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DIVISIBILIZING NORMAL-VALUED LATTICE-ORDERED GROUPS

W. Charles Holland

Dedicated to Professor Ján Jakubík on the occasion of his 80th birthday

ABSTRACT. The generalized wreath product of ordered permutation groups is used to prove that every normal-valued lattice-ordered group can be embedded in a divisible normal-valued lattice-ordered group.

A group G (written multiplicatively) is *divisible* if for all $g \in G$ and $n \in \mathbb{Z}^+$, there exists $x \in G$ such that $x^n = g$. A *lattice-ordered group* or *ℓ-group* is a group whose elements are partially ordered so that each pair of elements x, y have a least upper bound $x \vee y$ and so that for any elements w, z, $w(x \vee y)z = (wxz \vee wyz)$. An ℓ-group G is *normal valued* if for all $e \leq x, y \in G$, $xy \leq y^2x^2$. Normal-valued ℓ-groups constitute the largest proper variety (equationally defined class) of ℓ-groups, and contain among others all abelian ℓ-groups and all totally ordered groups. For these definitions and other background about ℓ-groups, see [2] or [1].

It is easy to prove that every Abelian ℓ-group can be embedded in a divisible Abelian ℓ-group. It is known (Holland [4]) that every ℓ-group can be embedded in a divisible ℓ-group. It is our purpose in this paper to prove the following (Theorem 6):

Theorem. *Every normal-valued ℓ-group can be embedded in a divisible normal-valued ℓ-group.*

Our main tool is the generalized wreath product first described in [5]. We summarize the main features of the special case we need here. Let Γ be a totally ordered index set, and for each $\gamma \in \Gamma$ let $(G_\gamma, \Omega_\gamma)$ be a permutation group (the group G_γ acting on the set Ω_γ). Choose an arbitrary element $0 \in \prod_{\gamma\in\Gamma} \Omega_\gamma$. For any element $x \in \prod_{\gamma\in\Gamma} \Omega_\gamma$, denote the γth coordinate of x by x_γ, and let

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$\mathrm{Supp}(x) = \{\gamma\colon x_\gamma \neq 0_\gamma\}$. Let

$$\Omega = \bigl\{x\colon \mathrm{Supp}(x) \text{ is inversely well ordered}\bigr\}.$$

For each $\gamma \in \Gamma$, there are natural equivalence relations on Ω whereby $x \equiv_\gamma y$ iff $x_\alpha = y_\alpha$ for each $\alpha \geq \gamma$, and $x \equiv^\gamma y$ iff $x_\alpha = y_\alpha$ for each $\alpha > \gamma$. Let $\overline{W}$ be the subgroup of the symmetric group on Ω consisting of all those permutations f such that both f and f^{-1} respect all $\equiv_\gamma$ (and hence all $\equiv^\gamma$). For each $\gamma \in \Gamma$ and $x \in \Omega$, $x|\gamma$ is the restriction of x to $\{\beta \in \Gamma\colon \gamma < \beta\}$.

For each $\gamma \in \Gamma$, $x \in \Omega$, and $f \in \overline{W}$, f is determined by coordinates $f_{\gamma,x|\gamma}$ in the symmetric group on Ω_γ, where $(xf)_\gamma = x_\gamma f_{\gamma,x|\gamma}$. It follows that

$$(fg)_{\gamma,x|\gamma} = f_{\gamma,x|\gamma} g_{\gamma,(xf)|\gamma}. \tag{1}$$

Then the wreath product $(W,\Omega) = \mathrm{Wr}_{\gamma\in\Gamma}(G_\gamma,\Omega_\gamma)$ is that subgroup of $(\overline{W},\Omega)$ consisting of all those f such that for each $\gamma \in \Gamma$ and $x \in \Omega$, $f_{\gamma,x|\gamma} \in G_\gamma$. It follows that each of the relations $\equiv_\gamma$ and $\equiv^\gamma$ is a W-congruence.

