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A KOTZIG TYPE THEOREM FOR LARGE MAPS ON SURFACES

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Dedicated to Professor Ján Jakubík on the occasion of his 80th birthday

ABSTRACT. An edge e of an embedded multigraph G with A and B as end-vertices is called an (a,b)-*edge* if $\deg_G(A) = a$ and $\deg_G(B) = b$. A. Kotzig in 1955 proved that every 3-connected planar graph G contains an (a,b)-edge with $a+b \le 13$; the bound being tight. The goal of this paper is to prove the following theorem:

Let G be a normal map on a surface $\mathscr{M}$ of Euler characteristic $\chi(\mathscr{M}) \le 0$ with the number of vertices $n > 26|\chi(\mathscr{M})|$. Then G contains an (a,b)-edge such that:

$$\begin{array}{rl} \text{(i)} & a = 3 \quad \text{and} \quad 3 \le b \le 12, \text{ or} \\ \text{(ii)} & a = 4 \quad \text{and} \quad 4 \le b \le 8, \text{ or} \\ \text{(iii)} & 5 \le a \le 6 \quad \text{and} \quad 5 \le b \le 6. \end{array}$$

The bounds 12, 8, 6 are tight.

1. Introduction

An orientable compact 2-manifold $\mathscr{S}_g$ or orientable surface $\mathscr{S}_g$ (see [R]) of genus g is obtained from the sphere by adding g handles. Correspondingly, a non-orientable compact 2-manifold $\mathscr{N}_q$ of genus q is obtained from the sphere by adding q crosscaps. The Euler characteristic is defined by

$$\chi(\mathscr{S}_g) = 2 - 2g \quad \text{and} \quad \chi(\mathscr{N}_q) = 2 - q\,.$$

Compact 2-manifolds are shortly called *surfaces* throughout the paper. Pseudograph may have loops or multiple edges. If a pseudograph G is embedded in a surface $\mathscr{M}$ then the connected components $\mathscr{M} - G$ are called the *faces* of G. If each face is an open disc then the embedding is called a *2-cell embedding.* The *facial walk* of a face α in a 2-cell embedding is the shortest closed walk induced

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by all the edges incident to α. The *degree* of a face α of a 2-cell embedding is the length of its facial walk. Vertices and faces of degree i are called *i-vertices* and *i-faces*, respectively. Vertices of degree $\geq k$ will be denoted by *$\geq k$-vertices*. We will denote the number of i-vertices in G by $n_i(G)$ or n_i. We call an edge e an *(a,b)-edge* if the endvertices of e are an a-vertex and a b-vertex.

If G is a 2-cell embedding of a pseudograph in a surface $\mathscr{M}$ and each vertex has valence at least three then G is called a *map* in $\mathscr{M}$. If, in addition, G is 2-connected, the embedding has "representativity" at least two, and each face has valence at least three, then G is called a *normal map*. A map, in which each face is bounded by three edges, can contain noncontractible loops. In a normal map this is excluded. A normal map G is called a *semitriangulation* if each face is a 3-face. (Note that multiple edges can appear in semitriangulation, otherwise G is called a *triangulation*). Let us recall that the representativity $\operatorname{rep}(G,\mathscr{M})$ (or face width) of (a 2-cell) embedded graph G into a surface $\mathscr{M}$ is equal to the smallest number k such that $\mathscr{M}$ contains a noncontractible closed curve which intersects the pseudograph G in k points (see, e.g., [RV]). It is obvious that a normal map contains no loops. For a map G let $V(G), E(G)$ and $F(G)$ be the vertex set, the edge set and the face set of G, respectively.

For a 2-cell embedding in a surface $\mathscr{M}$ Euler's formula holds:

$$\left|V(G)\right| - \left|E(G)\right| + \left|F(G)\right| = \chi(\mathscr{M}).$$

The degree of a vertex A in G is denoted by $\deg_G(A)$ or $\deg(A)$, if G is known from the context. Analogously, $\deg_G(\alpha)$ or $\deg(\alpha)$ denotes the degree of a face α.

In 1955 A. Kotzig [K] proved that every convex 3-polytope (i.e., every 3-connected planar graph) contains an (a,b)-edge with $a+b \leq 13$ in general and with $a+b \leq 11$ in the absence of faces of degree 3. Here we formulate it in a sligtly stronger form (for a simple proof see, e.g., [Je2]).

