Tatra Mt. Math. Publ. **27** (2003), 139–152

TATRA MOUNTAINS
Mathematical Publications

ON A RELATION BETWEEN MONOUNARY ALGEBRAS AND UNORIENTED GRAPHS

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Dedicated to my father, Professor Ján Jakubík, on the occasion of his 80th birthday

ABSTRACT. In this paper we deal with a question concerning the relations between monounary algebras and unoriented graphs. An analogous open problem on the relation between finite lattices and unoriented graphs was proposed by G. Birkhoff. To each monounary algebra $\mathcal{A}$ we assign in a natural way an unoriented graph $\mathcal{G}(\mathcal{A})$; the edges of $\mathcal{G}(\mathcal{A})$ are the elements of $\mathcal{A}$. We characterize all monounary algebras $\mathcal{A}$ having the property that each automorphism of $\mathcal{G}(\mathcal{A})$ is an automorphism of $\mathcal{A}$.

In the monograph [1], G. Birkhoff formulated the following two questions concerning the relation between lattices and unoriented graphs:

PROBLEM 5. *To find all finite lattices L such that each lattice L_1 whose unoriented graph is isomorphic to the unoriented graph of L, is lattice-isomorphic to L.*

PROBLEM 6. *To find all finite lattices L such that each automorphism of the unoriented graph of L is an automorphism of L.*

A list of papers on lattices, semilattices and multilattices which were inspired by Problem 5 was given by Ratanaprasert [3].

To each monounary algebra $\mathcal{A}$ we can assign an unoriented graph $\mathcal{G}(\mathcal{A})$. This enables to formulate for monounary algebras the questions analogous to Birkhoff's Problem 5 and Problem 6.

The question analogous to Problem 5 was investigated in the author's paper [2]. In the present paper we deal with the question analogous to Problem 6.

We recall the basic definitions. As usual, by an unoriented graph without loops and multiple edges we understand a pair (V, E), in which E is a collection

2000 Mathematics Subject Classification: 08A60.
Keywords: monounary algebra, unoriented graph, automorphism.

Supported by Grant VEGA 1/7468/20.

of two-element subsets of V. Let (A,f) be a monounary algebra. An unoriented graph without loops and multiple edges $\mathcal{G}(A,f) = (V,E)$ will be called the graph corresponding to (A,f), if $V = A$ and, whenever $x, y \in A$, $x \neq y$,

$$\{x,y\} \in E \iff f(x) = y \quad \text{or} \quad f(y) = x\,.$$

We will consider the following classes of monounary algebras:

$\mathcal{K}$ – the class of all monounary algebras (A,f) such that, whenever (B,g) is a monounary algebra with $\mathcal{G}(A,f) \cong \mathcal{G}(B,g)$, then $(A,f) \cong (B,g)$;

$\mathcal{M}$ – the class of all monounary algebras (A,f) such that each automorphism of the graph $\mathcal{G}(A,f)$ is an automorphism of the algebra (A,f);

$\mathcal{L}$ – the class of all monounary algebras (A,f) such that, whenever (B,g) is a monounary algebra and φ is a graph-isomorphism of $\mathcal{G}(A,f)$ onto $\mathcal{G}(B,g)$, then φ is an isomorphism of (A,f) onto (B,g).

In [2] there were found internal conditions characterizing monounary algebras belonging to the class $\mathcal{K}$. Monounary algebras from the class $\mathcal{L}$ will be described in Section 2 below. The main goal of the present paper is to characterize monounary algebras belonging to the class $\mathcal{M}$.

Let us notice that if we admit loops and/or multiple edges, then the definition of the graph corresponding to (A,f) can differ from the above definition; in such case our investigation would be analogous.

1. Preliminaries

In 1.1–1.7 we suppose that (A,f) is a connected monounary algebra such that (A,f) contains no cycle having more than 2 elements. We use the following notation, cf. [2]:

1.1. Notation. For $a \in A$ and $x \in A$ we define an integer $\alpha_a(x) \in \mathbb{N} \cup \{0,-1\}$ by putting

$$\alpha_a(x) = \begin{cases} n & \text{if } n \in \mathbb{N}\cup\{0\},\ x = f^n(a),\ x \neq f^m(a) \text{ for each} \\ & m \in \mathbb{N}\cup\{0\},\ m < n, \\ -1 & \text{if } f^n(a) \neq x \text{ for each } n \in \mathbb{N}\cup\{0\}. \end{cases}$$