Then for each $a \in G_\gamma$ and $x \in \Omega$, there exists $f \in W$ such that $f_{\gamma,x|\gamma} = a$ and for all $\alpha > \gamma$, $f_{\alpha,x|\alpha} = e$. The wreath product depends on the choice of $0 \in \prod \Omega_\gamma$, but in the case where each (G_γ,Ω_γ) is transitive, it does not.

When each Ω_γ is totally ordered, then Ω may be totally ordered anti-lexicographically; and when (G_γ,Ω_γ) is an ℓ-permutation group (a subgroup and sublattice of the pointwise-ordered group of order-preserving permutations of the totally ordered set Ω_γ), then so is (W,Ω), and the coordinates of $(f \vee e)$ are given by

$$(f \vee e)_{\gamma,x|\gamma} = \begin{cases} f_{\gamma,x|\gamma} & \text{if } \exists\beta > \gamma \text{ with } x_\beta f_{\beta,x|\beta} > x_\beta \text{ and } xf \equiv^\beta x\,, \\ e & \text{if } \exists\beta > \gamma \text{ with } x_\beta f_{\beta,x|\beta} < x_\beta \text{ and } xf \equiv^\beta x\,, \\ f_{\gamma,x|\gamma} \vee e & \text{otherwise; that is, } xf \equiv^\gamma x\,. \end{cases} \tag{2}$$

We will need two more properties of the wreath product. The first is that for the wreath product of two groups (that is, where the index set Γ has just two elements), $(W,\Omega) = (G_1,\Omega_1)\,\mathrm{Wr}\,(G_2,\Omega_2)$, then W is a semidirect product $W = \left(\prod_{y\in\Omega_2} G_y\right) \rtimes G_2$ where each $G_y \cong G_1$. The next is a form of associative law (see [5]). If $\Gamma = \Gamma_1 \cup \Gamma_2$ is an ordinal sum, every element of Γ_1 less than every element of Γ_2, then

$$\mathrm{Wr}_{\gamma\in\Gamma}(G_\gamma,\Omega_\gamma) = \bigl(\mathrm{Wr}_{\gamma\in\Gamma_1}(G_\gamma,\Omega_\gamma)\bigr)\,\mathrm{Wr}\,\bigl(\mathrm{Wr}_{\gamma\in\Gamma_2}(G_\gamma,\Omega_\gamma)\bigr).$$

Theorem 1. (Holland [3]) *Every ℓ-group is a subdirect product of transitive ℓ-permutation groups.*

Let (G,Ω) be a transitive ℓ-permutation group. A *G-congruence* is an equivalence relation on Ω which is respected by G and whose classes are convex. If $\mathcal{C}$

and $\mathcal{K}$ are G-congruences with $\mathcal{C} \subset \mathcal{K}$, and with no G-congruences properly between the two, then (G,Ω) induces an ℓ-permutation group (G^*,Ω^*) where Ω^* is the quotient of one of the $\mathcal{K}$-classes by $\mathcal{C}$. Then (G^*,Ω^*) has no proper congruence, and is called a *primitive component* of (G,Ω). The set of G-congruences is totally ordered by inclusion, and thus the set of primitive components has a natural total order.

Theorem 2. (Holland and McCleary [6]) *If (G,Ω) is a transitive ℓ-permutation group, then (G,Ω) can be embedded in the wreath product of its primitive components.*

Theorem 3. (J. Read [7]) *An ℓ-permutation group (G,Ω) is normal valued iff each primitive component of (G,Ω) is an ℓ-permutation subgroup of $(\mathbb{R},\mathbb{R})$ (the reals with the regular action).*

Theorem 4. *Let Γ be a totally ordered index set. For each $\gamma \in \Gamma$ let (G_γ,Ω_γ), $(H_\gamma,\Lambda_\gamma)$ be ℓ-permutation groups and suppose that for each $\gamma \in \Gamma$ there is an embedding $\varphi_\gamma\colon (G_\gamma,\Omega_\gamma) \to (H_\gamma,\Lambda_\gamma)$. Then there is an embedding*

$$\varphi\colon \mathrm{Wr}_{\gamma\in\Gamma}(G_\gamma,\Omega_\gamma) \to \mathrm{Wr}_{\gamma\in\Gamma}(H_\gamma,\Lambda_\gamma).$$

