Tatra Mt. Math. Publ. **27** (2003), 125–130

TATRA MOUNTAINS
Mathematical Publications

EMBEDDING STANDARD BL-ALGEBRAS INTO NON-COMMUTATIVE PSEUDO-BL-ALGEBRAS

Petr Hájek

Dedicated to Professor Ján Jakubík on the occasion of his 80th birthday

ABSTRACT. Each BL-algebra given by a continuous t-norm is a subalgebra of a non-commutative pseudo-BL-algebra on a "non-standard" interval [0,1].

Introduction

BL-algebras are algebras of truth functions of the basic fuzzy logic BL [5]. Each continuous t-norm defines a standard BL-algebra (or t-algebra) on the real interval $[0,1]$ (with its standard ordering). As proved in [3], BL-tautologies (propositional formulas being tautologies over each BL-algebra) are the same as t-tautologies (standard BL-tautologies). Speaking algebraically, the variety of BL-algebra is generated by the class of t-algebras.

Di Nola, Georgescu and Iorgulescu [1, 2] introduced and studied pseudo-BL-algebras (briefly, psBL-algebras), a generalization of BL-algebras not assuming commutativity of the semigroup operation (truth function of conjunction). The corresponding propositional logic was established in [6, 7].

Recall the definition of a psBL-algebra [1]: A pseudo-BL algebra (psBL-algebra) is a structure

$$\mathbf{A} = (A, \wedge, \vee, *, \rightarrow, \rightsquigarrow, 0, 1)\,,$$

where

(C1) $(A, \vee, \wedge, 1)$ is a bounded lattice,

(C2) $*$ is associative and 1 is its unit element, i.e., $1*x = x*1 = x$ (for all x),

(C3) $x * y \leq z$ iff $x \leq y \rightarrow z$ iff $y \leq x \rightsquigarrow z$ (for all x, y, z),

2000 Mathematics Subject Classification: 06D35, 06D30.
Keywords: BL-algebra, pseudo-BL-algebra, tautology.

Partial support of the grant No A1030004/00 of the Grant Agency of the Academy of Sciences of the Czech republic is acknowledged. Thanks are due to the anonymous referee for careful correction of several misprints and suggestions of improvement of some formulations.

(C4) $x \wedge y = (x \to y) * x = x * (x \rightsquigarrow y)$ (for all x, y),

(C5) $(x \to y) \vee (y \to x) = 1 = (x \rightsquigarrow y) \vee (y \rightsquigarrow x)$ (for all x, y).

(C2) is the axiom of monoid, (C3) of residuation, (C4) of divisibility, (C5) of prelinearity.

As shown in [4], there are no non-commutative standard psBL-algebras, i.e., psBL-algebras whose lattice reduct is the standard real interval $[0,1]$.[1]

In [7] I gave an example of a non-commutative psBL-algebra on the "non-standard" unit interval in which each standard element of $[0,1]$ has continuum of "infinitely near" non-standard elements; $\mathrm{NS}[0,1]$ is the set of pairs

$$\begin{aligned}&\bigl\{(0,y)\colon\ y \in Re,\ y \geq 0\bigr\} \cup \\ &\bigl\{(x,y)\colon\ x \in Re,\ 0 < x < 1,\ y \in Re\bigr\} \cup \\ &\bigl\{(1,y)\colon\ y \in Re,\ y \leq 0\bigr\}\end{aligned}$$

with lexicographic order (standard elements being the pairs $(x,0)$). The example is a pseudo-MV-algebra in the terminology of [1] and its standard elements form a standard BL-algebra (modulo the representation of $x \in [0,1]$ by the pair $(x,0)$). I asked at the end of [7] if each standard BL-algebra is embeddable in this way into a non-commutative psBL-algebra on the non-standard unit interval $\mathrm{NS}[0,1]$. Our result is the following:

Theorem 1. *For each continuous t-norm $*$ having at least one non-idempotent element there is a non-commutative psBL-algebra $\mathbf{A}$ on $\mathrm{NS}[0,1]$ whose reduct to $[0,1] \times \{0\}$ is isomorphic to the standard BL-algebra $[0,1]_*$ via the identification of $x \in [0,1]$ with the pair $(x,0)$.*

To prove this theorem we show in Section 2 that it holds for the Łukasiewicz t-norm as well as for the product t-norm. In Section 3 we recall the construction of each continuous t-norm from copies of Łukasiewicz and/or product t-norm and from idempotents; then we show the obvious generalization—construction of a psBL-chain as ordinal sum of psBL-chains. In Section 4 we obtain a proof of the theorem and formulate some open problems.