1.2. Notation. Let $a \in A$. We denote by $L_a = (A, l_a)$ the monounary algebra such that if $x \in A$, then

$$l_a(x) = \begin{cases} a & \text{if } \alpha_a(x) = 0, \\ f^{n-1}(a) & \text{if } \alpha_a(x) = n > 0, \\ f(x) & \text{otherwise}\,. \end{cases}$$

1.3. Notation. Let $a \in A$ be such that $f(a) \neq a$. We denote by $D_a = (A, d_a)$ the monounary algebra such that if $x \in A$, then

$$d_a(x) = \begin{cases} f^{n-1}(a) & \text{if } \alpha_a(x) = n > 0, \\ f(x) & \text{otherwise}. \end{cases}$$

1.4. Notation. Let $S = \{a_i\}_{i \in \mathbb{Z}}$ be a system of mutually distinct elements of A satisfying one of the following conditions:

(i) $f(a_i) = \begin{cases} a_{i+1} & \text{if } i \in \mathbb{Z},\ i < 0, \\ a_{i-1} & \text{if } i \in \mathbb{N}. \end{cases}$

(ii) $f(a_i) = a_{i-1}$ for each $i \in \mathbb{Z}$.

Then S will be called a *path* of (A, f). If (i) is valid, then the path S is said to be *broken*. Further, if $S = \{a_i\}_{i \in \mathbb{Z}}$ is a path of (A, f), then we define the sets $[a_i]$ for $i \in \mathbb{Z}$ as follows. If $i \neq 0$, then

$$[a_i] = \Big\{x \in A \colon (\exists m \in \mathbb{N} \cup \{0\})\big((f^m(x) = a_i) \mathbin{\&} (\forall l \in \mathbb{N} \cup \{0\},\ l < m)(f^l \notin S)\big)\Big\}.$$

Further,

$$\begin{aligned}[a_0] = &\Big\{x \in A \colon (\exists m \in \mathbb{N} \cup \{0\})\big((f^m(x) = a_0) \mathbin{\&} (\forall l \in \mathbb{N} \cup \{0\},\ l < m)(f^l \notin S)\big)\Big\} \\ &\cup \big\{x \in A \colon (\forall n \in \mathbb{N} \cup \{0\})(f^n(x) \notin S)\big\}.\end{aligned}$$

1.5. Notation. Suppose that $R = \{a_n\}_{n \subset \mathbb{N} \cup \{0\}}$ is a sequence of mutually distinct elements of A such that

$$f(a_n) = a_{n-1} \quad \text{for each } n \in \mathbb{N};$$

R will be called a *chain*. We denote by $P_R = (A, p_R)$ the monounary algebra such that if $x \in A$, then

$$p_R(x) = \begin{cases} a_{n+1} & \text{if } x = a_n,\ n \in \mathbb{N} \cup \{0\}, \\ f^{n-1}(a_0) & \text{if } \alpha_{a_0}(x) = n \in \mathbb{N}, \\ f(x) & \text{otherwise}. \end{cases}$$

1.6. Lemma. *Let $S = \{a_i\}_{i \in \mathbb{Z}}$ be a path of (A, f), $R = \{a_i\}_{i \in \mathbb{N} \cup \{0\}}$. If $i \in \mathbb{Z}$, then $[a_i]$ is a connected partial subalgebra of the monounary algebra $P_R = (A, p_R)$.*

Proof. The assertion follows immediately from 1.4 and 1.5. □

1.7. Definition. Let $k \in \mathbb{N}$. Suppose that $S = \{a_i\}_{i\in\mathbb{Z}}$ is a path of (A,f) and let $R = \{a_i\}_{i\in\mathbb{N}\cup\{0\}}$. We will say that (A,f) is *k-periodic along S*, if for each $i \in \mathbb{Z}$ the partial algebras $\bigl([a_i], p_R\bigr)$ and $\bigl([a_{i+k}], p_R\bigr)$ are isomorphic. If (A,f) is k-periodic (along S) for some $k \in \mathbb{N}$, then (A,f) is called *periodic* (along S).

1.8. Definition. Let (A,f) be a connected monounary algebra with a cycle $\{c\}$. Suppose that there are connected partial subalgebras U_1, U_2 of (A,f) with at least 2 elements and such that $U_1 \cong U_2$ and $U_1 \cap U_2 = \{c\}$. Then (A,f) will be called *symmetric*.

