Tatra Mt. Math. Publ. **26** (2003), 125–131

TATRA MOUNTAINS
Mathematical Publications

THE MSE OF THE BLUP IN A FINITE DISCRETE SPECTRUM LRM

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ABSTRACT. The problems of prediction in finite discrete spectrum linear regression models (FDSLRMs) are studied and an expression for the mean squared error (MSE) of the best linear unbiased predictor (BLUP) is derived. This expression allows to compare these regression models from the point of view of predictions. An example is given where as regressors the goniometric functions are used.

1. Introduction

There are many possibilities for prediction of time series. Well known and commonly used are methods are based on Box–Jenkins (1976) methodology using ARMA and ARIMA models. An other approach is based on modeling time series by *linear regression models* and finding the *best linear unbiased predictor* (BLUP). This method is known in the engineering literature as *kriging*, see Goldberger (1962), Brockwell and Davis (1987, 1996), Christensen (1991), Stein (1999), and Štulajter (2002).

The kriging theory is based on the following assumption. Let $X = X\big((1), \dots, X(n)\big)'$ be a finite observation of a time series $X(\cdot) = \{X(t) \colon t = 1, 2, \dots\}$ and let U be a predicted random variable (usually $U = X(s)$, $s > n$). The *prediction linear regression model* in the kriging theory is given by

$$X = F\beta + \varepsilon; \qquad E[\varepsilon] = 0\,, \quad \mathrm{Cov}_\nu(X) = \Sigma_\nu\,,$$
$$E[U] = f'\beta; \qquad \mathrm{Cov}_\nu(U, X) = r_\nu\,, \quad \nu \in \Upsilon\,,$$

where F is a known $n \times k$ matrix, f a known $k \times 1$ vector, $\beta = (\beta_1, \beta_2, \dots, \beta_k)' \in E^k$ are unknown regression parameters and ν are variance-covariance parameters of the model. If r_ν and Σ_ν are known, F is of full rank and Σ_ν is positive

2000 Mathematics Subject Classification: 62M10.
Keywords: time series, finite discrete spectrum linear regression model, best linear unbiased predictor, mean squared error of predictor.

Supported by the VEGA grant of the Slovak Grant Agency No. 1/4196/97.

definite, then the *best linear unbiased predictor* (BLUP) U_ν^* of U is given by

$$U_\nu^* = f'\beta_\nu^* + r_\nu' \Sigma_\nu^{-1}(X - F\beta_\nu^*)$$

and has the *mean squared error* (MSE)

$$E_\nu[U_\nu^* - U]^2 = D_\nu[U] - r_\nu' \Sigma_\nu^{-1} r_\nu + \left\| f - F'\Sigma_\nu^{-1} r_\nu \right\|^2_{\Sigma_{\beta_\nu^*}} ,$$

where $D_\nu[U]$ is the variance of U, $\|a\|_A^2 = a'Aa$, $\beta_\nu^*(X) = (F'\Sigma_\nu^{-1}F)^{-1}F'\Sigma_\nu^{-1}X$ is the BLUE of β and $\Sigma_{\beta_\nu^*} = (F'\Sigma_\nu^{-1}F)^{-1}$ is the covariance matrix of β_ν^* by the given ν.

For any time series $X(\cdot) = \{X(t)\colon t = 1,2,\dots\}$ we can choose different linear regression models for its mean value as well as for its covariance function which influence predicted values of this time series. Now we shall study a special type of a model for the observed time series and derive the MSE of the BLUP in this model.

2. The MSE of the BLUP in an FDSLRM

In some practical applications of time series theory we can use for an observed time series $X(\cdot)$ the *finite discrete spectrum linear regression model* (FDSLRM) given by

$$X(t) = \sum_{i=1}^{k} \beta_i f_i(t) + \sum_{j=1}^{l} Y_j v_j(t) + w(t); \qquad t = 1,2,\dots ,$$

where $f_i(\cdot), v_j(\cdot)$, $i = 1,2,\dots,k$, $j = 1,2,\dots,l$, are known functions, $w(\cdot)$ is a white noise process with the variance $D\big[w(t)\big] = \sigma^2$. In this model it is assumed that the random vector $Y = (Y_1, Y_2, \dots, Y_l)'$ has the expectation equal to zero, and that Y_j; $j = 1,2,\dots,l$, are uncorrelated random variables with variances σ_j^2; $j = 1,2,\dots,l$. Thus we have

$$E[Y] = 0, \quad \operatorname{Cov}(Y) = \operatorname{diag}(\sigma_j^2).$$

It is also assumed that $w(\cdot)$ is uncorrelated with Y.

