Tatra Mt. Math. Publ. **27** (2003), 231–236

TATRA MOUNTAINS
*Mathematical Publications*

# ON THE CONJUGACY OF BOOLEAN ALGEBRAS AND MV-ALGEBRAS

Beloslav Riečan

*Dedicated to Professor Ján Jakubík on the occasion of his 80th birthday*

ABSTRACT. The well-known Halmos–von Neumann theorem states that any non-atomic, separable probability Boolean algebra is conjugated to the Lebesgue measure algebra of the real unit interval. Recently an analogue to the theorem has been proved for MV-algebras and the family of Lebesgue integrable functions from the unit interval into itself. In the paper there is proved that the Boolean algebra conjugacy theorem follows from the MV-algebra conjugacy theorem.

## 1. Probability algebra conjugacy theorem

A probability algebra is a couple $(A,\mu)$, where $A$ is a Boolean $\sigma$-algebra with a positive measure $\mu$ (i.e., $\mu(a) = 0 \Rightarrow a = 0$) such that $\mu(1) = 1$.

As an example consider the family $\mathcal{L}$ of all Lebesgue measurable subsets of [0,1]. Let $\lambda_0$ be the Lebesque measure on $\mathcal{L}$. Then the couple $(\mathcal{L}/\lambda_0, \lambda)$, where $\mathcal{L}/\lambda_0$ is the factor $\sigma$-algebra (with respect to the $\sigma$-ideal of all sets $A$ such that $\lambda_0(A) = 0$) and $\lambda$ is the induced Lebesgue measure on $\mathcal{L}/\lambda_0$, is a probability algebra. One of the nicest theorem of measure theory characterizes algebraically the measure algebra $(\mathcal{L}/\lambda_0, \lambda)$.

To any probability algebra $(A,\mu)$, a metric space $(A,\rho)$ can be constructed by the equality $\rho(a,b) = \mu(a \triangle b)$. $(A,\mu)$ is separable if the corresponding metric space $(A,\rho)$ is separable. $(A,\mu)$ is non-atomic, if to any $a \in A$ with $\mu(a) > 0$ there exists $b < a$ such that $0 < \mu(b) < \mu(a)$.

Two probability algebras $(A,\mu)$, $(B,\nu)$ are conjugated if there exists a measure preserving $\sigma$-isomorphism $\varphi\colon (A,\mu) \to (B,\nu)$, i.e., such a bijection $\varphi\colon A \to B$ that $\varphi(\bigvee a_n) = \bigvee \varphi(a_n)$, $\varphi(\neg a) = \neg\varphi(a)$ and $\nu(\varphi(a)) = \mu(a)$ for any $a \in A$, $a_n \in A$ $(n = 1, 2, \ldots)$.

---

2000 Mathematics Subject Classification: 06D35.
Keywords: conjugacy theorem, MV-algebra, Boolean algebra.

The paper was supported by Grant VEGA 1/9056/02.

The following theorem will be mentioned as the Probability algebra conjugacy theorem.

**THEOREM 1.** *Any separable, nonatomic probability algebra* $(A,\mu)$ *is conjugated to* $(\mathcal{L}/\lambda_0, \lambda)$.

The aim of this paper is to present an alternative proof of the previous theorem. Besides the Probability MV-algebra conjugacy theorem we need in the proof of Theorem 1 the well-known Loomis–Sikorski theorem:

**THEOREM 2.** *For every Boolean* $\sigma$*-algebra* $A$ *there exists a* $\sigma$*-algebra* $\mathcal{S}$ *of subsets of a set* $\Omega$ *and a Boolean* $\sigma$*-homomorphism from* $\mathcal{S}$ *onto* $A$.

## 2. Probability MV-algebras

Recall that MV-algebra $M = (M, 0, 1, \neg, \oplus, \odot)$ is a system where $\oplus$ is associative and commutative with neutral element $0$, $\neg 0 = 1$, $\neg 1 = 0$, $x \oplus 1 = 1$, $x \odot y = \neg(\neg x \oplus \neg y)$, and $y \oplus \neg(y \oplus \neg x) = x \oplus \neg(x \oplus \neg y)$ for all $x, y \in M$.