2. Two embeddings

As shown in [1, 2], particular psBL-chains (psMV-chains, pseudo-product chains) can be obtained from linearly ordered groups similarly as MV-chains and product chains can be obtained from linearly ordered Abelian groups. In

[1] But note that there are standard non-commutative pseudo-MTL-algebras, or "weak psBL-algebras" in the terminology of [4]; for the logic of psMTL-algebras see [7].

particular, we take the famous (Birkhoff's) example of non-commutative linearly ordered group on $B = \{(x,y)\colon x,y \in Re,\ x > 0\}$ with the operation

$$(x_1, y_1) \cdot (x_2, y_2) = (x_1 \cdot x_2,\ y_1 \cdot x_2 + y_2)$$

and with lexicographic order. Its unit element is $(1,0)$; the interval $C = \left[(\frac{1}{2}, 0), (1,0)\right]$ is made to a psMV-algebra by defining, for $\alpha, \beta \in C$,

$$\alpha * \beta = \max\left(\left(\frac{1}{2}, 0\right),\ \alpha \cdot \beta\right),$$

$$\alpha \to \beta = \beta \cdot \alpha^{-1},\ \alpha \rightsquigarrow \beta = \alpha^{-1} \cdot \beta \qquad \text{for } \alpha \geq \beta,$$

$$\alpha \to \beta = \alpha \rightsquigarrow \beta = (1,0) \qquad \text{for } \alpha \leq \beta.$$

In [7] I suggested to transform the algebra from C to $\mathrm{NS}[0,1]$ and observed that by doing this we get a psBL-algebra on $\mathrm{NS}[0,1)$ whose standard part $[0,1] \times \{0\}$ is a continuous t-norm (after the obvious identification of $[0,1] \times \{0\}$ with $[0,1]$). Observe now that if we take a better transformation we get just Łukasiewicz t-norm:

Lemma 1. *Let $f(a,b) = (2^{a-1}, b)$. Then $f^{-1}(u,v) = (1 + \log_2 u, v)$, f is a bijection of $\mathrm{NS}[0,1]$ to the interval C above and f induces on $\mathrm{NS}[0,1]$ the operation*

$$(a,b) *_{\mathrm{psL}} (c,d) = \max\big((0,0), (a + c - 1,\ b \cdot 2^{c-1} + d)\big)$$

isomorphic with the above operation $*$ on C. Similarly, f induces the operations $\to_{\mathrm{psL}}, \rightsquigarrow_{\mathrm{psL}}$ isomorphic to the above $\to, \rightsquigarrow$ on C. This defines a psBL-algebra on $\mathrm{NS}[0,1]$; its standard part is just the t-algebra given by Łukasiewicz t-norm.

Proof. By trivial checking. □

Definition 1. The psBL-algebra on $\mathrm{NS}[0,1]$ from the preceding lemma will be called the *pseudo-Lukasiewicz algebra* on $\mathrm{NS}[0,1]$ and denoted by $\mathrm{NS}[0,1]_{\mathrm{psL}}$.