Under these assumptions the covariance functions $R_\nu(\cdot,\cdot)$ of $X(\cdot)$ are given by

$$R_\nu(s,t) = \sigma^2 \delta_{s,t} + \sum_{j=1}^{l} \sigma_j^2 v_j(s) v_j(t); \qquad s,t = 1,2,\dots,$$

where the parameters ν of $R_\nu(\cdot,\cdot)$ are $\nu = (\sigma^2, \sigma_1^2, \dots, \sigma_l^2)'$ and $\Upsilon = (0,\infty)^{l+1}$ is the parametric space. It follows from the spectral theory that $X(\cdot)$ is a finite discrete spectrum time series.

Spectral theory of time series is considered in Brockwell and Davis (1987), Priestley (1981) and similar models can be found also in Štulajter (2002).

For this model we get for a finite observation $X = \big(X(1), \dots, X(n)\big)'$ of $X(\cdot)$ the linear regression model

$$X = F\beta + \varepsilon\,,$$

where $F = (f_1 f_2 \dots f_k)$ is the $n \times k$ design matrix of this model.

In this model $E[\varepsilon] = 0$, and the covariance matrices $\Sigma_\nu; \nu \in \Upsilon$, of X are positive definite and are given by

$$\Sigma_\nu = \sigma^2 I + \sum_{j=1}^{l} \sigma_j^2 v_j v_j' = \sum_{j=0}^{l} \sigma_j^2 V_j\,,$$

where $V_0 = I$, $V_j = v_j v_j'$, with ranks $r(V_j) = 1$; $j = 1, 2, \dots, l$, and $\sigma_0^2 = \sigma^2$. We shall call this model also FDSLRM.

To compute the BLUP, $X_\nu^*(n+d)$ of $X(n+d)$, $d > 0$, and its mean squared error we need an expression for the matrix Σ_ν^{-1}. If we do not have any assumptions on the vectors v_j, $j = 1, 2, \dots, l$, this expression is rather complicated. For orthogonal vectors v_j, $j = 1, 2, \dots, l$, i.e., for vectors v_j; $j = 1, 2, \dots, l$, such that $(v_i, v_j) = v_i' v_j = \sum_{t=1}^{n} v_i(t) v_j(t) = 0$ for $i \neq j$, the following lemma is true.

Lemma 2.1. *Let the $n \times n$ covariance matrix Σ_ν be given by*

$$\Sigma_\nu = \sigma^2 I + \sum_{j=1}^{l} \sigma_j^2\, v_j v_j'\,,$$

where the vectors v_j; $j = 1, 2, \dots, l$, are orthogonal. Then

$$\Sigma_\nu^{-1} = \frac{1}{\sigma^2}\left(I - \sum_{j=1}^{l} \frac{1}{\sigma^2/\sigma_j^2 + \|v_j\|^2}\, v_j v_j'\right).$$

Proof. By a direct computation we easily get that

$$\left(\sigma^2 I + \sum_{i=1}^{l} \sigma_i^2 v_i v_i'\right) \frac{1}{\sigma^2}\left(I - \sum_{j=1}^{l} \frac{1}{\sigma^2/\sigma_j^2 + \|v_j\|^2}\, v_j v_j'\right)$$
$$= \frac{1}{\sigma^2}\left(I - \sum_{j=1}^{l} \frac{1}{\sigma^2/\sigma_j^2 + \|v_j\|^2}\, v_j v_j'\right)\left(\sigma^2 I + \sum_{i=1}^{l} \sigma_i^2\, v_i v_i'\right) = I$$

if v_j; $j = 1, 2, \dots, l$, are orthogonal vectors.

Using this result we get, after some computation, for orthogonal v_j; $j = 1, 2, \dots, l$,

$$\begin{aligned} r'_\nu \Sigma_\nu^{-1} &= \sum_{j=1}^{l} \sigma_j^2 v_j(n+d) v'_j \frac{1}{\sigma^2}\left(I - \sum_{i=1}^{l} \frac{1}{\sigma^2/\sigma_j^2 + \|v_i\|^2} v_i v'_i\right) \\ &= \frac{1}{\sigma^2} \sum_{j=1}^{l} \sigma_j^2 v_j(n+d) v'_j \left(1 - \frac{\|v_j\|^2}{\sigma^2/\sigma_j^2 + \|v_j\|^2}\right) \\ &= \sum_{j=1}^{l} \frac{v_j(n+d)}{\sigma^2/\sigma_j^2 + \|v_j\|^2} v'_j \end{aligned}$$

and

$$r'_\nu \Sigma_\nu^{-1} r_\nu = \sum_{j=1}^{l} \sigma_j^2 v_j^2(n+d) \frac{\|v_j\|^2}{\sigma^2/\sigma_j^2 + \|v_j\|^2}\,.$$