In every MV-algebra the binary operation $\leq$ is defined by $a \leq b$ if and only if $a \odot \neg b = 0$. $M$ makes with respect to $\leq$ a distributive lattice. An MV-algebra $M$ is $\sigma$-complete if its underlying lattice is $\sigma$-complete.

A probability MV-algebra is a couple $(M, m)$, where $M$ is an $MV\sigma$-algebra and $m\colon M \to [0,1]$ satisfies the following conditions:

$$m(1) = 1\,; \tag{i}$$

$$a \odot b = 0 \implies m(a \oplus b) = m(a) + m(b)\,; \tag{ii}$$

$$a_n \nearrow a \implies m(a_n) \nearrow m(a)\,; \tag{iii}$$

$$m(a) = 0 \iff a = 0\,. \tag{iv}$$

Consider the family $F$ of all Lebesgue measurable functions $f\colon [0,1] \to [0,1]$. Denote by $L$ the mapping $L\colon F \to [0,1]$ defined by $L(f) = \int_{[0,1]} f(x)\,dx$, the Lebesgue integral of $f$. Denote further by $\mathcal{F}$ the corresponding factor $MV$-algebra

$$\mathcal{F} = F/L = \big\{[f] \colon f \in F\big\}\,,$$

where $[f] = \big\{g \in F \colon L(|f-g|) = 0\big\}$, $[f] \oplus [g] = \big[\min(f+g, 1)\big]$, $[f] \odot [g] = \big[\max(f+g-1, 0)\big]$, $\neg[f] = [1-f]$, and $\mathbf{L}$ the corresponding functional

$$\mathbf{L}\left([f]\right) = \int_{[0,1]} f(x)\,dx\,.$$

Then $(\mathcal{F}, \mathbf{L})$ is a probability MV-algebra.

## 3. Probability MV-algebra conjugacy theorem

Two probability MV-algebras $(M_1, m_1), (M_2, m_2)$ are conjugated if there exists a bijection $\varphi\colon M_1 \to M_2$ such that

$$\varphi(a \oplus b) = \varphi(a) \oplus \varphi(b)\,,$$

$$\varphi(\neg a) = \neg\varphi(a)\,,$$

$$\varphi\left(\bigvee_{n=1}^{\infty} a_n\right) = \bigvee_{n=1}^{\infty} \varphi(a_n)\,,$$

$$m_2\bigl(\varphi(a)\bigr) = m_1(a)\,,$$

for any $a, b, a_n \in M$.

We need three properties of probability MV-algebras to characterize the conjugacy with the probability MV-algebra $(\mathcal{F}, \mathbf{L})$.

As before we can consider the metric space $(M, \rho)$, where

$$\rho(a, b) = m(|a - b|)\,.$$

Here $|a - b| = (a - b)^+ + (a - b)^- = (a - b) \vee (b - a)$. We say that $(M, m)$ is separable, if the metric space $(M, \rho)$ is separable.

The second property is nonatomicity, of course not in terms of the MV-algebra $M$, but in terms of the corresponding Boolean algebra. Denote by $A(M)$ this $\sigma$-algebra, i.e.,

$$A(M) = \{a \in M\colon\, a \oplus a = a\}\,.$$

The probability MV-algebra $(M, m)$ is nonatomic, if the probability Boolean algebra $\bigl(A(M), m|A(M)\bigr)$ is nonatomic, i.e., to any $a \in A(M)$ with $m(a) > 0$ there is $b \le a$, $b \in A(M)$ such that $0 < m(b) < m(a)$.

The third property is divisibility. An MV-algebra $M$ is divisible, if to any $a \in M$ and any $n \in \mathbb{N}$ there exists $b \in M$ such that $nb = a$.

