Tatra Mt. Math. Publ. **27** (2003), 23–43

TATRA MOUNTAINS
Mathematical Publications

AN MV-ALGEBRAIC INVARIANT FOR BOOLEAN ALGEBRAS WITH A FINITE-ORBIT AUTOMORPHISM

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Dedicated to Professor Ján Jakubík on the occasion of his 80th birthday

ABSTRACT. We classify boolean algebras equipped with a distinguished automorphism having no infinite orbit. For any object in this category, its corresponding invariant is a locally finite MV-algebra—the colimit (or, in universal algebra terms, the direct limit) of finite MV-algebras. Locally finite MV-algebras are an infinite-valued generalization of boolean algebras. All locally finite MV-algebras arise in this classification. However, this invariant does not determine an equivalence of categories. We shall use multisets as an intermediate invariant, and then apply the duality between multisets and locally finite MV-algebras. The latter was recently established by the present authors, and is an extension of Stone's duality between boolean spaces and boolean algebras. Our results show that an important class of MV-algebras is essentially the same as the class of boolean algebras equipped with a finite-orbit automorphism. Given the role of MV-algebras in infinite-valued logic, our results suggest that logical multi-valuedness can be developed within the framework of classical logic.

1. Locally finite boolean automorphisms

By a *boolean algebra with a distinguished automorphism* we shall understand a pair (B, β) such that B is a boolean algebra[1] and β is an automorphism of B, considered as a new unary operation. Accordingly, a *subalgebra* of (B, β) is a pair (S, β') consisting of a boolean subalgebra S of B which is closed under β, equipped with the restriction β' of β to S. By a *homomorphism from*

2000 Mathematics Subject Classification: 06D35, 06E25, 06E15.
Keywords: locally finite MV-algebra, boolean algebra with a locally finite automorphism, automorphism with no infinite orbit, MV-algebra, multiset, Stone duality supernatural number.

Partially supported by Project of Scientific and Technological Cooperation between Argentina and Italy, on "Many-valued Logic", Code 33I.

[1] Each boolean algebra B considered in this paper shall be assumed to be non-trivial (i.e., $0 \neq 1$ in B). All boolean algebras arising from our functors are easily shown to be non-trivial.

(B, β) *to* (C, γ), we understand a boolean homomorphism $h\colon B \to C$ such that $h(\beta(b)) = \gamma(h(b))$ for all $b \in B$.

Notation: Throughout this paper we shall use the notation

$$\mathbb{BA}\delta$$

for the category whose objects are boolean algebras with a distinguished automorphism, and whose morphisms are the above defined homomorphisms.[2]

Definition 1.1. Let (B, β) be an object in $\mathbb{BA}\delta$. The *orbit* of an element $b \in B$ is the set $\operatorname{orb}_\beta(b) = \{\beta^n(b)\colon n \geq 0\}$.[3] We say that b is *periodic* iff $\operatorname{orb}_\beta(b)$ is finite. In this case the least integer $n \geq 1$ such that $\beta^n(b) = b$ is called the *period of* b, and is denoted by

$$\kappa_b$$

so long as the automorphism β is clear from the context. If $\operatorname{orb}_\beta(b)$ is infinite, we write $\kappa_b = \infty$. We say that β is *locally finite* iff it has no infinite orbit.

If the set $\{\kappa_b\colon b \in B\}$ has a finite upper bound, and κ is the least common multiple of the numbers $\{\kappa_b\colon b \in B\}$, then we say that β is *κ-cyclic*. Objects (B, β) in the category $\mathbb{BA}\delta$ such that β is κ-cyclic are known in the literature as *κ-cyclic boolean algebras*. These algebras were first investigated by Moisil, and their theory was further developed by A. Monteiro (see [13] and references therein). When A is finite, for each $\gamma \in \operatorname{Aut}(A)$ there trivially exists an integer $\kappa \geq 1$ such that γ is κ-cyclic.

Notation: Throughout this paper we shall use the notation

$$\mathbb{LFBA}\delta$$

for the full subcategory of $\mathbb{BA}\delta$ given by all pairs (A, α) such that A is a boolean algebra and α is a locally finite automorphism. Further, we shall denote by

$$\mathbb{FBA}\delta$$

the subcategory of $\mathbb{BA}\delta$ given by *finite* boolean algebras with a distinguished automorphism.

Remark 1.2. *If* $(A, \gamma) \in \mathbb{LFBA}\delta$*, then all its finite subalgebras* $S \subseteq A$ (*as boolean algebras*) *closed under* γ *are in* $\mathbb{FBA}\delta$. Indeed, let S be a finite subalgebra of A closed under γ, and let $a \in S$. There is $n \geq 1$ such that $\gamma^n(a) = a$. Hence $a = \gamma(\gamma^{n-1}(a))$, and since $\gamma^{n-1}(a) \in S$, then γ maps S onto S. Therefore, the restriction of γ to S is an automorphism of S. One then sees that γ is a locally finite automorphism of a boolean algebra A iff all finitely generated subalgebras of (A, γ) are finite. Thus, γ is locally finite iff the algebra (A, γ) is locally finite in the usual algebraic sense.

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[2] The class of boolean algebras B with a distinguished automorphism β becomes a variety (also called an equational class) provided we also equip B with the unary operation β^{-1}.

[3] As usual, β^0 is the identity automorphism.

Corollary 1.3. *A boolean algebra with a distinguished automorphism is locally finite if and only if it is the filtered colimit of a direct system of finite boolean algebras with a distinguished automorphism.*

Here, by *colimit* we mean what in universal algebra terms is usually called *direct limit*. See the Appendix for details and terminology about limits and colimits.

2. Locally finite MV-algebras and multisets

Locally finite MV-algebras

We assume familiarity with MV-algebras [14], [6].[4] An MV-algebra is said to be *locally finite* iff all its finitely generated subalgebras are finite.[5] Every boolean algebra is a locally finite MV-algebra.

In the next result, we collect some characterizations of locally finite MV-algebras. See [7] for a proof:

lemma 2.1. *The following conditions are equivalent for any MV-algebra A:*

(i) *A is locally finite.*

(ii) *For each prime ideal J of A, the quotient A/J is isomorphic to a subalgebra of $\boldsymbol{Q} \cap [0,1]$, where $\boldsymbol{Q}$ denotes the rational numbers.*

(iii) *There is a set I such that A is isomorphic to a subalgebra of the MV-algebra $\left(\boldsymbol{Q} \cap [0,1]\right)^I$ formed by functions of finite range.*

(iv) *A is the filtered colimit of a direct system of finite MV-algebras.*

Notation: For an MV-algebra A, $\mathcal{X}, (A)$ will denote the set of homomorphisms from A into the MV-algebra $[0,1]$.

Supernatural numbers

For our duality we need the following generalization of natural numbers:

[4]Each MV-algebra A considered in this paper shall be assumed to be non-trivial (i.e., $0 \neq 1$ in A).

[5]A word of caution is necessary here. Following tradition, we are giving to "locally finite" its usual algebraic meaning, but in the MV-algebraic literature, the term "locally finite" is also (misleadingly) used as a synonym of "simple", i.e., "without proper homomorphic images". This may lead to confusion, because of the existence of infinite singly generated simple MV-algebras (see, for instance, [6, Corollary 7.2.7]).

Definition 2.2. A *supernatural number* ([15]) is a function $\nu\colon \boldsymbol{P} \to \{0, 1, \dots, \infty\}$, where $\boldsymbol{P}$ denotes the sequence of prime numbers. For any two supernatural numbers ν and μ, we write $\nu \leq \mu$ iff $\nu(p) \leq \mu(p)$ for all $p \in \boldsymbol{P}$.