Next we have

$$r'_\nu \Sigma_\nu^{-1} F = \sum_{j=1}^{l} \frac{v_j(n+d)}{\sigma^2/\sigma_j^2 + \|v_j\|^2} \,(v'_j f_1, v'_j f_2, \dots, v'_j f_k)$$

and, under the assumption that the *vectors* v_j *and* f_i *are mutually orthogonal*, i.e., $(v_j, f_i) = 0$; $i = 1, 2, \dots, k$, $j = 1, 2, \dots, l$, we get

$$r'_\nu \Sigma_\nu^{-1} F = 0\,.$$

For orthogonal vectors v_j and f_i, we get

$$\beta_\nu^*(X) = \hat\beta(X) = (F'F)^{-1} F'X$$

and

$$\operatorname{Cov}(\beta_\nu^*) = \operatorname{Cov}(\hat\beta) = \sigma^2 (F'F)^{-1}.$$

□

Using these results we get the following theorem.

THEOREM 2.1. *Let us consider the linear prediction regression model*

$$X = F\beta + \varepsilon\,, \quad E[\varepsilon] = 0\,, \quad \operatorname{Cov}_\nu(X) = \sigma^2 I + \sum_{j=1}^{l} \sigma_j^2 v_j v'_j\,,$$

$$E\big[X(n+d)\big] = f'\beta\,, \quad \operatorname{Cov}_\nu\big(X(n+d); X\big) = \sum_{j=1}^{l} \sigma_j^2 v_j(n+d) v_j\,; \quad \nu \in \Upsilon\,,$$

where $F = (f_1 f_2 \dots f_k)$, f_i; $i = 1,2,\dots,k$, *and* v_j, $j = 1,2,\dots,l$, *are given vectors and where* $f = \big(f_1(n+d),\dots,f_k(n+d)\big)'$. *Let* $f_i \perp v_j$ *for all* $i = 1,2,\dots,k$, $j = 1,2,\dots,l$, *and let* $v_i \perp v_j$ *for all* $i,j = 1,2,\dots,l$, $i \neq j$. *Then the* BLUP, $X_\nu^*(n+d)$, *is given by*

$$X_\nu^*(n+d) = f'(F'F)^{-1}F'X + \sum_{j=1}^{l} \frac{v_j(n+d)}{\sigma^2/\sigma_j^2 + \|v_j\|^2}\, v_j' X$$

and

$$\mathrm{MSE}_\nu\big[X_\nu^*(n+d)\big] = \sigma^2 \left(1 + \sum_{j=1}^{l} \frac{v_j^2(n+d)}{\sigma^2/\sigma_j^2 + \|v_j\|^2} + f'(F'F)^{-1}f\right).$$

Remarks. 1. From this expression it is easy to see that it is an increasing function of σ^2 and σ_j^2; $j = 1,2,\dots,l$. For many reasonable models, $\|v_j\|^2$; $j = 1,2,\dots,l$, are large numbers for n large, and thus in this case the influence of σ_j^2, $j = 1,2,\dots,l$, on this term of the MSE is insignificant if σ_j^2 are not extremely small and σ^2 is not extremely large.

2. To fulfill the conditions of the theorem it is necessary to assume that $k+l \le n$.

3. If $\sigma_j^2 \to \infty$, then

$$\mathrm{MSE}_\nu\big[X^*(n+d)\big] \to \sigma^2 \left(1 + \sum_{j=1}^{l} v_j^2(n+d)/\|v_j\|^2 + f'(F'F)^{-1}f\right).$$

4. If the vectors f_i, $i = 1,2,\dots,k$, are also mutually orthogonal, then

$$f'(F'F)^{-1}F'X = \sum_{i=1}^{k} \big(f_i(n+d)/\|f_i\|^2\big) f_i' X$$

and

$$f'(F'F)^{-1}f = \sum_{i=1}^{k} f_i^2(n+d)/\|f_i\|^2 .$$

In the following example we give a model which can be used in some economic time series applications where a seasonal effect can be expected. This seasonal effect can be modeled as a fixed effect by sine and cosine functions with a given, usually low, frequency. The error time series in this example is assumed to be a special finite discrete spectrum time series modeled by sine and cosine functions with given, usually high, frequencies.

