, \xi_i)$, quasi-random if its covariance matrix $D(\xi)$ is $\tilde{\sigma}^2 I + \tau J$, where $\sigma^2 = \tilde{\sigma}^2 + \tau$, $\tau = \mathrm{Cov}(\xi_i, \xi_j)$, and factor ξ is called fixed if $D(\xi) = h\big(I - \frac{1}{n} J\big)$, here $i, j = 1, \ldots, n$, and n is the number of levels of ξ.

The motivation for searching new ways for characterizing factors in a mixed linear model has arisen from intuitive understanding of common nature of random and fixed factors. In the present paper we have proposed an approach that enables to consider factors within a common framework. We also suggest to use the covariance matrix as a tool of reparameterization of factors in mixed linear models. The problem of characterizing interaction effects between different types of factors is still an open question.

REFERENCES

[1] HARVILLE, D. A.: *Alternative formulations and procedures for the two-way mixed model*, Biometrics **34** (1978), 441–453.

[2] KHURI, A.—MATHEW, T.—SINHA, B. K.: *Statistical Tests for Mixed Linear Models*, Wiley, New York, 1998.

[3] SAS Institute Inc., SAS/STAT Software: Changes and Enhancements through Release 6.11, Cary, NC: SAS Institute Inc. (1996).

[4] SCHEFFÈ, H.: *The Analysis of Variance*, Wiley, New York, 1959.

[5] SEARLE, S. R.: *Linear Models*, Wiley, New York, 1971.

[6] SEELY, J. F.—RADY, E. H.: *When can random effects be treated as fixed effects for computing a test statistic for a linear hypothesis*, Comm. Statist. Theory Methods **17** (1988), 1089–1109.

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