Endowed with this order relation, the set of supernatural numbers forms a lattice denoted $\boldsymbol{G}$: indeed, $\boldsymbol{G}$ is a locale, (i.e., a complete lattice where finite meets distribute over arbitrary joins), whence $\boldsymbol{G}$ is also a complete Heyting algebra. We shall not make explicit use of these facts in this paper.

Under various denominations, supernatural numbers are frequently used in the literature as classifiers ([15], [3], [8], [9, Chapter XIII, § 85]).

A supernatural number ν is said to be *finite* iff ∞ does not belong to the range of ν, and $\nu(x)$ is zero for all $x \in \boldsymbol{P}$, up to a finite number of possible exceptions. We denote by $\boldsymbol{G}_{\mathrm{fin}}$ the sublattice of $\boldsymbol{G}$ given by finite supernatural numbers. Regarding ν as a list of exponents for the sequence of prime numbers, the fundamental theorem of arithmetic allows us to identify each natural number n with a finite supernatural number ν_n. In this way, $\nu_1(p) = 0$ for all prime numbers p. Under the identification

$$n \leftrightarrow \nu_n \tag{2.1}$$

the ordered set $\boldsymbol{G}_{\mathrm{fin}}$ coincides with the distributive lattice $\boldsymbol{N}_{\mathrm{d}} = \{1, 2, 3, \dots\}$ of natural numbers equipped with divisibility.

A first property of supernatural numbers is that $\boldsymbol{G}$ provides a concrete realization of the lattice $\mathcal{I}(\boldsymbol{N}_{\mathrm{d}})$ of ideals of $\boldsymbol{N}_{\mathrm{d}}$:[6]

Lemma 2.3. *There is an order-isomorphism between $\mathcal{I}(\boldsymbol{N}_{\mathrm{d}})$ and $\boldsymbol{G}$. Under this isomorphism, natural numbers (strictly speaking, principal ideals of $\boldsymbol{N}_{\mathrm{d}}$) correspond to finite supernatural numbers.*

Proof. Routine (see [7] for details). □

A second main property of supernatural numbers is that they come equipped with a useful topology. As a matter of fact, recalling the identification (2.1), we endow $\boldsymbol{G}$ with the topology having as an open basis all sets of the form

$$U_n = \{\nu \in \boldsymbol{G}\colon \nu \geq \nu_n\}, \qquad n = 1, 2, 3, \dots$$

By abuse of notation, $\boldsymbol{G}$ shall also stand for the topological space of supernatural numbers.

Multisets

The following is a generalization of the notion of finite multiset:[7]

[6] Our visualization of each ideal of $\boldsymbol{N}_{\mathrm{d}}$ as a supernatural number follows a time-honored tradition of regarding "ideals" as "ideal numbers".

[7] While finite multisets are commonplace in mathematics [1], infinite multisets, and morphisms between multisets are not so frequently considered.

Definition 2.4. Let $\mathbb{S}$ denote the category of *Stone spaces* (i.e., boolean spaces, i.e., totally disconnected compact Hausdorff spaces) and continuous maps. Then the category $\mathbb{C}$ of *multisets* is defined as follows:

Objects: All pairs (S, σ) such that S is a Stone space and $\sigma\colon S \to \boldsymbol{G}$ is continuous.

Morphisms: Whenever (T, τ) and (U, υ) are objects in $\mathbb{C}$, then a morphism $\varphi\colon (T, \tau) \to (U, \upsilon)$ is a continuous function $\varphi\colon T \to U$ such that $\tau \geq \upsilon\varphi$.

Stone spaces form a full reflexive subcategory of multisets by identifying a Stone space S with the multiset (S, o), where $o(x) = \nu_1 \ \forall x \in S$, that is, $o(x)(p) = 0 \ \forall x \in S,\ p \in \mathbf{P}$. The left adjoint is given by the forgetful functor.

We denote by $\mathbb{M}$ the full subcategory of multisets given by *finite multisets*, i.e., those pairs (X, σ) where X is a discrete finite set and $\sigma\colon X \to \boldsymbol{N}_{\mathrm{d}}$ assigns a (finite super)natural number to each $x \in X$.

Under the identification 2.1 we have by definition:

Proposition 2.5. *A morphism $\varphi\colon (T, \tau) \to (U, \upsilon)$ in $\mathbb{M}$ is a multiplicity decreasing function, in the sense that for all $x \in T$, (the natural number corresponding to) $\upsilon\big(\varphi(x)\big)$ is a divisor of $\tau(x)$.*

Stone's duality can be extended to a duality between multisets and locally finite MV-algebras:

Theorem 2.6. *There is a natural equivalence between the opposite of the category $\mathbb{C}$ of multisets and the category $\mathbb{LFMV}$ of locally finite MV-algebras, and under this equivalence finite multisets correspond to finite MV-algebras.*

This equivalence extends Stone's duality between Stone spaces and boolean algebras in the sense that multisets of the form (S, o) correspond to boolean algebras.

Proof. A main result of [7]. □

It is relatively simple to construct inverse limits (in particular products) of multisets as well as of locally finite MV-algebras (see [7]). It follows then that the categories of locally finite MV-algebras and of multisets have all colimits (in particular, coproducts), a fact which is not immediate in either case. We remark that these colimits do not coincide with the constructions made in the larger categories of MV-algebras or topological spaces respectively.

From Theorem 2.6 (and Lemma 2.1 (iv)) it also follows immediately:

Corollary 2.7. *For every multiset $(X, \sigma) \in \mathbb{C}$ there is an inverse system*

$$\big\{(X_i, \sigma_i)\colon i \in J\big\}, \quad \big\{\phi_{ij}\colon (X_j, \sigma_j) \to (X_i, \sigma_i)\colon i, j \in J\,,\ \ j \geq i\big\}$$

of finite multisets (X_i, σ_i), indexed by an upward directed poset J, such that (X, σ) is isomorphic to the filtered limit ("inverse" limit, in universal algebraic terms) of this system.

3. Finite MV-algebras and finite multisets

In this section we construct an equivalence between the categories of finite MV-algebras and finite multisets. This equivalence is actually a restriction of the general duality between locally finite MV-algebras and multisets (Theorem 2.6 above) established in [7], but it is interesting to restrict attention to the finite case.

We begin by constructing a functor $\mathcal{M}$ from $\mathbb{M}^{\mathrm{op}}$, the opposite of the category of finite multisets, into $\mathbb{FMV}$, the category of finite MV-algebras and homomorphisms. As usual, for each $n = 2, 3, 4, \dots$, the MV-algebra $\boldsymbol{L}_n$ is defined as the subalgebra of $[0, 1]$

$$\boldsymbol{L}_n = \{0, 1/(n-1), 2/(n-1), \dots, (n-2)/(n-1), 1\}.$$

We shall tacitly and repeatedly use the well known facts that these are the only finite subalgebras of $[0, 1]$ and that $\boldsymbol{L}_n$ is embeddable in $\boldsymbol{L}_m$ iff $m-1$ is a multiple of $n-1$ (see, for instance, the remark following Proposition 3.5.3 in [6]). Furthermore, for $m-1$ a multiple of $n-1$ there is a *unique* homomorphism $\xi_{nm}\colon \boldsymbol{L}_n \to \boldsymbol{L}_m$, which is given by the stipulation

$$\xi_{nm}\left(\frac{j}{n-1}\right) = \frac{n-1}{m-1} \times \frac{j}{n-1}, \qquad 0 \le j \le n-1.$$

For each finite multiset (S, σ) we define $\mathcal{M}(S, \sigma)$ to be the finite MV-algebra

$$\mathcal{M}(S, \sigma) = \prod_{s \in S} \boldsymbol{L}_{\sigma(s)+1}.$$

Let $(S, \sigma) \xrightarrow{\varphi} (T, \tau)$ be a morphism in $\mathbb{M}$. To each element $f \in \mathcal{M}(T, \tau)$ we associate the element $\tilde{f} \in \mathcal{M}(S, \sigma)$ defined by

$$\tilde{f}(s) = \xi_{\tau(\varphi(s))+1\,\sigma(s)+1}\big(f(\varphi(s))\big), \qquad \text{for each } s \in S.$$

The correspondence $f \mapsto \tilde{f}$ is a homomorphism from $\mathcal{M}(T, \tau)$ into $\mathcal{M}(S, \sigma)$, and the stipulation

$$\mathcal{M}(\varphi)(f) = \tilde{f}, \qquad \text{for each } f \in \mathcal{M}(T, \tau),$$

yields a functor $\mathcal{M}$ from $\mathbb{M}^{\mathrm{op}}$ into $\mathbb{FMV}$.