**THEOREM 3.** *If a probability MV-algebra $(M, m)$ is separable, nonatomic and divisible, then $(M, m)$ is conjugated to the probability MV-algebra $(\mathcal{F}, \mathbf{L})$.*

**THEOREM 4.** *Theorem 1 follows by Theorems 3 and 2.*

## 4. Proof

Let $(A, \mu)$ be a separable, nonatomic probability algebra. We want to prove that $(A, \mu)$ is conjugated to $(\mathcal{L}/\lambda_0, \lambda)$.

By Theorem 2 there exists a measurable space $(\Omega, \mathcal{S})$ and a $\sigma$-epimorphism $\psi\colon \mathcal{S} \to A$. Define a mapping $\nu\colon \mathcal{S} \to [0,1]$ by the formula

$$\nu(C) = \mu\big(\psi(C)\big)\,.$$

The definition of $\nu$ does not depend on the choice of $C$. Indeed, let $\psi(C_1) = \psi(C_2) = c$. Then $\mu\big(\psi(C_1)\big) = \mu\big(\psi(C_1 \setminus C_1 \cap C_2)\big) + \mu\big(\psi(C_1 \cap C_2)\big) = \mu\big(\psi(C_1) - \psi(C_1) \wedge \psi(C_2)\big) + \mu\big(\psi(C_1) \wedge \psi(C_2)\big) = \mu(c - c) + \mu(c) = \mu(c)$ and similarly $\mu\big(\psi(C_2)\big) = \mu(c)$.

It is not difficult to prove that $\nu\colon \mathcal{S} \to [0,1]$ is a probability measure, hence $(\Omega, \mathcal{S}, P)$ is a probability space. Let $G$ be the family of all $\mathcal{S}$-measurable functions from $\Omega$ to $[0,1]$, $K\colon G \to [0,1]$, $K(g) = \int_\Omega g\, d\nu$,

$$\mathcal{G} = G/K = \big\{[g]\colon g \in G\big\}\,,$$

where $[g] = \big\{h \in G\colon\ \int_\Omega |g - h| d\nu = 0\big\}$. Define $\mathcal{K}\colon \mathcal{G} \to [0,1]$ by the formula

$$\mathcal{K}\big([g]\big) = \int\limits_\Omega g\, d\nu\,.$$

Then $(\mathcal{G}, \mathcal{K})$ is a probability MV-algebra. We shall prove that $(\mathcal{G}, \mathcal{K})$ satisfies the assumptions of Theorem 3.

$(\mathcal{G}, \mathcal{K})$ is evidently divisible.

To prove separability of $(\mathcal{G}, \mathcal{K})$ we need to construct a countable dense subset of $\mathcal{G}$ with respect to the metric $\rho$ given by the formula

$$\rho\big([f],[g]\big) = \int\limits_\Omega |f - g|\, d\nu\,.$$

Let $D = \{d_1, d_2, d_3, \dots\}$ be a countable dense subset of $(A, d)$, where $d(a,b) = \mu(a \bigtriangleup b)$. Choose $D_k \in \mathcal{S}$ such that $\psi(D_k) = d_k$, $k = 1, 2, \dots$, and put $\mathcal{D} = \{D_k\colon k = 1, 2, \dots\}$. Then the family

$$\left\{\left[\sum_{i=1}^{n} \alpha_i \chi_{E_i}\right]\colon \alpha_i \in Q\,,\ E_i \in \mathcal{D},\ n \in \mathbb{N}\right\}$$

is a countable dense set in $(\mathcal{G}, \rho)$.

If we want to investigate the nonatomicity of $(\mathcal{G}, \mathcal{K})$, we must consider the underlying Boolean algebra

$$\begin{aligned} A(\mathcal{G}) &= \big\{[f] \in \mathcal{G}\colon [f] \oplus [f] = [f]\big\} \\ &= \{[f] \in \mathcal{G}\colon f \oplus f = f \nu\text{-a.e.}\} \\ &= \big\{[\chi_C]\colon C \in \mathcal{S}\big\} = A\,, \end{aligned}$$

hence $A(\mathcal{G})$ is nonatomic, and $(\mathcal{G},\mathcal{K})$ is nonatomic, too, by the definition.