Proof. Each $\varphi_\gamma\colon (G_\gamma,\Omega_\gamma) \to (H_\gamma,\Lambda_\gamma)$ is a pair of maps which we do not distinguish notationally: an injective ℓ-homomorphism $\varphi_\gamma\colon G_\gamma \to H_\gamma$ and an order embedding $\varphi_\gamma\colon \Omega_\gamma \to \Lambda_\gamma$; and for each $x \in \Omega_\gamma$ and $g \in G_\gamma$, $(xg)\varphi_\gamma = x\varphi_\gamma g\varphi_\gamma$. We assume that each of the maps $\varphi_\gamma\colon \Omega_\gamma \to \Lambda_\gamma$ takes the zero of Ω_γ to the zero of Λ_γ, and we denote all of these by the symbol 0. If $(W,\Omega) = \mathrm{Wr}_{\gamma\in\Gamma}(G_\gamma,\Omega_\gamma)$ and $(V,\Lambda) = \mathrm{Wr}_{\gamma\in\Gamma}(H_\gamma,\Lambda_\gamma)$ then we define a corresponding pair of maps $\varphi\colon W \to V$ and $\varphi\colon \Omega \to \Lambda$ as follows. For $x \in \Omega$ and $\gamma \in \Gamma$, let

$$(x\varphi)_\gamma = x_\gamma\varphi_\gamma. \tag{3}$$

Since $\varphi_\gamma\colon \Omega_\gamma \to \Lambda_\gamma$ is an order embedding such that $0\varphi_\gamma = 0$, then $\mathrm{Supp}(x) = \mathrm{Supp}(x\varphi)$, so $x\varphi \in \Lambda$; and if $u,v \in \Omega$ with $u < v$, then $u_\gamma < v_\gamma$ where $\max\{\alpha\colon u_\alpha \neq v_\alpha\} = \gamma = \max\{\alpha\colon (u\varphi)_\alpha \neq (v\varphi)_\alpha\}$. Hence, $(u\varphi)_\gamma = u_\gamma\varphi_\gamma < v_\gamma\varphi_\gamma = (v\varphi)_\gamma$. Therefore, $u\varphi < v\varphi$.

Thus

$$\varphi \text{ is an order embedding of } \Omega \text{ into } \Lambda. \tag{4}$$

For $f \in W$, $\gamma \in \Gamma$, and $y \in \Lambda$, define

$$(f\varphi)_{\gamma,y|\gamma} = \begin{cases} (f_{\gamma,x|\gamma})\varphi_\gamma & \text{if there exists } x \in \Omega \text{ with } y|\gamma = (x\varphi)|\gamma, \\ e = \text{the identity} & \text{otherwise}. \end{cases} \tag{5}$$

Then for each $x \in \Omega$ and $\gamma \in \Gamma$,

$$\begin{aligned}\big((xf)\varphi\big)_\gamma &= \big((xf)_\gamma\big)\varphi_\gamma \\ &= \big(x_\gamma f_{\gamma,x|\gamma}\big)\varphi_\gamma \\ &= (x_\gamma\varphi_\gamma)\big(f_{\gamma,x|\gamma}\varphi_\gamma\big) \\ &= (x\varphi)_\gamma(f\varphi)_{\gamma,(x\varphi)|\gamma} \\ &= \big((x\varphi)(f\varphi)\big)_\gamma,\end{aligned}$$

and so

$$(xf)\varphi = (x\varphi)(f\varphi). \tag{6}$$

It remains to show that $\varphi\colon W \to V$ is one-to-one and an ℓ-homomorphism.