We shall now construct a functor $\mathcal{N}$ from $\mathbb{FMV}$ to $\mathbb{M}^{\mathrm{op}}$. For each finite MV-algebra, let $\sigma_A\colon \mathcal{X}(A) \to \mathbf{N}$ be defined by stipulating that for each $\chi \in \mathcal{X}(A)$, $\sigma_A(\chi) = \#\big(\chi(A)\big) - 1$, where $\#\big(\chi(A)\big)$ denotes the cardinal of the finite simple algebra $\chi(A)$. Define $\mathcal{N}(A) = \big(\mathcal{X}(A), \sigma_A\big)$. Suppose that B is also a finite MV-algebra and that $h\colon A \to B$ is a homomorphism. Since for each $\chi \in \mathcal{X}(B)$, $\mathcal{X}(h)(\chi)(A) = \chi\big(h(A)\big) \subseteq \chi(B)$, we get that $\sigma_B(\chi)$ is a multiple of

$\sigma_A(\mathcal{X}(h)(\chi))$, for all $\chi \in \mathcal{X}(B)$. Therefore, by defining $\mathcal{N}(h) = \mathcal{X}(h)$ we obtain a functor from $\mathbb{FMV}$ to $\mathbb{M}^{\mathrm{op}}$.

Our next aim is to relate the functors $\mathcal{M}$ and $\mathcal{N}$.

For each finite multiset (S, σ), the evaluation $s \mapsto f(s)$ defines an isomorphism $\eta_\sigma \colon S \to \prod_{s \in S} \mathbf{L}_{\sigma(s)+1} = \mathcal{N}(\mathcal{M}((S,\sigma)))$. Moreover, for each morphism $(S,\sigma) \xrightarrow{\varphi} (T,\tau)$ in $\mathbb{M}$, the following diagram commutes:

$$\begin{tikzcd}
(S,\sigma) \arrow[r, "\varphi"] \arrow[d, "\eta_{(S,\sigma)}"'] & (T,\tau) \arrow[d, "\eta_{(T,\tau)}"] \\
\mathcal{N}(\mathcal{M}(S,\sigma)) \arrow[r, "\mathcal{N}(\mathcal{M}(h))"] & \mathcal{N}(\mathcal{M}(T,\tau)).
\end{tikzcd} \tag{3.2}$$

Let A be a finite MV-algebra. For each $a \in A$, let $\hat{a} \in \prod_{\chi \in \mathcal{X}(A)} \chi(A)$ be defined by $\hat{a}(\chi) = \chi(a)$. Writing $\mu_A(a) = \hat{a}$ for each $a \in A$, we get an isomorphism μ_A from A onto $\mathcal{M}(\mathcal{N}(A))$. For each homomorphism $h \colon A \to B$ we have the following commutative diagram:

$$\begin{tikzcd}
A \arrow[r, "h"] \arrow[d, "\mu_A"'] & B \arrow[d, "\mu_B"] \\
\mathcal{M}(\mathcal{N}(A)) \arrow[r, "\mathcal{M}(\mathcal{N}(h))"] & \mathcal{M}(\mathcal{N}(B)).
\end{tikzcd} \tag{3.3}$$

From (3.2) and (3.3) it follows that the functors $\mathcal{N}M$ and $\mathcal{M}N$ are naturally equivalent to the identity functors of $\mathbb{M}^{\mathrm{op}}$ and of $\mathbb{FMV}$, respectively. Summing up we have

Theorem 3.1. *The quadruple $\langle \mathcal{M}, \mathcal{N}, \eta, \mu \rangle$ defines a natural equivalence between the categories $\mathbb{M}^{\mathrm{op}}$ and $\mathbb{FMV}$.*

4. Finite boolean algebras with an automorphism, and finite multisets

For each finite boolean algebra A, we can safely identify its Stone space $\mathcal{X}(A)$ with the set of atoms of A, endowed with the discrete topology. Under this identification, $\mathcal{X}(A)$ is closed under every automorphism of A, and each automorphism β of A is uniquely determined by its restriction to $\mathcal{X}(A)$. Specifically, for each $a \in A$, we can write

$$\beta(a) = \bigvee_{\substack{x \in \mathcal{X}(A) \\ x \leq a}} \beta(x). \tag{4.4}$$

From $\mathbb{FBA}\delta$ to $\mathbb{M}^{\mathrm{op}}$.

For each finite Boolean algebra B and each automorphism β of B, let us define the binary relation $\equiv_\beta$ on $\mathcal{X}(B)$ by the stipulation $x \equiv_\beta y$ iff there is $\kappa \geq 1$ such that $y = \beta^\kappa(x)$. Then $\equiv_\beta$ is an equivalence relation, and the equivalence class of an element $x \in \mathcal{X}(B)$ is its orbit.

For each $s \in \mathcal{X}(B)/\equiv_\beta$, letting $\sigma_\beta(s)$ denote the number of elements of the orbit s, it follows that $(\mathcal{X}(B)/\equiv_\beta, \sigma_\beta)$ is a finite multiset, i.e., an object in the category $\mathbb{M}$.

Let now (A,α) and (B,β) be objects in $\mathbb{FBA}\delta$, and $h\colon (A,\alpha) \to (B,\beta)$ be a morphism. The boolean homomorphism $h\colon A \to B$ determines a map $\mathcal{X}(h)\colon \mathcal{X}(B) \to \mathcal{X}(A)$. Explicitly, recalling our identification between maximal ideals and atoms, $\mathcal{X}(h)$ is defined as follows, for each $y \in \mathcal{X}(B)$ and each $a \in A$:

$$\mathcal{X}(h)(y) \leq a \quad \text{iff} \quad y \leq h(a).$$

For each $y \in \mathcal{X}(B)$ and $a \in A$ we then have $\mathcal{X}(h)\big(\beta(y)\big) \leq a$ iff $\beta(y) \leq h(a)$ iff $y \leq \beta^{-1}\big(h(a)\big) = h\big(\alpha^{-1}(a)\big)$ iff $\mathcal{X}(h)(y) \leq \alpha^{-1}(a)$ iff $\alpha\big(\mathcal{X}(h)(y)\big) \leq a$. Therefore, $\mathcal{X}(h)\big(\beta(y)\big) = \alpha\big(\mathcal{X}(h)(y)\big)$ for all $y \in \mathcal{X}(B)$. Further, $\mathcal{X}(h)\big(\mathrm{orb}_\beta(y)\big) = \mathrm{orb}_\alpha\big(\mathcal{X}(h)(y)\big)$. From $\alpha^{\kappa_y}\big(\mathcal{X}(h)(y)\big) = \mathcal{X}(h)\big(\beta^{\kappa_y}(y)\big) = \mathcal{X}(h)(y)$ it follows that the number of elements in $\mathrm{orb}_\beta(y)$ must be a multiple of the number of elements of $\mathrm{orb}_\alpha\big(\mathcal{X}(h)(y)\big)$. Thus, by Proposition 2.5, the stipulation $\breve{h}\big(\mathrm{orb}_\beta(y)\big) = \mathrm{orb}_\alpha\big(\mathcal{X}(h)(y)\big)$ defines a morphism