Theorem 1. *Every 3-connected planar graph contains a $(3,a)$-edge, $3 \leq a \leq 10$, or a $(4,b)$-edge, $4 \leq b \leq 7$, or a $(5,c)$-edge, $5 \leq c \leq 6$. The bounds 10, 7 and 6 are best possible.*

This theorem has been generalized and/or extended in many directions. For recent surveys see [JV1, JV4]. Ivančo [I] generalized the theorem in the following way:

Theorem 2 [I]. *Let G be a connected graph of orientable genus g and minimum degree $\delta(G) \geq 3$. Then G contains an (a,b)-edge with*

$$a+b \leq \begin{cases} 2g+13 & \text{if } 0 \leq g \leq 3, \\ 4g+7 & \text{if } g \geq 3 \end{cases}$$

and if G does not contain 3-cycles,

$$a+b \le \begin{cases} 8 & \text{if } g=0\,, \\ 4g+5 & \text{if } g \ge 1\,. \end{cases}$$

The bounds are best possible.

The aim of this paper is to show that the result of Ivančo can be essentially improved if the embedded graph G has a large number of vertices.

THEOREM 3. *Let G be a normal map on a surface $\mathscr{M}$ of Euler characteristic $\chi(\mathscr{M}) \le 0$ and let n be the number of vertices of G. If*

(a) $\sum\limits_{\deg_G(X)\ge 7} \big(\deg_G(X)-6\big) > 48\big|\chi(\mathscr{M})\big|$, *or*

(b) $n > 26\,\big|\chi(\mathscr{M})\big|$

then G contains an (a,b)-edge so that

$$\begin{array}{rl} \text{(i)} & a=3 \quad \text{and} \quad 3\le b\le 12\,, \ \text{or} \\ \text{(ii)} & a=4 \quad \text{and} \quad 4\le b\le 8\,, \ \text{or} \\ \text{(iii)} & 5\le a\le 6 \quad \text{and} \quad 5\le b\le 6\,. \end{array}$$

The bounds 12, 8, and 6 are best possible.

If in the case of a large "positive charge" we suppress 3- and 4-vertices we obtain the sharper bound 11 (see [JV2]).

THEOREM 4 **[JV2].** *Let G be a 2-cell embedding of a graph in a surface $\mathscr{M}$ of Euler characteristic $\chi(\mathscr{M}) \le 0$ with minimum degree $\delta(G) \ge 5$. If $\sum\limits_{\deg_G(X)\ge 7} \big(\deg_G(X)-6\big) > 16\big|\chi(\mathscr{M})\big|$ then G contains an (a,b)-edge so that $a+b\le 11$.*

For the torus, G r ü n b a u m and S h e p a r d [GS] proved the sharp bound $a+b\le 15$ which holds for all polyhedral maps of the torus.

The bound for the family of all polyhedral maps $\mathscr{M}$ has another structure, namely, a precise bound is $\max\{a,b\} \le 2\left\lfloor \frac{5+\sqrt{49-24\chi(\mathscr{M})}}{2} \right\rfloor$ (see [JV5]). For the family of all triangulations of orientable surfaces Z a k s [Z] obtained a sharp bound: $a+b \le 3+2\left\lfloor \frac{5+\sqrt{49-24\chi(\mathscr{M})}}{2} \right\rfloor$. In the family of all loopless 3-connected pseudographs a precise bound is: $\max\{a,b\} \le 12\Big(1+\frac{|\chi(\mathscr{M})|}{3}\Big)$ (see [JV6]).

2. Proof of the Theorem 3—the upper bounds

Let G be a counterexample with n vertices, which has among all counterexamples with n vertices the maximum number of edges.

Notation. The vertex X of G is called *minor*, if the degree of X is at most 5. The vertex X of G is called *major*, if the degree of X is at least 7. Let us denote by $p(X,A)$ the number of edges joining the vertices X and A in the normal map G. The number $p(X,A)$ is called the multiplicity of the edge XA. If $p(X,A)=1$ then the edge XA is called the *simple edge*.