By Theorem 3 there exists a bijection $\varphi\colon \mathcal{G} \to \mathcal{F}$ such that $\varphi\big([f]\oplus[g]\big) = \varphi([f]) \oplus \varphi([g])$, $\varphi(\neg[f]) = \neg\varphi([f])$, $\varphi\left(\bigvee\limits_{i=1}^{\infty}[f_i]\right) = \bigvee\limits_{i=1}^{\infty}\varphi([f_i])$, $\mathbf{L}(\varphi([f])) = \int\limits_{\Omega} f\,d\nu$.

We shall construct an isomorphism $\kappa\colon (A,m) \to (\mathcal{L}/\lambda_0,\lambda)$. Take $c \in A$, and $C \in \mathcal{S}$ such that $\psi(C) = c$. Since $[\chi_C]$ is idempotent, $\varphi\big([\chi_C]\big)$ is idempotent, too, $\varphi\big([\chi_C]\big) = [\chi_D]$. We want to define $\kappa(c) = [D] \in \mathcal{L}/\lambda_0$. Of course, we must prove that $\kappa(c)$ does not depend on the choice of $C$.

Let $\psi(C_1) = \psi(C_2) = c$. then $\psi(C_1 \triangle C_2) = 0$, hence $\nu(C_1 \triangle C_2) = 0$, and $\chi_{C_1} = \chi_{C_2}$, $\nu$-a.e. Therefore $[\chi_{D_1}] = \varphi\big([\chi_{C_1}]\big) = \varphi\big([\chi_{C_2}]\big) = [\chi_{D_2}]$,

$$\int\limits_{[0,1]} \left|\chi_{D_1} - \chi_{D_2}\right| d\lambda_0 = 0\,,$$

hence $\lambda_0(D_1 \triangle D_2) = 0$, and $[D_1] = [D_2]$. Therefore the definition of $\kappa$ by $\kappa(c) = [D]$, with $\psi(C) = c, \varphi\big([\chi_C]\big) = [\chi_D]$, is correct.

If $\kappa(c_1) = \kappa(c_2)$, then $\chi_{D_1} = \chi_{D_2}$, $\lambda_0$-a.e. It follows $\chi_{C_1} = \chi_{C_2}$, $\nu$-a.e., hence $\nu(C_1 \triangle C_2) = 0$ and $c_1 = \psi(C_1) = \psi(C_2) = c_2$. We have proved that $\kappa\colon A \to \mathcal{L}/\lambda_0$ is an injection.

Let $[D] \in \mathcal{L}/\lambda_0$. Choose $C \in \mathcal{S}$ such that $\varphi\big([\chi_C]\big) = [\chi_D]$ and put $c = \psi(C)$. Then $\kappa(c) = [D]$, hence $\kappa$ is surjective.

If $c_1, c_2 \in A$, then there are $C_1, C_2 \in \mathcal{S}$ such that $\psi(C_1) = c_1$, $\psi(C_2) = c_2$, $\psi(C_1 \cup C_2) = c_1 \cup c_2$. Therefore $\chi_{C_1\cup C_2} = \chi_{C_1} \oplus \chi_{C_2}$,

$$\begin{aligned}\varphi\big([\chi_{C_1\cup C_2}]\big) &= \varphi\big([\chi_{C_1}]\big) \oplus \varphi\big([\chi_{C_2}]\big)\\ &= [\chi_{D_1}] \oplus [\chi_{D_2}] = [\chi_{D_1\cup D_2}]\,,\end{aligned}$$

$$\kappa(c_1 \cup c_2) = [D_1 \cup D_2] = [D_1] \vee [D_2] = \kappa(c_1) \vee \kappa(c_2)\,,$$

hence $\kappa$ preserves the operation $\vee$.