$$\breve{h}\colon \big(\mathcal{X}(B)/{\equiv_\beta},\, \sigma_{(B,\beta)}\big) \to \big(\mathcal{X}(A)/{\equiv_\alpha},\, \sigma_{(A,\alpha)}\big)$$

in the category $\mathbb{M}$. Summing up, for each $(A,\alpha) \in \mathbb{FBA}\delta$, let us define

$$\mathcal{C}(A,\alpha) = \big(\mathcal{X}(A)/{\equiv_\alpha},\, \sigma_\alpha\big)\,,$$

and for each morphism $h\colon (A,\alpha) \to (B,\beta)$ in $\mathbb{FBA}\delta$ let us write

$$\mathcal{C}(h) = \breve{h}\,.$$

We then have

Proposition 4.1. *$\mathcal{C}$ is a functor from $\mathbb{FBA}\delta$ to $\mathbb{M}^{\mathrm{op}}$, the opposite of the category of finite multisets.*

From $\mathbb{M}^{\mathrm{op}}$ to $\mathbb{FBA}\delta$

Let $(S,\sigma) \in \mathbb{M}$ be a finite multiset. For each $s \in S$, let $X_s = \{1, 2, \ldots, \sigma(s)\}$, and let $\pi_s\colon X_s \to X_s$ be the cyclic permutation $1 \mapsto 2 \mapsto \cdots \mapsto \sigma(s) \mapsto 1$. Define X_σ as the disjoint union of the sets X_s, and define $\pi_\sigma\colon X_\sigma \to X_\sigma$ in such a way that the restriction of π_σ to X_t coincides with π_t. Let A_σ be the (finite) boolean algebra of all subsets of X_σ and let α_σ be the automorphism of A_σ determined by π_σ. Let us define

$$\mathcal{B}(S,\sigma) = (A_\sigma, \alpha_\sigma)\,. \tag{4.5}$$

To complete the definition of functor $\mathcal{B}$, suppose that $(T,\tau) \xrightarrow{\varphi} (S,\sigma)$ is a morphism in $\mathbb{M}$. For each $t \in T$, let $f_t\colon X_t \to X_{\varphi(t)}$ be defined by the stipulation $f_t(\pi_t^i(1)) = \pi_{\varphi(t)}^i(1)$. Since $\tau(t)$ is a multiple of $\sigma(\varphi(t))$, f_t is well defined, and there is a unique function $f\colon X_\tau \to X_\sigma$ such that the restriction of f to X_t coincides with f_t. Since for each $z \in X_\tau$, $f(\pi_\tau(z)) = \pi_\sigma(f(z))$, f determines a unique homomorphism $h\colon \mathcal{B}(\sigma) \to \mathcal{B}(\tau)$. Let us define

$$\mathcal{B}(\varphi) = h\,.$$

Then, recalling (4.5) we have

PROPOSITION 4.2. *$\mathcal{B}$ is a functor from $\mathbb{M}^{\mathrm{op}}$ to $\mathbb{FBA}\delta$.*

We shall now relate the two functors $\mathcal{B}$ and $\mathcal{C}$. Observe that $\mathcal{C}(A_\sigma,\alpha_\sigma) = (X_\sigma/{\equiv_{\alpha_\sigma}}, \sigma_{\alpha_\sigma})$. Let $\rho\colon X_\sigma \to X_\sigma/{\equiv_{\alpha_\sigma}}$ be the natural projection, and let $f\colon X_\sigma \to S$ be the function defined by the stipulation $f(z) = s$ iff $z \in X_s$. Then there is a unique bijection $\nu_\sigma\colon X_\sigma/{\equiv_{\alpha_\sigma}} \to S$ such that $f(z) = \nu_\sigma(\rho(z))$, for each $z \in X_\sigma$. Moreover, we have the identity $\sigma(\nu_\sigma(u)) = \sigma_{\alpha_\sigma}(u)$ for each $u \in X_\sigma/{\equiv_{\alpha_\sigma}}$. Therefore, ν_σ is an isomorphism from (S,σ) onto $\mathcal{C}(\mathcal{B}(S,\sigma))$. For each morphism $\varphi\colon (S,\sigma)(T,\tau)$, we also have the commutative diagram

$$\begin{tikzcd}
(S,\sigma) \arrow[r, "\varphi"] \arrow[d, "\nu_{(S,\sigma)}"'] & (T,\tau) \arrow[d, "\nu_{(T,\tau)}"] \\
\mathcal{C}(\mathcal{B}(S\sigma)) \arrow[r, "\mathcal{C}(\mathcal{B}(\varphi))"] & \mathcal{C}(\mathcal{B}(T,\tau))\,.
\end{tikzcd} \tag{4.6}$$

We have proved:

THEOREM 4.3. *The family ν of isomorphisms defines a natural equivalence from $\mathcal{I}_{\mathbb{M}^{\mathrm{op}}}$, the identity functor of $\mathbb{M}^{\mathrm{op}}$, to $\mathcal{CB}$.*

COROLLARY 4.4. *The functor $\mathcal{C}$ is full, and each finite multiset (S,σ) is isomorphic to $\mathcal{C}(A,\alpha)$ for some object (A,α) in $\mathbb{FBA}\delta$. The functor $\mathcal{B}$ is faithful.*

P r o o f. As a consequence of Theorem 4.3 the following diagram commutes:

$$\begin{tikzcd}
\big[(S,\sigma),(T,\tau)\big] \arrow[rr, "B"] \arrow[dr, "\nu_*\circ\nu^{-1}"'] & & \big[\mathcal{B}(S,\sigma),\mathcal{B}(T,\tau)\big] \\
& \big[\mathcal{CB}(S,\sigma),\mathcal{CB}(T,\tau)\big]\,. \arrow[ur, "\mathcal{C}"'] &
\end{tikzcd}$$

Here, square brackets denote the set of morphisms in the appropriate categories, and $\nu_* \circ \nu^{-1*}$ is the bijection defined by $\varphi \mapsto \nu\varphi\nu^{-1}$. It follows that $\mathcal{B}$ is injective and $\mathcal{C}$ is surjective. That is, $\mathcal{B}$ is full and $\mathcal{C}$ is faithful. The rest follows at once from Theorem 4.3 $\square$

Remark 4.5. It is not difficult to see that each object (A, α) in $\mathbb{FBA}\delta$ is isomorphic to $\mathcal{B}(\mathcal{C}(A, \alpha))$. But $\mathcal{BC}$ is not naturally equivalent to $\mathcal{I}_{\mathbb{FBA}\delta}$. Specifically, $\mathcal{C}$ is not faithful and $\mathcal{B}$ is not full. Indeed, by way of contradiction, suppose there exists a natural equivalence from $\mathcal{BC}$ to $\mathcal{I}_{\mathbb{FBA}\delta}$. From Proposition 4.3, it would follow that the pair $(\mathcal{C}, \mathcal{B})$ is an equivalence of categories, whence in particular, $\mathcal{C}$ would preserve coproducts. That this is not the case, is shown by the following:

Counter-example: Let B be a finite boolean algebra with two atoms, let α be the non-trivial automorphism of B, and (C, β) be the coproduct $(B, \alpha) \coprod (B, \alpha)$ in $\mathbb{FBA}\delta$. Then $S(B, \alpha)$ is a singleton, say $\{s\}$, and $\sigma_{(S,\alpha)}(s) = 2$. By contrast, $S(C, \beta)$ has two distinct elements, and $\sigma_{(C,\beta)}$ is the constant function 2. As a matter of fact, coproducts in $\mathbb{M}^{\mathrm{op}}$ are the same as products in $\mathbb{M}$; the latter exist by Corollary 2.7. A straightforward calculation yields $\mathcal{C}(B, \alpha) \times \mathcal{C}(B, \alpha) = \mathcal{C}(B, \alpha) \neq \mathcal{C}(C, \beta)$.