Claim. *G is a semitriangulation.*

Proof of the Claim. Suppose G has an *r-gon* α, $r \geq 4$. Let A, B, C and D be consecutive vertices incident with the face α. If α is incident with a vertex of degree at most 6, say B, then both neighbours A, C of B incident with α have degrees at least 7. We insert the edge AC into α. The edge AC has endvertices of degrees at least 8. If α is incident only with vertices of degree at least 7, then we can insert the edge AC. In both cases the new edge is no loop and the insertion of the new edge into α leads to a counterexample with one edge more than in G. □

We consider only 2-cell embeddings of graphs. Hence Euler's formula implies:

$$\sum_{A\in V(G)} \big(\deg_G(A)-6\big) + 2\sum_{\alpha\in F(G)} \big(\deg_G(\alpha)-3\big) = 6\,\big|\chi(\mathscr{M})\big| \tag{1}$$

because $\chi(\mathscr{M}) \leq 0$.

Since G is a semitriangulation, $\sum\limits_{\alpha\in F(G)} \big(\deg_G(\alpha)-3\big) = 0$.

In the following we use the Discharging method. We assign to each vertex A the charge $c(A) = \deg_G(A) - 6$ and to each face α the charge $c(\alpha) = 2\deg_G(\alpha) - 6$, respectively. These charges of the vertices and faces will be locally redistributed to charges $h(A)$ and $h(\alpha)$ of vertices and faces, respectively, by the following rules. First we apply Rule **R1**, then Rule **R2** is applied.

Let $k := \deg_G(X)$.

Rule R1.

(a) If $7 \leq k \leq 12$ then X sends $\frac{3}{2} \times \frac{k-6}{k}$ to each minor neighbour A through each connecting edge, i.e., the vertex X sends the total charge $p(X,A) \times \frac{3}{2} \times \frac{k-6}{k}$ to A.

(b) If $k \geq 13$ then X sends 1 to each minor neighbour A through each connecting edge, i.e., the vertex X sends the total charge $p(X,A) \times 1$ to A.

We determine now a lower bound for the new charge $h(X)$ for each vertex $X \in V(G)$.

Lemma 1. *After applying Rule* **R1** *to every vertex* X *of* G, *the new charge is*

$$h(X) \geq \begin{cases} 0 & \text{if } k = 3, 4, 6\,, \\ \frac{1}{14}, \frac{5}{14}, \frac{1}{2}, 1, 1, \frac{35}{22}, \frac{3}{2} & \text{if } k = 5, 7, 8, 9, 10, 11, 12\,, \text{ respectively}\,, \\ (k-6) - \lfloor \frac{k}{2} \rfloor & \text{if } k \geq 13\,, \end{cases}$$

i.e., $h(X) \geq \frac{1}{8}(k-6)$.

Proof of Lemma 1. We distinguish several cases.

1. $k = 3$. The vertex X is only adjacent with ≥ 13-vertices. Then by **R1**(b) the vertex X receives from the neighbours the total charge 3. Hence $h(X) = 0$.

2. $k = 4$. In this case all neighbours of X are ≥ 9-vertices and $c(X) = -2$. In order to strengthen this result we consider two subcases.

2.1. Let the vertex X be adjacent only with vertices of degree l, $9 \leq l \leq 12$. Then X receives a charge $\geq \frac{3}{2} \times \frac{1}{3}$ from an l-vertex by Rule **R1**(a). The totally received charge is $\geq 4 \times \frac{1}{2}$ and thus $h(X) \geq 0$.

2.2. If X has two ≥ 13-neighbours or one ≥ 13-neighbour A with $p(X, A) \geq 2$ then X receives from its neighbourhood a charge $\geq 2 + 2 \times \frac{3}{2} \times \frac{1}{3} = 3$, and hence $h(X) \geq 1$. If X has exactly one neighbour A of degree ≥ 13 and $p(X, A) = 1$ then X receives the charge $\geq 1 + 3 \times \frac{3}{2} \times \frac{1}{3} = \frac{5}{2}$ and so $h(X) \geq \frac{1}{2}$.

3. $k = 5$. In this case X has only neighbours of degree ≥ 7 and $c(X) = -1$. Applying the Rule **R1**(a) to the vertex X, it receives at least the charge $5 \times \frac{3}{2} \times \frac{1}{7} = \frac{15}{14}$. Hence $h(X) \geq -1 + \frac{15}{14} = \frac{1}{14} > 0$.

4. $k = 6$. Obviously, $h(X) = 0$.

5. $7 \leq k \leq 12$.

Each vertex X sends the charge $\frac{3}{2} \times \frac{k-6}{k}$ to every minor vertex via each simple edge by Rule **R1**(a).