We note that if $y \in \Lambda$ and $y|\gamma = (x\varphi)|\gamma$, then for each $\beta > \gamma$,

$$\begin{aligned}\big(y(f\varphi)\big)_\beta &= y_\beta(f\varphi)_{\beta,y|\beta} \\ &= (x\varphi)_\beta(f\varphi)_{\beta,(x\varphi)|\beta} \\ &= \big((x\varphi)(f\varphi)\big)_\beta \\ &= \big((xf)\varphi\big)_\beta\end{aligned}$$

and so

$$y|\gamma = (x\varphi)|\gamma \quad \text{implies} \quad \big(y(f\varphi)\big)|\gamma = \big((xf)\varphi\big)|\gamma. \tag{7}$$

Let $y \in \Lambda$, $\gamma \in \Gamma$, and $f, g \in W$. Suppose first that there is some $x \in \Omega$ such that $y|\gamma = (x\varphi)|\gamma$. Then

$$\begin{aligned}\big((fg)\varphi\big)_{\gamma,y|\gamma} &= \big((fg)_{\gamma,x|\gamma}\big)\varphi_\gamma && \text{(by (5))} \\ &= \big(f_{\gamma,x|\gamma}g_{\gamma,(xf)|\gamma}\big)\varphi_\gamma && \text{(by (1))} \\ &= (f_{\gamma,x|\gamma})\varphi_\gamma\big(g_{\gamma,(xf)|\gamma}\big)\varphi_\gamma && \text{(because } \varphi_\gamma \text{ is a homomorphism)} \\ &= (f\varphi)_{\gamma,y|\gamma}(g\varphi)_{\gamma,(y(f\varphi))|\gamma} && \text{(by (5) and (7))} \\ &= \big((f\varphi)(g\varphi)\big)_{\gamma,y|\gamma} && \text{(by (1))}.\end{aligned}$$

On the other hand, if there is no $x \in \Omega$ such that $(x\varphi)|\gamma = y|\gamma$ then

$$\begin{aligned}\big((fg)\varphi\big)_{\gamma,y|\gamma} &= e && \text{(by (5))} \\ &= e \cdot e \\ &= (f\varphi)_{\gamma,y|\gamma}(g\varphi)_{\gamma,(y(f\varphi))|\gamma} && \text{(by (5) and (7))} \\ &= \big((f\varphi)(g\varphi)\big)_{\gamma,y|\gamma} && \text{(by (1))}.\end{aligned}$$

Therefore,

$$(fg)\varphi = f\varphi g\varphi\,. \tag{8}$$

Suppose that $e \neq f \in W$. Then for some $x \in \Omega$, $xf \neq x$. By (4) and (6), $x\varphi \neq (xf)\varphi = x\varphi f\varphi$. Hence, $f\varphi \neq e$. Thus,

$$\varphi\colon W \to V \text{ is one-to-one}\,. \tag{9}$$

Now we show that $\varphi\colon W \to V$ preserves the lattice operations. It suffices to show that $(g \vee e)\varphi = (g\varphi) \vee e$. Let $y \in \Lambda$ and $\gamma \in \Gamma$. We must show that $\big[(g \vee e)\varphi\big]_{\gamma,y|\gamma} = \big[(g\varphi) \vee e\big]_{\gamma,y|\gamma}$.

1. Suppose first that there exists $x \in \Omega$ such that $y|\gamma = (x\varphi)|\gamma$. It follows that for each $\delta \geq \gamma$, $y|\delta = (x\varphi)|\delta$. We then have

$$\begin{aligned} y_\delta(g\varphi)_{\delta,y|\delta} &= x_\delta\varphi_\delta(g_{\delta,x|\delta})\varphi_\delta \\ &= (x_\delta g_{\delta,x|\delta})\varphi_\delta \\ &\begin{cases} > x_\delta\varphi_\delta = y_\delta & \text{if } x_\delta g_{\delta,x|\delta} > x_\delta\,, \\ < x_\delta\varphi_\delta = y_\delta & \text{if } x_\delta g_{\delta,x|\delta} < x_\delta\,, \\ = x_\delta\varphi_\delta = y_\delta & \text{if } x_\delta g_{\delta,x|\delta} = x_\delta\,. \end{cases} \end{aligned} \tag{10}$$

Referring to (2), we will deal with each of the three cases.