5. Classification: The general case

From Theorems 3.1 and 4.3 and Corollary 4.4 it follows that the composite map $\mathcal{BN}$ is a faithful functor from $\mathbb{FMV}$ to $\mathbb{FBA}\delta$, and $\mathcal{MC}$ is a full functor from $\mathbb{FBA}\delta$ to $\mathbb{FMV}$. Our aim in this section is to extend these functors to the corresponding categories of locally finite algebras. We shall thus apply the theory of Ind-objects developed in the Appendix together with the duality established in [7]. For terminology and notation the reader is referred to the Appendix.

Recall first that a variety (or equational class) $\mathbf{V}$ is said to be *congruence-distributive* provided that the congruence lattice of each algebra in $\mathbf{V}$ is distributive. A well known result of B a k e r [4, 5] asserts that a finite algebra A in a congruence-distributive variety $\mathbf{V}$ is finitely axiomatizable. This means that A is characterized by a finite set of equations $\sigma_1 = \tau_1, \dots, \sigma_k = \tau_k$, where σ_i, τ_i are terms in the language of $\mathbf{V}$ for $1 \leq i \leq k$. Suppose that A has n generators, and let $F(k)$ and $F(n)$ be the free algebras in $\mathbf{V}$ with k and n generators, respectively. Define homomorphisms $f\colon F(k) \to F(n)$ and $g\colon F(k) \to F(n)$ by the prescriptions $f(y_i) = \sigma_i^{F(n)}$ and $g(y_i) = \tau_i^{F(n)}$, where $y_1, \dots, y_k$ are free generators of $F(k)$, and $\sigma_i^{F(n)}$ and $\tau_i^{F(n)}$ denote the evaluations of the terms σ_i and τ_i in $F(n)$, for $1 \leq i \leq k$. If $h\colon F(n) \to A$ is the surjective homomorphism obtained by sending the generators of $F(n)$ onto the generators of A, it follows that h is a coequalizer of the pair f, g, and hence A is finitely presented according to the definition given in 6.7. Summing up, we have that *finite algebras in congruence-distributive varieties are finitely presented.*

Proposition 5.1. *The inclusion* $\mathbb{FMV} \overset{i}{\hookrightarrow} \mathbb{LFMV}$ *determines an equivalence* $\mathcal{U}$ *of categories:*

$$\begin{tikzcd} \mathbb{FMV} \arrow[r, hook, "h"] \arrow[dr, hook, "i"'] & \mathrm{Ind}(\mathbb{FMV}) \arrow[d, "\mathcal{U}"] \\ & \mathbb{LFMV}\,, \end{tikzcd} \qquad \mathcal{U} \circ h = i\,.$$

P r o o f . Since MV-algebras admit a lattice structure, it follows from [5, Theorem 12.3] that they form a congruence-distributive variety. Hence finite MV-algebras are finitely presented, and the proof follows from Lemma 2.1 together with Example 6.7. ☐

Proposition 5.2. *The inclusion* $\mathbb{M} \overset{i}{\hookrightarrow} \mathbb{C}$ *determines an equivalence* $\mathcal{V}$ *of categories:*

$$\begin{tikzcd} \mathbb{M} \arrow[r, hook, "h"] \arrow[dr, hook, "i"'] & \mathrm{Pro}(\mathbb{M}) \arrow[d, "\mathcal{V}"] \\ & \mathbb{C}\,, \end{tikzcd} \qquad \mathcal{V} \circ h = i\,,$$

where $\mathrm{Pro}(\mathbb{M}) := \big(\mathrm{Ind}(\mathbb{M}^{\mathrm{op}})\big)^{\mathrm{op}}$.

P r o o f . It follows by the above proposition, jointly with Theorems 3.1 and 2.6. ☐

This proposition (together with 2.7) means that multisets are *free* filtered inverse limits of finite multisets (that is, the category $\mathbb{C}$ is the *filtered colimit completion* of $\mathbb{M}$). In particular, it follows that Stone spaces are free filtered inverse limits of finite sets.

Proposition 5.3. *The inclusion* $\mathbb{FBV}\delta \overset{i}{\hookrightarrow} \mathbb{LFBA}\delta$ *determines an equivalence* $\mathcal{W}$ *of categories:*

$$\begin{tikzcd} \mathbb{FBA}\delta \arrow[r, hook, "h"] \arrow[dr, hook, "i"'] & \mathrm{Ind}(\mathbb{FBA}\delta) \arrow[d, "\mathcal{W}"] \\ & \mathbb{LFBA}\delta\,, \end{tikzcd} \qquad \mathcal{W} \circ h = i\,.$$

P r o o f . As remarked in Footnote 2, boolean algebras with a distinguished automorphism become a variety $\mathbf{W}$ if we add the inverse automorphism as a new unary operation. By [5. Theorem 12.3], $\mathbf{W}$ is a congruence-distributive variety. Hence finite boolean algebras with a distinguished automorphism are boolean algebras with a distinguished automorphism which are finitely presented. Then the proof follows by Corollary 1.3 and Example 6.7. $\square$

Consider the functors $\mathcal{B}$ and $\mathcal{C}$ defined in Propositions 4.2 and 4.1. We have the following commutative diagram:

$$\begin{tikzcd}
\mathbb{M}^{\mathrm{op}} \arrow[r, "h"] \arrow[d, "\mathcal{B}"] & \mathrm{Ind}(\mathbb{M}^{\mathrm{op}}) \arrow[r, "\mathcal{V}"] \arrow[d, "\mathrm{Ind}(\mathcal{B})"] & \mathbb{C}^{\mathrm{op}} \arrow[d, "\mathcal{G}"] \\
\mathbb{FBA}\delta \arrow[r, "h"] \arrow[d, "\mathcal{C}"] & \mathrm{Ind}(\mathbb{FBA}\delta) \arrow[r, "\mathcal{W}"] \arrow[d, "\mathrm{Ind}(\mathcal{C})"] & \mathbb{LFBA}\delta \arrow[d, "\mathcal{F}"] \\
\mathbb{M}^{\mathrm{op}} \arrow[r, "h"] & \mathrm{Ind}(\mathbb{M}^{\mathrm{op}}) \arrow[r, "\mathcal{V}"] & \mathbb{C}^{\mathrm{op}} .
\end{tikzcd}$$

Here the functors $\mathrm{Ind}(\mathcal{B})$ and $\mathrm{Ind}(\mathcal{C})$ are given by Corollary 6.5, and the functors $\mathcal{G}$ and $\mathcal{F}$ are determined by the fact that $\mathcal{V}$ and $\mathcal{W}$ are equivalences of categories.

From the natural equivalence $\mathcal{C} \circ \mathcal{B} \cong \mathrm{id}$ given by Theorem 4.3 it follows that there is a natural equivalence $\mathrm{Ind}(\mathcal{C}) \circ \mathrm{Ind}(\mathcal{B}) \cong \mathrm{id}$, which in turn induces a natural equivalence $\mathcal{G} \circ \mathcal{F} \cong \mathrm{id}$.