Now let us analyze the construction of a pseudo-product algebra from a linearly ordered group presented in [2] 2.21 and apply it to the group B. Similarly as in the commutative case, taking $B^- = \{(x,y) \in B\colon (x,y) \leq (1,0)\}$ with the multiplication of B, adding $(0,0)$ as a least and zero element and defining appropriately the implications $\to, \rightsquigarrow$ we get a pseudo-product algebra. But we want the operations to act on the whole of $\mathrm{NS}[0,1]$; this is achieved by making all elements $(0,y)$ of $\mathrm{NS}[0,1]$ to idempotents. In detail,

Definition 2. For $(a,b), (c,d) \in \mathrm{NS}[0,1]$, let

$$(a,b) *_{\mathrm{psR}} (c,d) = (ac, bc + d) \quad \text{if } a, c > 0,$$

$$(a,b) *_{\mathrm{psR}} (c,d) = \min\big((a,b), (c,d)\big) \quad \text{if } a = 0 \text{ or } c = 0.$$

$$(a,b) \to_{\mathrm{psR}} (c,d) = (a,b) \rightsquigarrow_{\mathrm{psR}} (c,d) = 1 \quad \text{for } (a,b) \leq (c,d).$$

Otherwise:

$$(a,b) \rightarrow_{\mathrm{psR}} (c,d) = \Big(\frac{c}{a}, \frac{d-b}{a}\Big),$$
$$(a,b) \rightsquigarrow_{\mathrm{psR}} (c,d) = \Big(\frac{c}{a}, d - \frac{bc}{a}\Big) \qquad \text{for } a,c>0\,,$$
$$(a,b) \rightarrow_{\mathrm{psR}} (0,d) = (a,b) \rightsquigarrow_{\mathrm{psR}} (0,d) = (0,d)\,.$$

Denote this structure by $\mathrm{NS}[0,1]_{\mathrm{psR}}$.

LEMMA 2. $\mathrm{NS}[0,1]_{\mathrm{psR}}$ *is a psBL-algebra; its standard part is the product t-algebra.*

P r o o f. By trivial checking. □

Note that $\mathrm{NS}[0,1]_{\mathrm{psR}}$ is *not* a pseudo-product algebra in the sense of [2]; one cannot cancel by any element $(0,y)$. But we prefer to have the algebra defined on the whole of $\mathrm{NS}[0,1]$.

3. Ordered sums

Recall the structure of continuous t-norms (M o s t e r t–S h i e l d s, see, e.g., [5]). Let $*$ be a continuous t-norm, let E be the (closed) set of its idempotents. For $e \in E$, let e^+ be the smallest idempotent bigger than e if there is such, $e^+ = e$ otherwise. If $e^+ > e$ then the restriction of $*$ to the interval $[e,e^+]$ is isomorphic either to Łukasiewicz t-norm on $[0,1]$ or to the product t-norm on $[0,1]$. The isomorphism preserves also the residuum $x \Rightarrow y$ for $x,y \in [e,e^+]$, $x>y$. For x,y not belonging to the same $[e,e^+]$, $x*y = \min(x,y)$ and if $x>y$ then $x \Rightarrow y = y$. Conversely, if $(E \leq)$ is a linearly ordered index set with least and greatest element, define e^+ as above and assume that

(1) for each $e \in E$, $\mathbf{A}_e$ is a psBL-algebra with the least element e and largest element e^+ (thus if $e = e^+$ then A_e is singleton);

(2) there is a linear order $\leq_A$ of $A = \bigcup\limits_{e\in E} A_e$ extending $\leq_E$ such that A_e is the interval $[e,e^+]$ and the order of A_e is just the restriction of $\leq_A$ to A_e. The largest element of A is denoted by 1.

Define operations on A as follows:

$$x*y = x *_e y \quad \text{if } x,y \in A_e \text{ for some } e\,,$$
$$x*y = \min(x,y) \quad \text{otherwise}\,;$$
$$x \rightarrow y = x \rightsquigarrow y = 1 \quad \text{if } x \leq y\,; \quad \text{for } x>y\,,$$
$$x \rightarrow y = x \rightarrow_e y \quad \text{and} \quad x \rightsquigarrow y = x \rightsquigarrow_e y \text{ if } x,y \in A_e \text{ for some } e\,,$$
$$x \rightarrow y = x \rightsquigarrow y = y \quad \text{otherwise}\,.$$

Denote this structure by $\bigoplus_{e\in E} \mathbf{A}_e$ and call it the ordered sum of the A'_es.