EXAMPLE 2.1. Let $X(\cdot)$ be a time series given by the model

$$X(t) = \beta + \beta_1 \cos\lambda_1 t + \beta_2 \sin\lambda_1 t + \sum_{j=2}^{l} (Y_j \cos\lambda_j t + Z_j \sin\lambda_j t) + w(t)\,; \;\; t = 1, 2, \ldots,$$

where λ_j; $j = 1, 2, \ldots, l$, are *Fourier frequencies* for some even n, $\lambda_j = (2\pi/n) \times j$; $j = 1, 2, \ldots, l$, and assume that the number n of observations of time series $X(\cdot)$ is larger than $2l+2$. Next we assume that $Y_2, \ldots, Y_l, Z_2, \ldots, Z_l$ are uncorrelated random variables. Then the assumptions of Theorem 3.1. are fulfilled. The covariance function $R_\nu(\cdot,\cdot)$ of this time series is given by

$$R_\nu(s,t) = \sigma^2 \delta_{s,t} + \sum_{j=1}^{l} (\sigma_j^2 \cos\lambda_j s \cos\lambda_j t + \kappa_j^2 \sin\lambda_j s \sin\lambda_j t)\,; \quad s, t = 1, 2, \ldots,$$

where $\sigma_j^2 = D[Y_j]$ and $\kappa_j^2 = D[Z_j]$; $j = 2, 3, \ldots, l$.

We remark that the time series $X(\cdot)$ *is covariance stationary* if $D[Y_j] = D[Z_j] = \sigma_j^2$, $j = 2, 3, \ldots, l$, and in this case

$$R_\nu(s,t) = \sigma^2 \delta_{s,t} + \sum_{j=1}^{l} \sigma_j^2 \cos\lambda_j (s - t)\,; \qquad s, t = 1, 2, \ldots$$

Usually we estimate the frequencies λ_j, $j = 1, 2, \ldots, l$ using the periodogram computed in Fourier frequencies, and it is impossible to determine whether the local maxima of the periodogram are due to the periodicities in mean value or due to the finite discrete spectrum structure of the error time series. Thus the model considered in this example is one of possible reasonable models for time series data with a seasonal effect.

The BLUP, $X_\nu^*(n+d)$, in this model, based on the finite observation $X = \big(X(1), \ldots X(n)\big)'$ of the length n, is given by

$$\begin{aligned} X_\nu^*(n+d) &= f'(F'F)^{-1}F'X \\ &\quad + \sum_{j=2}^{l} \left(\frac{\cos\lambda_j(n+d)}{\sigma^2/\sigma_j^2 + n/2} X' \cos\lambda_j + \frac{\sin\lambda_j(n+d)}{\sigma^2/\kappa_j^2 + n/2} X' \sin\lambda_j \right), \end{aligned}$$

where

$$F = \begin{pmatrix} 1 & \cos\lambda_1 1 & \sin\lambda_1 1 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 1 & \cos\lambda_1 n & \sin\lambda_1 n \end{pmatrix},$$

$$(F'F)^{-1} = \begin{pmatrix} 1/n & 0 & 0 \\ 0 & 2/n & 0 \\ 0 & 0 & 2/n \end{pmatrix},$$

$f = \big(1, \cos\lambda_1(n+d), \sin\lambda_1(n+d)\big)'$, $\cos\lambda_j. = (\cos\lambda_j 1, \dots, \cos\lambda_j n)'$ and $\sin\lambda_j. = (\sin\lambda_j 1, \dots, \sin\lambda_j n)'$.

The MSE of this predictor is given by

$$\begin{aligned}\mathrm{MSE}_\nu\big[X^*_\nu(n+d)\big] &= \sigma^2\left[1 + \sum_{j=2}^{l}\left(\frac{\cos^2\lambda_j(n+d)}{\sigma^2/\sigma_j^2 + n/2} + \frac{\sin^2\lambda_j(n+d)}{\sigma^2/\kappa_j^2 + n/2}\right)\right] \\ &\quad + \sigma^2\big[1/n + (2/n)\cos^2\lambda_1(n+d) + (2/n)\sin^2\lambda_1(n+d)\big] \\ &= \sigma^2\left[1 + 3/n + \sum_{j=2}^{l}\left(\frac{\cos^2\lambda_j(n+d)}{\sigma^2/\sigma_j^2 + n/2} + \frac{\sin^2\lambda_j(n+d)}{\sigma^2/\kappa_j^2 + n/2}\right)\right].\end{aligned}$$

In the case when $X(\cdot)$ is covariance stationary we get

$$\mathrm{MSE}_\nu\big[X^*_\nu(n+d)\big] = \sigma^2\left[1 + 3/n + \sum_{j=2}^{l}\frac{1}{\sigma^2/\sigma_j^2 + n/2}\right].$$

For practical applications of the results derived, the problem of estimation of parameter ν is important. This problem will be studied in an independent article.

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Received March 26, 2002

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