Summing up, we have proved

Theorem 5.4. *There are functors $\mathcal{G}$ from $\mathbb{C}^{\mathrm{op}}$ to $\mathbb{LFBA}\delta$, and $\mathcal{F}$ from $\mathbb{LFBA}\delta$ to $\mathbb{C}^{\mathrm{op}}$ such that the composite functor $\mathcal{FG}$ is naturally equivalent to the identity functor of $\mathbb{C}^{\mathrm{op}}$.*

From this theorem, in the light of the duality between $\mathbb{C}^{\mathrm{op}}$ and $\mathbb{LFMV}$ established in [7] we get (where, by abuse of notation, functors maintain the same denotations):

Theorem 5.5. *There are functors $\mathcal{G}$ from $\mathbb{LFMV}$ to $\mathbb{LFBA}\delta$, and $\mathcal{F}$ from $\mathbb{LFBA}\delta$ to $\mathbb{LFMV}$, such that the composite functor $\mathcal{FG}$ is naturally equivalent to the identity functor of $\mathbb{LFMV}$.*

Notice that we have given explicit constructions for all the functors in the above two theorems.

Our promised classification is achieved in the following

Corollary 5.6. *The functor $\mathcal{G}$ is faithful, the functor $\mathcal{F}$ is full, and each locally finite MV-algebra A is isomorphic to $\mathcal{F}(B,\beta)$ for some object (B,β) in $\mathbb{LFBA}\delta$.*

EXAMPLE 5.7. The following example shows how our duality can be applied to prove non-isomorphism results for boolean algebras with a distinguished locally finite automorphism. Let p be a prime number. For each $n \geq 1$ let $X_{p,n} = \{1, \ldots, p^n\}$ and let $\pi_{p,n} \colon X_{p,n} \to X_{p,n}$ be defined by the stipulation:

$$\pi_{p,n}(k) = \begin{cases} k+1 & \text{if } 1 \leq k \leq p^n - 1, \\ 1 & \text{if } k = p^n. \end{cases}$$

For each $1 \leq m \leq n$, let $f_{m,n} \colon X_{p,n} \to X_{p,m}$ be defined by $f_{mn}(k) = [k]_{p^m}$ for each $1 \leq k \leq p^n$, where $[k]_q$ denotes the least integer ≥ 1 which is congruent to k modulo q. Denote by $A_{p,n}$ the boolean algebra of all subsets of $X_{p,n}$, and by $\alpha_{p,n}$ the automorphism of $A_{p,n}$ determined by $\pi_{p,n}$. Since the functions $f_{m,n}$ are surjective, they define embeddings $h_{m,n} \colon (A_{p,m}, \alpha_{p,m}) \to (A_{p,n}, \alpha_{p,n})$ for each $1 \leq m \leq n$. Let

$$(A_p, \alpha_p) = \lim_{n \geq 1} (A_{p,n}, \alpha_{p,n}).$$

Since $\mathcal{F}(A_{p,n}, \alpha_{p,n}) = \boldsymbol{L}_{p^n+1}$, for each $n \geq 1$, it follows that $\mathcal{F}(A_p, \alpha_p)$ is the MV-algebra M_p formed by the fractions of the form $\frac{j}{p^k}$ with $0 \leq j \leq p^k$, $k = 1, 2, \ldots$.

Each algebra (A_p, α_p) is conveniently described by means of its dual space. The latter is a pair (X_p, π_p), where X_p is a Stone space and $\pi_p \colon X_p \to X_p$ is a surjective homeomorphism. For each p, the space X_p is the limit of the inverse system formed by the sets $X_{p,n}$ endowed with the discrete topology, together with the maps $f_{m,n}$. Hence X_p is the set of all sequences $x \colon \boldsymbol{N} \to \boldsymbol{N}$ satisfying the following conditions, for each $n \geq 1$:

$$1 \leq x(n) \leq p^n$$

and

$$x(n+1) \equiv x(n) \pmod{p^n},$$

endowed with the topology inherited from the product $\prod_{n \geq 1} X_{p,n}$. The homeomorphism π_p is defined by the formula

$$\big(\pi_p(x)\big)(n) = \pi_{p,n}\big(x(n)\big), \quad \text{for each} \quad x \in X_p, \quad \text{and} \quad n \geq 1.$$

The elements of A_p are the clopen subsets of X_p, and for each $a \in A_p$, $\alpha_p(a) = \pi_p^{-1}(a)$. Hence the elements of A can be represented in the form

$$a = \{x \in X_p \colon x(n_1) = k_1, \ldots, x(n_r) = k_r\},$$

where $1 \leq k_i \leq p^{n_i}$, $1 \leq i \leq r$, and whenever $m = \max(n_1, \ldots, n_r)$ then $\alpha_p^{m+1}(a) = a$.

If the primes p and q are distinct, then the MV-algebras M_p and M_q are not isomorphic [6, Corollary 7.2.6]. Therefore the algebras (A_p, α_p) and (A_q, α_q) are not isomorphic.

6. Appendix: Ind-objects

The purpose of this appendix is to fix notation and terminology, and to make some constructions and results more accessible to those readers who are not familiar with category theory. We shall sketch the basics of Grothendieck's theory of Ind-Objects which is used in this paper. Our standard reference is [2, Ex I, 8].

We begin by recalling some routine material about limits indexed by a category J. A special case is when J is a poset (that is, given any $i, j \in J$, there is a unique arrow $i \to j$ when $i \leq j$, and none otherwise). Skipping some technicalities, we shall treat the index category J as if it were a poset, and, unless strictly necessary, we shall not label the arrows $i \to j$. Accordingly, the reader may read $i \to j$ simply as notation for $i \leq j$.

Given a category $\mathbb{C}$, a (*direct*) *system* in $\mathbb{C}$ indexed by J is a family $(C_i)_{i\in J}$ together with a *transition morphism* $C_i \to C_j$, whenever $i \to j$ in J. These data are subject to the usual equations expressing that the correspondence $i \mapsto C_i$ is a functor $J \to \mathbb{C}$.

A *cone* in $\mathbb{C}$ is a family of morphisms $C_i \xrightarrow{\varphi_i} C$, $i \in J$, such that for all $i \to j$ the following diagram is commutative:

$$\begin{tikzcd} & & C \\ C_i \arrow[r] \arrow[urr, "\varphi_i"] & C_j \arrow[ur, "\varphi_j"'] & \end{tikzcd} .$$

A *colimit* (also known as *direct limit*) of the (direct) system in $\mathbb{C}$, denoted $L = \mathrm{colim}_{i\in J} C_i$, is a universal cone. That is, it is a cone, given by *"injections"* $C_i \xrightarrow{\lambda_i} L$, such that for any cone $C_i \xrightarrow{\varphi_i} C$, there exists a unique morphism $L \xrightarrow{\psi} C$ compatible with the cones, in the sense that the following diagram is commutative:

$$\begin{tikzcd} & L \arrow[r, "\psi"] & C \\ C_i \arrow[ur, "\lambda_i"] \arrow[urr, "\varphi_i"'] & & \end{tikzcd} .$$

An (*inverse*) *system* in $\mathbb{C}$ is a contravariant functor $J^{\mathrm{op}} \to \mathbb{C}$ (that is, given $i \to j$ in J, the transition morphism goes in the opposite direction $C_i \leftarrow C_j$). By reversing the arrows in the above definition, we obtain the definition of *limit* (also known as *inverse limit*) of the (inverse) system, denoted $L = \lim_{i\in J} C_i$, the cone being given by *"projections"* $L \xrightarrow{\pi_i} C_i$.

The reader should be aware that these definitions are largely matter of convention. We could, e.g., also consider colimits of inverse systems, or limits of direct systems. As a matter of fact, there is only one concept, which expresses itself in different forms by taking the dual categories in the appropriate places.