1.1 Suppose first that there exists $\beta > \gamma$ such that $x_\beta g_{\beta,x|\beta} > x_\beta$ and $xg \equiv^\beta x$. By (10), β has the same property for y and $g\varphi$, and so by (2) and (3),

$$\begin{aligned} \big[(g \vee e)\varphi\big]_{\gamma,y|\gamma} &= \big[(g \vee e)_{\gamma,x|\gamma}\big]\varphi_\gamma \\ &= g_{\gamma,x|\gamma}\varphi_\gamma \\ &= (g\varphi)_{\gamma,y|\gamma} \\ &= \big[(g\varphi) \vee e\big]_{\gamma,y|\gamma}\,. \end{aligned} \tag{11}$$

1.2 Next suppose that there exists $\beta > \gamma$ such that $x_\beta g_{\beta,x|\beta} < x_\beta$ and $xg \equiv^\beta x$. As in the previous case, β has the same property for y and $g\varphi$, and so by (2) and (3),

$$\begin{aligned} \big[(g \vee e)\varphi\big]_{\gamma,y|\gamma} &= \big[(g \vee e)_{\gamma,x|\gamma}\big]\varphi_\gamma \\ &= e\varphi_\gamma \\ &= e \\ &= \big[(g\varphi) \vee e\big]_{\gamma,y|\gamma}\,. \end{aligned} \tag{12}$$

1.3 Finally, suppose that $xg \equiv^{\gamma} x$. Then as in the previous cases, γ has the same property for y and $g\varphi$, and so by (2) and (3),

$$\begin{aligned} \big[(g \vee e)\varphi\big]_{\gamma,y|\gamma} &= \big[(g \vee e)_{\gamma,x|\gamma}\big]\varphi_{\gamma} \\ &= (g_{\gamma,x|\gamma} \vee e)\varphi_{\gamma} \\ &= g_{\gamma,x|\gamma}\varphi_{\gamma} \vee e \\ &= (g\varphi)_{\gamma,y|\gamma} \vee e \\ &= \big[(g\varphi) \vee e\big]_{\gamma,y|\gamma} . \end{aligned} \tag{13}$$

From (11), (12), and (13), we have that

$$\big[(g \vee e)\varphi\big]_{\gamma,y|\gamma} = \big[(g\varphi) \vee e\big]_{\gamma,y|\gamma}$$

in this case.

2. Now we suppose that there is no $x \in \Omega$ such that $y|\gamma = (x\varphi)|\gamma$. Then by (5), $(g\varphi)_{\gamma,y|\gamma} = e = \big[(g \vee e)\varphi\big]_{\gamma,y|\gamma}$. Therefore, each of the three possibilities for $[g\varphi \vee e]_{\gamma,y|\gamma}$, namely, $(g\varphi)_{\gamma,y|\gamma}$, e, and $(g\varphi)_{\gamma,y|\gamma} \vee e$ is e. Hence, here, too,

$$\big[(g \vee e)\varphi\big]_{\gamma,y|\gamma} = [g\varphi \vee e]_{\gamma,y|\gamma} ,$$

proving that

$$\varphi \text{ preserves the lattice operations}. \tag{14}$$

From (14), (8), (9), (6), and (4) the theorem is proved. □

Lemma 1. *Let* $(W,\Omega) = (G_1,\Omega_1)\,\mathrm{Wr}(G_2,\Omega_2)$, *so that* $W = \Big(\prod_{u\in\Omega_2} G_u\Big) \rtimes G_2$ *where* $G_u \cong G_1$, *and let* $g = \big((g_u)_{u\in\Omega_2}, \bar g\big) \in W$. *Suppose further that each orbit* $\{u\bar g^i\}_{i\in\mathbb{Z}}$ *is infinite, and that there is an element* $\bar f \in G_2$ *such that* $\bar f^n = \bar g$ *for some* $n \in \mathbb{Z}^+$. *Then there is* $f = \big((f_u)_{u\in\Omega_2}, \bar f\big) \in W$ *such that* $f^n = g$.