Obviously, each face α of our semitriangulation G is incident with three different vertices.

5.1. Let X be incident only with simple edges. Then X is adjacent with at most $\lfloor \frac{k}{2} \rfloor$ minor vertices and none two of them are adjacent. If X sends a charge to $\lfloor \frac{k}{2} \rfloor$ neighbours then its value is $\leq \lfloor \frac{k}{2} \rfloor \times \frac{3}{2} \times \frac{k-6}{k}$.

Thus $h(X) \geq k - 6 - (\lfloor \frac{k}{2} \rfloor \times \frac{3}{2} \times \frac{k-6}{k}) \geq \frac{1}{4} \times (k-6) > \frac{1}{8}(k-6) \geq \frac{1}{8}$ and, more special, $h(X) \geq \frac{5}{14}, \frac{1}{2}, 1, 1, \frac{35}{22}, \frac{3}{2}$. for $k = 7, 8, 9, 10, 11, 12$, respectively.

5.2. If X is joined to a minor vertex Y with a multiple edge then $p(X, Y) = 2$ because Y has a degree ≤ 5. It is easy to see that at most $\lfloor \frac{k}{2} \rfloor$ edges incident with X transfer a charge from X to its minor neighbours. So we can use the same counting arguments as in case **5.1**, and therefore $h(X) \geq \frac{5}{14}, \frac{1}{2}, 1, 1, \frac{35}{22}, \frac{3}{2}$

for $k = 7, 8, 9, 10, 11, 12$, respectively. Consequently, $h(X) > \frac{1}{8}(k-6) \geq \frac{1}{8}$.

6. $k \geq 13$.

6.1. Let X be incident only with simple edges. This case is similar with the case **5.1**, hence X can be adjacent with at most $\lfloor \frac{k}{2} \rfloor$ minor vertices, and none two of them are adjacent. Then $h(X) \geq k - 6 - \lfloor \frac{k}{2} \rfloor \geq \frac{1}{8}(k-6)$.

6.2. If X is joined with a minor vertex Y via multiple edges XY then $p(X,Y) = 2$ and we proceed analogously as in case **5.2**. Again we arrive at $h(X) \geq k - 6 - \lfloor \frac{k}{2} \rfloor \geq \frac{1}{8}(k-6)$. In both cases $h(X) \geq 1, 1, 2, 2, 3, 3, \dots$ for $k = 13, 14, 15, 16, 17, 18, \dots$, respectively. $\square$

Relation (1) together with Lemma 1 and the hypothesis

$$\sum_{\deg(X)\geq 7} \bigl(\deg_G(X) - 6\bigr) > 48\bigl|\chi(\mathscr{M})\bigr| \ \text{ yields}$$

$$\begin{aligned}
6\,\bigl|\chi(\mathscr{M})\bigr| &= \sum_{X\in V} c(X) + \sum_{\alpha\in F} c(\alpha) \\
&= \sum_{X\in V} h(X) \geq \sum_{\deg_G(X)\geq 7} h(X) \geq \frac{1}{8} \sum_{\deg(X)\geq 7} \bigl(\deg_G(X) - 6\bigr) \\
&> \frac{1}{8} 48\,\bigl|\chi(\mathscr{M})\bigr| = 6\,\bigl|\chi(\mathscr{M})\bigr|, \quad \text{a contradiction}.
\end{aligned}$$

After redistribution all vertices have a nonnegative charge by Lemma 1. Next we will discharge the vertices with positive charge $h(X)$ by Rule **R2**.

Rule R2.

Each vertex X sends its charge $h(X)$ equally to its incident faces. It is obvious that each face is incident with at least two major vertices.

Lemma 2. *By Rule* **R2** *each vertex X with $\deg_G(X) \geq 7$ or 8 sends a charge $\frac{h(X)}{\deg_G(X)} \geq \frac{5}{98}$ or $\frac{1}{16}$, respectively, to each neighbouring face.*

Lemma 3. *After applying Rule* **R2** *we have the new charge $h'(X)$ and $h'(\alpha)$:*

$$h'(X) = 0 \quad \text{for all vertices } X \in V(G),$$
$$h'(\alpha) > \frac{1}{9} \quad \text{for all faces } \alpha \in V(G).$$

P r o o f o f L e m m a 3. Let α be a face of the semitriangulation G. We denote the three different vertices of α by X, Y, and Z such that $\deg_G(X) \leq \deg_G(Y) \leq \deg_G(Z)$. If $\deg_G(X) \geq 7$ then X, Y, and Z send a total charge $\geq 3 \times \frac{5}{98} = \frac{15}{98} > \frac{1}{9}$ to α. Next let $\deg_G(X) \leq 6$. Then $7 \leq \deg_G(Y) \leq \deg_G(Z)$.