When the category J is discrete, limits indexed by J are called products, and denoted $L = \prod_i C_i$. When J is a *small* category, in the category of sets products are given by the usual set of J-tuples. For arbitrary small J, the limit is given by the subset of those J-tuples, $(x_i)_{i\in J}$, $x_i \in C_i$, consistent with the transition morphisms. That is, such that for all arrows $i \xrightarrow{u} j$ in J, $C(u)(x_j) = x_i$.

When the category J is discrete, colimits indexed by J are called *coproducts*, and denoted $L = \coprod_i C_i$. When J is a *small* category, in the category of sets coproducts are given by the disjoint union:

$$L = \coprod_{i\in J} C_i = \{(x, i) \colon x \in C_i,\ i \in J\}.$$

The fundamental concept of *germ* essentially consists in the construction of filtered colimits in the category of sets:

A nonempty category J is *filtered* if it satisfies the following two conditions:

F1) Given any two $i, j \in J$, there is $k \in J$ together with arrows $i \to k$, $j \to k$.

F2) Given any two parallel arrows $i \underset{v}{\overset{u}{\rightrightarrows}} j$, there is $k \in J$ and an arrow $j \xrightarrow{w} k$ such that $wu = wv$.

Equivalently, in diagram form we can write

$$F1) \quad \begin{tikzcd} i \arrow[dr, dashed] & \\ & k, \\ j \arrow[ur, dashed] & \end{tikzcd} \qquad F2) \quad \begin{tikzcd} i \arrow[r, shift left] \arrow[r, shift right] & j \arrow[r, dashed] & k. \end{tikzcd}$$

When J is a poset the second condition is redundant, and we have the usual notion of *directed* poset.

A system is called *filtered* when the index category is filtered. We have:

Proposition 6.1. *Given any filtered system of sets C_i, $i \in J$, the colimit $L = \mathrm{colim}_i\, C_i$, is given by the following construction: First take the disjoint union of the C_i, and then take the quotient by the following equivalence relation:*

$$(x, i) \equiv (y, j) \iff \exists\, k,\ i \xrightarrow{u} k,\ j \xrightarrow{v} k,\ \textit{such that } C(u)(x) = C(v)(y) \textit{ in } C_k.$$

P r o o f . The proof is straightforward but interesting, because it shows why and where the two conditions in the definition of filtered category are necessary. □

We shall adopt the following terminology: Whenever the elements of the $C_i^{\prime s}$ are, say, *jokers*, then the elements of the filtered colimit L shall be called *germs of jokers*.

Given a germ $\xi \in L$ and a representative of its equivalence class (x, i), by a traditional abuse of notation we shall write $\xi = (x, i)$. By definition, elements $x \in C_i$ and $y \in C_j$ determine the same germ iff they eventually coincide in some C_k.

We now briefly discuss Grothendieck's theory of Ind-Objects. Given a category $\mathbb{C}$, the category $\mathrm{Ind}(\mathbb{C})$ is a *syntactical construction* that freely adds to $\mathbb{C}$ colimits of small filtered systems. With the appropriate set-theoretical care the smallness assumption is not necessary in general (see [2]).

Definition 6.2. ([2], Ex I, 8.2.1.2 and 8.2.5.1) Given any category $\mathbb{C}$, an *Ind-Object* of $\mathbb{C}$ is a small filtered (direct) system $X = (C_i)_{i \in J}$. Ind-Objects form a category, denoted $\mathrm{Ind}(\mathbb{C})$. Given any other Ind-Object $Y = (D_\alpha)_{\alpha \in \Gamma}$, the set of morphisms in $\mathrm{Ind}(\mathbb{C})$ is defined by the following formula:

$$\hom(X,\, Y) = \lim_i \ \mathrm{colim}_\alpha \hom(C_i,\, D_\alpha)$$

(notice that because the hom functor is contravariant in the first variable and covariant in the second, the system of sets $\hom(C_i, D_\alpha)$ is a direct system in $\alpha \in \Gamma$ and an inverse system in $i \in J$).

In the following proposition we give a useful description of the morphisms of Ind-Objects. The equivalence between this description and the one in Definition 6.2 follows by the above construction of inverse limits and of filtered colimits of sets.

Proposition 6.3. *Given two Ind-Objects in $\mathbb{C}$, $X = (C_i)_{i \in J}$, $Y = (D_\alpha)_{\alpha \in \Gamma}$, a morphism $X \to Y$ in $\mathrm{Ind}(\mathbb{C})$ is a pair $(\varphi, (f_i)_{i \in J})$, where $J \xrightarrow{\varphi} \Gamma$ is a function between the objects of J and Γ, and the f_i are morphisms $C_i \xrightarrow{f_i} D_{\varphi(i)}$ in $\mathbb{C}$, such that the following condition is satisfied:*

a) *For all $i \to j$ in J, there is $\alpha \in \Gamma$ and arrows $\varphi(i) \to \alpha$, $\varphi(j) \to \alpha$, such that the following diagram commutes:*

$$\begin{tikzcd}
D_{\varphi(i)} \arrow[rr, bend left] & D_{\varphi(j)} \arrow[r] & D_\alpha \arrow[r] & \cdots \\
C_i \arrow[u, "f_i"] \arrow[r] & C_j \arrow[u, "f_j"] \arrow[r] & \cdots &
\end{tikzcd}$$

Furthermore, two morphisms $(\varphi, (f_i)_{i\in J})$, $(\psi, (g_i)_{i\in J})$ *are considered equal if:*

b) *For all* $i \in J$, *there is* $\alpha \in \Gamma$ *and arrows* $\varphi(i) \to \alpha$, $\psi(i) \to \alpha$, *such that the following diagram commutes:*

$$\begin{tikzcd} & D_{\varphi(i)} \arrow[dr] & \\ C_i \arrow[ur, "f_i"] \arrow[dr, "g_i"'] & & D_\alpha \\ & D_{\psi(i)} \arrow[ur] & \end{tikzcd} .$$

Notice that φ *is not assumed to be a functor. Thus, from* $i \to j$ *one need not have* $\varphi(i) \to \varphi(j)$.

P r o o f . Again, the proof is straightforward and interesting, showing how the two conditions a) and b) work in harmony with Proposition 6.1 and Definition 6.2 □

Intuitively, Condition a) means that a morphism is induced between the colimits of the systems, whenever these colimits exist. Condition b) means that the induced morphisms would be equal. In particular, if two systems are isomorphic in $\mathrm{Ind}(\mathbb{C})$, then they will be isomorphic wherever their colimits exist. These considerations explain why the extension defined in (UP) below is well defined and functorial.

It can be checked that the category $\mathrm{Ind}(\mathbb{C})$ has all small filtered colimits ([2] Ex I 8.5.1).

Any object of $\mathbb{C}$ can be considered as an Ind-Object, upon letting the index category be a singleton. This defines a canonical full and faithful functor $h\colon \mathbb{C} \to \mathrm{Ind}(\mathbb{C})$ which is injective on objects. Let us identify $\mathbb{C}$ with a full subcategory of $\mathrm{Ind}(\mathbb{C})$. Given $C \in \mathbb{C}$, let write $h(C) = C$, by abuse of notation.