1.9. Definition. Let (A,f) be a monounary algebra with a 2-element cycle. Assume that there are connected partial subalgebras U_1, U_2 of (A,f) with at least 2 elements and such that $U_1 \cong U_2$, $U_1 \cup U_2 = A$ and $U_1 \cap U_2 = \emptyset$. Then (A,f) will be called *symmetric*. (Notice that in this case U_i $(i = 1,2)$ contains exactly one of the cyclic elements of (A,f).)

1.10. Definition. A cycle C of a monounary algebra is called *large* if it has more than 2 elements.

Remark. As usual, if $n \in \mathbb{N}$, $j \in \mathbb{Z}$, then the symbol j_n denotes the corresponding class in $\mathbb{Z}_n$ (integers modulo n).

1.11. Notation. Let (A,f) be a monounary algebra and let C be a large cycle of (A,f) having n elements. For $x \in C$ we denote

$$[x] = \bigl\{y \in A\colon (\exists k \in \mathbb{N}\cup\{0\})(f^k(y) = x \text{ and } f^m(y) \notin C \text{ for each } m \in \mathbb{N}\cup\{0\},\ m < k)\bigr\}.$$

Take fixed $c \in C$. If $i \in \mathbb{N}$, then we denote

$$C^{(c)}_{i_n} = \bigl[f^i(c)\bigr].$$

Notice that this notion is correctly defined, since C is an n-element cycle. Further, $\bigl(C^{(c)}_{i_n}, f\bigr)$ for $i \in \mathbb{Z}$ is a connected partial monounary algebra.

1.12. Definition. Let (A,f) be a monounary algebra possessing a large cycle C with n elements. We will say that (A,f) is *symmetric*, if there is $c \in C$ such that either $C^{(c)}_{i_n} \cong C^{(c)}_{(n-i)_n}$ for each $i \in \mathbb{Z}$ or $C^{(c)}_{i_n} \cong C^{(c)}_{(1-i)_n}$ for each $i \in \mathbb{Z}$.

It is easy to see that the following assertion is valid:

1.13. Lemma. (a) *If (A,f) is a connected monounary algebra with a cycle $\{c\}$, $\varphi \in \operatorname{Aut}\mathcal{G}(A,f)$, $\varphi(c) = c$, then $\varphi \in \operatorname{Aut}(A,f)$.*

(b) *If (A,f) is a connected monounary algebra with a 2-element cycle $\{c_1,c_2\}$, $\varphi \in \operatorname{Aut}\mathcal{G}(A,f)$, $\varphi\bigl(\{c_1,c_2\}\bigr) = \{c_1,c_2\}$, then $\varphi \in \operatorname{Aut}(A,f)$.*

(c) *If* (A,f) *is a connected monounary algebra with a large cycle* C *and* $c \in C$, $\varphi \in \operatorname{Aut}\mathcal{G}(A,f)$, $\varphi\big(f(c)\big) = f\big(\varphi(c)\big)$, *then* $\varphi \in \operatorname{Aut}(A,f)$.

(d) *If* (A,f) *is a connected monounary algebra possessing no cycle,* $a \in A$, $\varphi \in \operatorname{Aut}\mathcal{G}(A,f)$ *such that* $\varphi\big(f^n(a)\big) = f^n\big(\varphi(a)\big)$ *for each* $n \in \mathbb{N}$, *then* $\varphi \in \operatorname{Aut}(A,f)$.

2. The class $\mathcal{L}$

By the definition, the class $\mathcal{L}$ is a subclass of $\mathcal{K}$. We have the following characterization of the monounary algebras belonging to the class $\mathcal{K}$:

2.1. Theorem. ([2], Thm. 4.3). *Let* (A,f) *be a monounary algebra. Then* $(A,f) \in \mathcal{K}$ *if and only if, whenever* K *is a connected component of* (A,f) *with* $\operatorname{card} K > 1$,

(i) *there is a cycle* C *of* (K,f) *with* $n > 2$ *elements, and*

(ii) *there is* $k \in \{0,1,\cdots,n-1\}$ *such that if* $i \in \{0,1,\cdots,n-1\}$, *then* $(C_{i_n}, f) \cong (C_{(k-i)_n}, f)$.