Let $c \in A$, $C \in \mathcal{S}$ such that $\psi(C) = c$, hence $\varphi\big([\chi_C]\big) = [\chi_D]$, $\kappa(c) = [D]$. Then $\psi(C') = c'$, $\varphi\big([\chi_{C'}]\big) = \neg\varphi\big([\chi_C]\big) = \neg\big([\chi_C]\big) = \neg[\chi_D] = [\chi_{D'}]$, hence $\kappa(c') = [D'] = [D]' = \kappa(c)'$.

If $c_n \in A$ $(n = 1,2,\dots)$, $c_n \nearrow c$, then there are $C_n \in \mathcal{S}$ such that $C_n \nearrow C$. It follows $\chi_{C_n} \nearrow \chi_C$, $\chi_{D_n} = \varphi\big([\chi_{C_n}]\big) \nearrow \varphi\big([\chi_c]\big) = [\chi_D]$, hence $\kappa(c_n) = [D_n] \nearrow [D] = \kappa(c)$.

Finally, if $c \in A$, $C \in \mathcal{S}$, $\psi(C) = c$, then $\mu(c) = \nu(C) = \int\limits_{\Omega} \chi_C\,d\nu = \mathbf{L}\Big(\varphi\big([\chi_C]\big)\Big) = \mathbf{L}\big([\chi_D]\big) = \int\limits_{[0,1]} \chi_D\,d\lambda_0 = \lambda\big([D]\big) = \lambda\big(\kappa(c)\big)$, hence $\kappa$ is measure preserving.

## 5. Conclusions

Denote by **PA** Theorem 1 (Probability algebra conjugacy theorem), by **PMVA** Theorem 3 (Probability MV-algebra conjugacy theorem), by **LS** Theorem 2 (Loomis–Sikorski theorem), by **DMLS** Dvurečenskij–Mundici–Loomis–Sikorski Theorem ([3], Theorem 7.1.22). We have proved that

$$\mathbf{PMVA} \wedge \mathbf{LS} \Longrightarrow \mathbf{PA}\,.$$

In [6] we have proved that

$$\mathbf{DMLS} \wedge \mathbf{PA} \Longrightarrow \mathbf{PMVA}\,.$$

Of course, there are independent proofs of **LS** and **DMLS**. We mean that the results presented in the paper could be interesting since there are deep relations between Boolean algebras and MV-algebras as well as among their applications.

REFERENCES

[1] CIGNOLI, R.—D'OTTAVIANO, I. M. L.—MUNDICI, D.: *Algebraic Foundations of Many-Valued Reasoning*, Kluwer Acad. Publ., Dordrecht, 1993.

[2] DVUREČENSKIJ, A. Loomis–Sikorski theorem for $\sigma$-complete MV-algebras and $l$-groups: J. Austral. Math. Soc. Ser. A **68** (2000), 261–277.

[3] DVUREČENSKIJ, A.—PULMANNOVÁ, S.: *New Trends in Quantum Structures*, Kluwer Acad. Publ., Dordrecht, 2000.

[4] HALMOS, P. R.—von NEUMANN, J.: *Operator methods in classical mechanics*, Ann. Math. **43** (1942), 332–350.

[5] MUNDICI, D.: *Tensor products and the Loomis-Sikorski theorem for MV-algebras*, Adv. in Appl. Math. **22** (1999), 227–248.

[6] RIEČAN, B.: *The conjugacy of probability MV-$\sigma$-algebras with the unit interval*, Atti Sem. Mat. Fis. Univ. Modena (to appear).

Received February 4, 2003

*Mathematical Institute*
*Slovak Academy of Sciences*
*Štefánikova 49*
*SK–814 73 Bratislava*
*SLOVAKIA*

*E-mail*: riecan@mat.savba.sk

*Department of Mathematics*
*Faculty of Natural Sciences*
*Matej Bel University*
*Tajovského 40*
*SK–974 01 Banská Bystrica*
*SLOVAKIA*

*E-mail*: riecan@fpv.umb.sk