Given any Ind-Object $X = (C_i)_{i\in J}$, the following formula holds in $\mathrm{Ind}(\mathbb{C})$:

$$X = \mathrm{colim}_i\, C_i\,.$$

That is, the system itself coincides with its own colimit in $\mathrm{Ind}(\mathbb{C})$. This means that by taking the colimit in $\mathrm{Ind}(\mathbb{C})$ one does not loose any information built in the system. Unsurprisingly enough, the map $h\colon \mathbb{C} \to \mathrm{Ind}(\mathbb{C})$ has the following universal property:

(UP). Given any functor $\mathbb{C} \xrightarrow{f} \mathbb{E}$ into a category having colimits of the form $\mathrm{colim}_i\, f(C_i)$ for any small filtered system in $\mathbb{C}$, there is an extension (which preserves filtered colimits) $\mathrm{Ind}(\mathbb{C}) \xrightarrow{\overline{f}} \mathbb{E}$. In symbols,

$$\begin{tikzcd} \mathbb{C} \arrow[r, "h"] \arrow[dr, "f"'] & \mathrm{Ind}(\mathbb{C}) \arrow[d, "\overline{f}"] \arrow[dl, phantom, "\cong"] \\ & \mathbb{E}\,, \end{tikzcd} \qquad\qquad \overline{f} \circ h \cong f\,.$$

This extension, is *well defined* by the formula $\overline{f}(X) = \operatorname{colim}_i f(C_i)$, and actually makes the above diagram (strictly) commutative. Recalling Proposition 6.3, one promptly checks that $\overline{f}$ is definable on morphisms of Ind-Objects in a functorial way. While the extension is not unique, it is characterized by the equation $\overline{f} \circ h \cong f$, up to a unique natural isomorphism. As a matter of fact, all 2-categorical aspects are completely taken care of by the following:

Proposition 6.4. ([2], Ex I, 8.7.3) *Let $\mathbb{E}$ be any category with small filtered colimits. Then composition with the functor $h\colon \mathbb{C} \to \operatorname{Ind}(\mathbb{C})$ induces an equivalence of categories:*

$$\operatorname{HOM}(\operatorname{Ind}(\mathbb{C}),\, \mathbb{E})_+ \xrightarrow{\cong} \operatorname{HOM}(\mathbb{C},\, \mathbb{E})\,, \qquad g \mapsto g \circ h\,.$$

A quasi-inverse for this equivalence is given by the assignment $f \mapsto \overline{f}$.

It follows that the category $\operatorname{Ind}(\mathbb{C})$ is characterized by this universal property up to equivalence of categories (and not up to isomorphism of categories).

(Here HOM stands for the category of functors and natural transformations, and "+" stands for the full subcategory of those functors that preserve filtered colimits).

Corollary 6.5. ([2], Ex I, 8.6.3) *Given any functor $F\colon \mathbb{C} \to \mathbb{D}$, the formula $\operatorname{Ind}(F)(C_i)_{i\in J} = (FC_i)_{i\in J}$ defines a filtered colimit preserving functor $\operatorname{Ind}(F)\colon \operatorname{Ind}(\mathbb{C}) \to \operatorname{Ind}(\mathbb{D})$, commuting with the canonical functors h, $h \circ F = \operatorname{Ind}(F) \circ h$.*

We have $\operatorname{Ind}(\mathrm{id}_{\mathbb{C}}) = \mathrm{id}_{\operatorname{Ind}(\mathbb{C})}$, and for any composition $F \circ G$, we can write $\operatorname{Ind}(F \circ G) = \operatorname{Ind}(F) \circ \operatorname{Ind}(G)$.

Proof. Consider the composite $f\colon \mathbb{C} \xrightarrow{F} \mathbb{D} \xrightarrow{h} \operatorname{Ind}(\mathbb{D})$ and apply Proposition 6.4. ☐

We now give a characterization of the category of Ind-Objects of a category $\mathbb{C}$, which is useful in practice to recognize when a given category $\mathbb{E}$ is actually equivalent to $\operatorname{Ind}(\mathbb{C})$.

Proposition 6.6. *Let $\mathbb{C} \xrightarrow{f} \mathbb{E}$ be a full and faithful functor, where $\mathbb{E}$ is a category with colimits of the form $\operatorname{colim}_i f(C_i)$ for any small filtered system in $\mathbb{C}$, (to simplify notation we shall abuse and omit to indicate f, as if $\mathbb{C}$ were a full subcategory of $\mathbb{E}$). Consider the functor $\operatorname{Ind}(\mathbb{C}) \xrightarrow{\overline{f}} \mathbb{E}$. Then:*

a) *$\overline{f}$ is full and faithful provided that, for any $C \in \mathbb{C}$ and $Y = (D_\alpha)_{\alpha\in\Gamma} \in \operatorname{Ind}(\mathbb{C})$, the following two conditions hold:*

i) For any arrow $C \to \operatorname{colim}_\alpha D_\alpha$ *in* $\mathbb{E}$, *there is a factorization:*

$$\begin{tikzcd} C \arrow[r] \arrow[dr, dashed] & \operatorname{colim}_\alpha D_\alpha \\ & D_\alpha \arrow[u] \end{tikzcd}$$

ii) For any two parallel arrows $C \rightrightarrows D_\alpha$ *that become equal in the colimit, there is* β *and a transition morphism* $\alpha \to \beta$ *such that the two arrows become equal already in* D_β:

$$\begin{tikzcd} C \arrow[r, shift left] \arrow[r, shift right] & D_\alpha \arrow[r] \arrow[dr, dashed] & \operatorname{colim}_\alpha D_\alpha \\ & & D_\beta \arrow[u] \end{tikzcd}$$

b) $\overline{f}$ *is an equivalence of categories if in addition every object in* $\mathbb{E}$ *is a small filtered colimit of objects* $D \in \mathbb{C}$.

Under these conditions, $\mathbb{E}$ *has all small filtered colimits.*

Proof. a) Given an arrow $\overline{f}(X) \to \overline{f}(Y)$ in $\mathbb{E}$, recall that $\overline{f}(X) = \operatorname{colim}_i C_{\underline{i}}$, and $\overline{f}(Y) = \operatorname{colim}_\alpha D_\alpha$. Consider, for each i, the composite map $C_i \to \overline{f}(X) \to \overline{f}(Y)$, use Proposition 6.3, and apply the two conditions in a) above.

b) Any full and faithful functor that is surjective on objects is an equivalence ([12], Ch IV). □

As an application consider an *equational theory in universal algebra*, and its corresponding variety (or equational class) of algebras. Recall that a *finitely presented algebra* is an algebra C that has a finite presentation. In other words, C can be presented as a coequalizer of two morphisms $F(k) \rightrightarrows F(n) \longrightarrow C$, where k and n are integers, and F denotes the free algebra in the appropriate number of generators (see our discussion before Proposition 5.1).

A morphism $C \to A$ *from* C *into any algebra* A *is determined by* n *elements of* A *satisfying* k *equations.*

Example 6.7. For any equational theory in universal algebra, let $\mathbb{C}$ be the full subcategory determined by any class of *finitely presented algebras*, let $\mathbb{E}$ be the full subcategory determined by the class of all algebras that are small filtered colimits of algebras in $\mathbb{C}$, and let f be the inclusion of full subcategories. Then the functor $\mathbb{C} \xrightarrow{f} \mathbb{E}$ satisfies the conditions in Proposition 6.6.

P r o o f . Condition b) is clear, so we have to check the two conditions in a). Let $D = \mathrm{colim}_\alpha D_\alpha$. The underlying set of D is the colimit of the underlying sets of the $D_\alpha^{\prime s}$. That is, the elements of D are germs (see Proposition 6.1). A morphism $C \to D$ is determined by a finite number of germs satisfying a finite number of equations. There is an index β such that they all have representatives in D_β, and these representatives will eventually satisfy the equations in D_α, for some index α, $\beta \to \alpha$. This determines a morphism $C \to D_\alpha$ that factors the morphism $C \to D$. The second condition in a) is proved in a similar way. □

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Received December 17, 2002

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