P r o o f. Since $\bar f^n = \bar g$, each orbit $\{u\bar f^i\}$ is also infinite. On each such orbit, arbitrarily choose $f_u, f_{u\bar f}, f_{u\bar f^2}, \dots, f_{u\bar f^{n-1}}$. Since we want $f^n = g$, we need $g_u = f_u f_{u\bar f} f_{u\bar f^2} \cdots f_{u\bar f^{n-1}} f_{u\bar f^n}$, and so we must have

$$f_{u\bar f^n} = f^{-1}_{u\bar f^{n-1}} \cdots f^{-1}_{u\bar f^2} f^{-1}_{u\bar f} f^{-1}_{u} g_u .$$

This in turn determines $f_{u\bar f^{n+1}}$, $f_{u\bar f^{n+2}}$, etc; in the same way, $f_{u\bar f^{-1}}$, $f_{u\bar f^{-2}}$, etc., are determined. The process is repeated independently on each orbit, and the result is an $f \in W$ such that $f^n = g$. □

Theorem 5. *If for each $\gamma \in \Gamma$, $(G_\gamma, \Omega_\gamma)$ is a divisible ℓ-permutation group, and $(W, \Omega) = \mathrm{Wr}_{\gamma\in\Gamma}(G_\gamma, \Omega_\gamma)$, then W is divisible.*

Proof. Let $g \in W$ and $n \in \mathbb{Z}^+$. We seek an $f \in W$ such that $f^n = g$. The ordered set Ω is partitioned into convex subsets, the *orbitals* of g, each of which is either a fixed point of g, or an interval on which g moves each point up, or an interval on which g moves each point down. Since g respects each of the natural equivalences $\equiv_\gamma$, it is clear that for each orbital I, and each natural equivalence class C, either $C \subseteq I$ or $I \subseteq C$. If for each orbital I we find $f_{(I)} \in W$ such that $(f_{(I)} \restriction I)^n = g \restriction I$, then the permutation f obtained by patching together all of the $f_{(I)}$'s preserves all natural equivalences and has each component $f_{\gamma,x|\gamma} \in G_\gamma$, the appropriate group. Therefore, $f \in W$; and it is clear that $f^n = g$.

So now consider the action of g on one of its non-trivial orbitals I. Restricting everything to I, we may assume that our wreath product has the form $(W, \Omega) = (W_1, \Omega_1)\,\mathrm{Wr}(W_2, \Omega_2)$ where $(W_1, \Omega_1) = \mathrm{Wr}_{\gamma<\beta}(G_\gamma, \Omega_\gamma)$ and $(W_2, \Omega_2) = (G_\beta, \Omega_\beta)$ where β is the largest member of Γ such that for any $x \in I$, $xg \not\equiv_\beta x$. Then we have the hypotheses of Lemma 1, and so there is $f \in W$ such that $f^n = g$. $\square$

Theorem 6. *Every normal-valued ℓ-group can be embedded in a divisible normal-valued ℓ-group.*

Proof. Let G be a normal-valued ℓ-group. By Theorem 1, G is a subdirect product of transitive ℓ-permutation groups. And since a direct product of divisible groups is divisible, we may assume that G is a transitive ℓ-permutation group. By Theorem 2, G can be embedded in the wreath product of its primitive components, and by Theorem 3, each of these is a subgroup of the reals with the regular action. Therefore, by Theorem 4, G can be embedded in a wreath product $W = \mathrm{Wr}_{\gamma\in\Gamma}(H_\gamma, \Lambda_\gamma)$ where each $(H_\gamma, \Lambda_\gamma)$ is the real numbers with the regular action, and hence is a divisible ℓ-permutation group. By Theorem 5, W is divisible. Moreover, by Theorem 3, W is normal valued. $\square$

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Department of Mathematics and Statistics
Bowling Green State University
Bowling Green
Ohio 43403
U. S. A.

E-mail: chollan@bgnet.bgsu.edu