2.2. Theorem. *A monounary algebra* (A,f) *belongs to* $\mathcal{L}$ *if and only if* $f(x) = x$ *for each* $x \in A$.

Proof. Let $(A,f) \in \mathcal{L}$. Then $(A,f) \in \mathcal{K}$. Assume that there is $x \in A$ such that $f(x) \neq x$, i.e., there is a connected component M of (A,f) with $\operatorname{card} M > 1$. Then (i) and (ii) of 2.1 is valid. Define a unary operation g on A as follows:

$$g(x) = \begin{cases} f(x) & \text{if } x \in A - C\,, \\ y & \text{if } x \in C \quad \text{and} \quad y \in C \cap f^{-1}(x)\,. \end{cases}$$

Then (A,g) is a monounary algebra and it is easy to verify that the identity mapping ι_A is an isomorphism of $\mathcal{G}(A,f)$ onto $\mathcal{G}(A,g)$ (cf. also [2], 3.5). We have $\operatorname{card} C > 2$, thus there are distinct elements $a,b,c \in C$ with $f(a) = b$, $f(b) = c$. This implies $g(b) = a$. The property $(A,f) \in \mathcal{L}$ yields that ι_A an isomorphism of (A,f) onto (A,g), hence

$$c = f(b) = \iota_A\big(f(b)\big) = g\big(\iota_A(b)\big) = g(b) = a \neq c\,,$$

which is a contradiction.

The converse implication is obvious. □

3. (A,f) connected, with a large cycle

We start to investigate the class $\mathcal{M}$.

In 3.1 and 3.2 we suppose that (A,f) is a connected monounary algebra possessing a large cycle C with $\operatorname{card} C = n$.

3.1. Lemma. *If (A,f) is symmetric, then $(A,f)\notin\mathcal{M}$.*

Proof. Suppose that (A,f) is symmetric. Then there is $c\in C$ such that either $C^{(c)}_{i_n}\cong C^{(c)}_{(n-i)_n}$ for each $i\in\mathbb{Z}$ or $C^{(c)}_{i_n}\cong C^{(c)}_{(1-i)_n}$ for each $i\in\mathbb{Z}$.

Assume that the first relation is valid. For each $i\in\mathbb{Z}$ there is an isomorphism φ_i of the partial monounary algebra $C^{(c)}_{i_n}$ onto $C^{(c)}_{(n-i)_n}$. Let φ be a mapping of A into A such that for $i\in\mathbb{Z}$ and $x\in C^{(c)}_{i_n}$, $\varphi(x)=\varphi_i(x)$. Notice that if $x\in C^{(c)}_{i_n}$, then $\varphi^2(x)=\varphi_{n-i}\big(\varphi_i(x)\big)\in C^{(c)}_{(n-(n-i))_n}=C^{(c)}_{i_n}$. Hence φ is surjective and it can be easily verified that $\varphi\in\operatorname{Aut}\mathcal{G}(A,f)$. Further, $f(c)\in C^{(c)}_{1_n}$, thus $\varphi\big(f(c)\big)\in C^{(c)}_{(n-1)_n}$, $\varphi(c)\in C^{(c)}_{0_n}$, thus $f\big(\varphi(c)\big)\in C^{(c)}_{1_n}\neq C^{(c)}_{(n-1)_n}$ for $n>2$, therefore $\varphi\notin\operatorname{Aut}(A,f)$. Hence $(A,f)\notin\mathcal{M}$.

Now suppose that the second relation holds. Analogously as above, let φ_i be an isomorphism of the partial monounary algebra $C^{(c)}_{i_n}$ onto $C^{(c)}_{(1-i)_n}$ and let φ be a mapping of A into A such that for $i\in\mathbb{Z}$ and $x\in C^{(c)}_{i_n}$, $\varphi(x)=\varphi_i(x)$. Then $f(c)\in C^{(c)}_{1_n}$, which implies $\varphi\big(f(c)\big)\in C^{(c)}_{0_n}$, $\varphi(c)\in C^{(c)}_{1_n}$, thus $f\big(\varphi(c)\big)\in C^{(c)}_{2_n}\neq C^{(c)}_{0_n}$. Therefore $\varphi\notin\operatorname{Aut}(A,f)$ and $(A,f)\notin\mathcal{M}$. ☐