Lemma 3. $\bigoplus_{e\in E}$ *is a psBL-algebra.*

Proof by checking. Note that for BL-algebras this occurs in [8]; for the case of two psBL-algebras see [2].

Proof of the theorem

Let a continuous t-norm $*$ be given; let E be its set of idempotents. If $e \in E$ and $e^+ > e$, let f_e be the isomorphism of $[0,1]_{\mathrm{L}}$ or $[0,1]_\Pi$ to the BL-algebra given by the restriction of $*$ to $[e, e^+]$. Extend it in the obvious way to an isomorphism of $\mathrm{NS}[0,1]_{\mathrm{psL}}$ or $\mathrm{NS}[0,1]_{\mathrm{psR}}$ to a psBL-algebra on the interval $[(e,0),(e^+,0)]$ in $\mathrm{NS}[0,1]$. Call it $\mathbf{A}_{(e,0)}$. Let $\hat{E}$ consist of all elements $(e,0)$ for $e \in E$ plus all elements $(x,y) \in \mathrm{NS}[0,1]$ not belonging to any interval $\big[(e,0),(e^+,0)\big]$ for $e \neq e^+$ (i.e., all (e,y), $y > 0$, if e is infimum of all idempotents bigger than e; and all (e,y), $y < 0$, if e is supremum of all idempotents less than e). If $e^+ > e$ then $\mathbf{A}_{(e,0)}$ has been defined; for all other $(e,y) \in \hat{E}$ let $\mathbf{A}_{e,y}$ be the one-element BL-algebra with the domain $\big\{(e,y)\big\}$. Now $\bigcup\big\{A_{(e,y)} \colon (e,y) \in \hat{E}\big\} = \mathrm{NS}[0,1]$ and the ordered sum $\bigoplus\big\{\mathbf{A}_{(e,y)} \colon (e,y) \in \hat{E}\big\}$ is defined as above. It is a psBL-chain with the field $\mathrm{NS}[0,1]$ and its subalgebra on $[0,1] \times \{0\}$ is isomorphic to the given continuous t-norm via the obvious identification of x with $(x,0)$. This completes the proof.

We could call these psBL-algebras (on $\mathrm{NS}[0,1]$, whose restriction to $[0,1] \times \{0\}$ is a continuous t-norm) *semistandard.* It is natural to ask, in an analogy to BL-algebras, if psBL^r-tautologies (see [7], tautologies of subdirectly representable psBL-algebras) coincide with semistandard psBL-tautologies. Note that [2] contains much material analogous to the material used to prove that BL-tautologies coincide with standard BL-tautologies (t-tautologies). But the problem seems open.

REFERENCES

[1] DI NOLA, A.—GEORGESCU, G.—IORGULESCU, A.: *Pseudo-BL-algebras: Part I,* Mult.-Valued Log. **8** (2002), 673–714.

[2] DI NOLA, A.—GEORGESCU, G.—IORGULESCU, A.: *Pseudo-BL-algebras: Part II,* Mult.- Valued Log. **8** (2002), 747–750.

[3] CIGNOLI, R.—ESTEVA, F.—GODO, L.—TORRENS, A.: *Basic fuzzy logic in the logic of continuous t-norms and their residua,* Soft Computing **4** (2000), 106–112.

[4] FLONDOR, P.—GEORGESCU, G.—IORGULESCU, A.: *Pseudo-t-norms and pseudo--BL-algebras*, Soft Computing **5** (2001), 355–371.
[5] HÁJEK, P.: *Metamathematics of Fuzzy Logic*, Kluwer, Dordrecht, 1998.
[6] HÁJEK, P.: *Fuzzy logics with non-commutative conjunction*, J. Logic Comput. (to appear).
[7] HÁJEK, P.: *Observations of non-commutative fuzzy logic*, Soft Computing (to appear).
[8] HÁJEK, P.: *Basic logic and BL-algebras*, Soft Computing **2** (1998), 124–128.

Received October 25, 2002

Institute of Computer Science
Academy of Sciences
Pod vodárenskou věží 4
CZ–182 07 Praha-Libeň
CZECH REPUBLIC
E-mail: hajek@cs.cas.cz