If $\deg_G(Z) \geq 8$ then Y and Z send the total charge $\geq \frac{5}{98} + \frac{1}{16} = \frac{89}{784} > \frac{1}{9}$ to α. Next let $\deg_G(Y) = \deg_G(Z) = 7$. Then $\deg_G(X) \in \{5, 6\}$. If $\deg_G(X) = 5$ then X sends $\frac{1}{70}$ to α, and α receives $\geq 2 \times \frac{5}{98} + \frac{1}{70} > \frac{1}{9}$ from X, Y, and Z. Next let $\deg_G(X) = 6$. Then Y has the 6-neighbour X. Hence the vertex Y is incident with at most two edges joining Y with a minor vertex. By Rule **R1**(a) the vertex Y sends $\leq 2 \times \frac{3}{2} \times \frac{1}{7} = \frac{3}{7}$ to its neighbourhood. Therefore, $h(Y) \geq 1 - \frac{3}{7} = \frac{4}{7}$. The same is true for the vertex Z, i.e., $h(Z) \geq \frac{4}{7}$. Consequently, Y and Z send a total charge $\geq 2 \times \frac{4}{7} \times \frac{1}{7} = \frac{8}{49} > \frac{1}{9}$ to α. In all cases $h'(\alpha) > \frac{1}{9}$. □

By Lemma 3 each vertex X has now the charge $h'(X) = 0$ and each 3-face α has the charge $h'(\alpha) > \frac{1}{9}$. Let f be the number of faces of the graph G. Since G is a semitriangulation, $f = 2(n + |\chi(\mathscr{M})|)$ holds.

Relation (1) together with Lemma 3 and the hypothesis $n > 26\,|\chi(\mathscr{M})|$ yields

$$6\,|\chi(\mathscr{M})| = \sum_{X \in V} c(X) + \sum_{\alpha \in F} c(\alpha) = \sum_{X \in V} h'(X) + \sum_{\alpha \in F} h'(\alpha) \geq \sum_{\alpha \in F(G)} h'(\alpha) \geq \frac{1}{9} f$$
$$= \frac{2}{9}\big(n + |\chi(\mathscr{M})|\big) > \frac{2}{9}\big(26\,|\chi(\mathscr{M})| + |\chi(\mathscr{M})|\big) = 6\,|\chi(\mathscr{M})|,$$

a contradiction.

This completes the proof of the upper bound in Theorem 3.

3. Proof of Theorem 3—the lower bounds

Let $\mathrm{ch}(G) := \sum_{\deg_G(X) \geq 7} \big(\deg_G(X) - 6\big)$; it is called the *positive charge* of G.

The main goal of this section is to prove that on every surface $\mathscr{M}$ with $\chi(\mathscr{M}) \leq 0$ there exist normal maps with $n > 26\,|\chi(\mathscr{M})|$ vertices which have only $(3, 12)$-edges, or only $(4, 8)$-edges or only $(6, 6)$-edges from the list of Theorem 3.

If $\mathrm{ch}(G) > 48\,|\chi(\mathscr{M})|$ then the same is true with the exception: in the third case we can only construct a sequence of normal maps of large positive charge containing a $(6, 6)$-edge and also other edges from the list (compare with Theorem 4).

We start our proof with Theorem 5 which has been proved (in a dual formulation) in [Je1] for orientable surfaces $\mathscr{M}$ and in [JV3] for nonorientable surfaces $\mathscr{M}$.

Theorem 5. *Let $\mathscr{M}$ be a surface of Euler characteristic $\chi(\mathscr{M}) \leq 0$. Let $(n_i\colon 3 \leq i \neq 6)$ be a sequence of nonnegative integers which satisfies the conditions*

(i) $\sum_{i \geq 3} (6 - i) n_i = 6\chi(\mathscr{M})$, *and*

(ii) $n_3 = n_4 = n_5 = 0$.

Then there exists a constant K such that for every $n_6 \geq K$ there is a triangulation G of $\mathscr{M}$ with

$$n_i(G) = n_i\,, \quad \textit{for all} \quad i \geq 3$$

and having $(6,6)$-edges.

From Theorem 5 it immediately follows that the bound 6 in Theorem 3(iii) is tight for normal maps with a large number of vertices. If we insert a vertex X into every 3-face α of the triangulation G of Theorem 5 and join X with every vertex of α we obtain a triangulation G^* that has only $(3,12)$-edges from our list. So the bound 12 in Theorem 3(i) is the best possible. Note that $\deg_{G^*}(Y) = 2\deg_G(Y)$ for every vertex $Y \in V(G)$. Every "new" vertex X has degree 3. The positive charge of G^* is $\mathrm{ch}(G^*) > 6K$.

If the number of vertices is large enough then G has two adjacent 6-vertices Z_1 and Z_2. If we insert a vertex X into every 3-face α of the triangulation G of Theorem 5 not incident with Z_1 or Z_2 and join X with every vertex incident with α then we obtain a triangulation G^{**} that has the $(6,6)$-edge Z_1Z_2 from the list, and the positive charge $\mathrm{ch}(G^{**}) > 6(K-10)$. The graph G^{**} has also other edges from the list.

To prove that also the bound 8 is not improvable we start with a result analogous to Theorem 5. The following theorem for orientable surfaces $\mathscr{M}$ is a consequence of a result by Jucovič [Ju]. For nonorientable surfaces it can be proved by a minor modification of the proof of Jucovič from [Ju] where instead of handles, crosscaps are introduced. A *quadrangulation* is a normal map all faces of which are 4-gons.

THEOREM 6. *Let $\mathscr{M}$ be a surface of Euler characteristic $\chi(\mathscr{M}) \leq 0$. Let $(n_i\colon 3 \leq i \neq 4)$ be a sequence of nonnegative integers which satisfies the conditions*

$$\sum_{i\geq 3}(4-i)n_i = 4\chi(\mathscr{M}) \quad \textit{and} \quad n_3 = 0\,.$$

Then there is a constant K such that for every $n_4 \geq K$ there is a quadrangulation Q of the surface $\mathscr{M}$ with

$$n_i(G) = n_i\,, \quad \textit{for all } i \geq 3\,,$$

and having $(4,4)$-edges.

If a new vertex X is introduced into every 4-gon β of the quadrangulation Q and it is joint with all four vertices incident with β, the result is a triangulation T having only $(4,8)$-edges from the list of our Theorem 3. Note that $\deg_T(Y) = 2\deg_Q(Y)$ for every "old" vertex Y and $\deg_T(X) = 4$ for every "new" vertex X. The positive charge of T is $\mathrm{ch}(T) > 2K$.

This completes the proof of Theorem 3.

4. Remarks

1. One can ask if the condition on the number of vertices $n > 26\bigl|\chi(\mathscr{M})\bigr|$ is optimal. A detailed look at the proof shows that 26 can be replaced by $\frac{101}{4}$. On the other hand, arguments below show that it cannot be 20 or less. It is known, see [MB] or [JNŠ], that there are infinitely many 7-valent regular triangulations. Let H be such a triangulation on a surface $\mathscr{M}$. The number e of edges of H is $e = \frac{7}{2}n$ and the number f of faces of H is $f = \frac{7}{3}n$. Thus the number n of vertices of H is $n = 6\bigl|\chi(\mathscr{M})\bigr|$. If a 3-valent vertex X is inserted into every 3-face of H (join it with all three vertices of the face), a new triangulation H^* is obtained. Every edge of H^* is either a $(3,14)$-edge or a $(14,14)$-edge. The number of inserted vertices is $f = \frac{7}{3}n$. Hence the number of all vertices of H^* is $n + f = n + \frac{7}{3}n = \frac{10}{3}n = 20\bigl|\chi(\mathscr{M})\bigr|$ and H^* does not contain any edge from the list of Theorem 3.

2. Two fundamental generalizations of Kotzig's theorem have been proved recently; first for paths on k vertices, $k \geq 1$, by Fabrici and Jendroľ [FJ], and the second for connected subgraphs of order k by Enomoto and Ota [EO]. These two results together with Kotzig's theorem served as starting points for the theory of light subgraphs, see [JV1, JV4].

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