3.2. Lemma. *If $(A,f)\notin\mathcal{M}$, then (A,f) is symmetric.*

Proof. Suppose that $(A,f)\notin\mathcal{M}$, i.e., there is $\varphi\in\operatorname{Aut}\mathcal{G}(A,f)$ such that $\varphi\notin\operatorname{Aut}(A,f)$. Let $a\in C$. The relation $\varphi\in\operatorname{Aut}\mathcal{G}(A,f)$ implies that $\big\{\varphi(a),\varphi\big(f(a)\big)\big\}$ is an edge in $\mathcal{G}(A,f)$, hence either (a) $f\big(\varphi(a)\big)=\varphi\big(f(a)\big)$ or (b) $\varphi(a)=f\Big(\varphi\big(f(a)\big)\Big)$. According to 1.13(c) we obtain that (b) is valid. There is $k\in\{0,1,\dots,n-1\}$ such that $\varphi(a)=f^k(a)$. Then

$$\varphi(C^{(a)}_{i_n})=C^{(a)}_{(k-i)_n}\quad\text{for each } i\in\mathbb{Z}$$

and

$$C^{(a)}_{i_n})\cong C^{(a)}_{(k-i)_n}\quad\text{for each } i\in\mathbb{Z}.$$

First let k be even. We denote $c=f^{k/2}(a)$. Then, for $i\in\mathbb{Z}$,

$$C^{(c)}_{i_n}=C^{(a)}_{(i+k/2)_n}\cong C^{(a)}_{(k-(i+k/2))_n}=C^{(a)}_{(k/2-i)_n}=C^{(c)}_{(-i)_n}=C^{(c)}_{(n-i)_n},$$

therefore (A,f) is symmetric.

Now suppose that k is odd and put $c=f^{(k-1)/2}(a)$. Then, for $i\in\mathbb{Z}$,

$$C^{(c)}_{i_n}=C^{(a)}_{(i+(k-1)/2)_n}\cong C^{(a)}_{(k-(i+(k-1)/2))_n}=C^{(a)}_{(1-i+(k-1)/2)_n}=C^{(c)}_{(1-i)_n},$$

hence (A,f) is symmetric. ☐

3.3. Proposition. *Let (A,f) be a connected monounary algebra which possesses a large cycle. Then $(A,f)\in\mathcal{M}$ if and only if (A,f) is not symmetric.*

Proof. It is a consequence of 2.1 and 2.2. ☐

4. Some necessary conditions

In this section we assume that (A, f) is a connected monounary algebra possessing no large cycle. Further, let $\varphi \in \operatorname{Aut} \mathcal{G}(A, f) - \operatorname{Aut}(A, f)$.

4.1. Lemma. *There exist $t \in A$ and $n \in \mathbb{N}$ such that either $f^n(\varphi(t)) = t$ or $f^n(t) = \varphi(t)$.*

P r o o f . If (A, f) contains a cycle, then we can take t of the cycle and the assertion is valid. Suppose that (A, f) contains no cycle. Let $u \in A$, $v = \varphi(u)$. There are $k, m \in \mathbb{N}$ such that $f^k(u) = f^m(v)$. Denote $p = f^k(u)$.

1) First assume that $k \leqq m$. We have

$$\{u, f(u)\}\,,\ \{f(u), f^2(u)\},\ \dots,\ \{f^{k-1}(u), p\} \in E\,.$$

Hence the relation $\varphi \in \operatorname{Aut} \mathcal{G}(A, f)$ yields

$$\{v, \varphi(f(u))\}\,,\ \{\varphi(f(u)), \varphi(f^2(u))\},\ \dots\,, \{\varphi(f^{k-1}(u)), \varphi(p)\} \in E\,,$$

i.e.,

$$\begin{aligned}
&f(v) = \varphi(f(u)) \ \text{ or } \ f\Big(\varphi(f(u))\Big) = v\,,\\
&f\Big(\varphi(f(u))\Big) = \varphi(f^2(u)) \ \text{ or } \ f\Big(\varphi(f^2(u))\Big) = \varphi(f(u)),\ \dots\,,\\
&f\Big(\varphi(f^{k-1}(u))\Big) = \varphi(p) \ \text{ or } \ f(\varphi(p)) = \varphi(f^{k-1}(u))\,.
\end{aligned}$$

This implies that $\varphi(p) \in f^{-i}(f^j(v))$ for some $i, j \in \mathbb{N} \cup \{0\}$ such that $i + j = k$ (there must be k edges between $\varphi(p)$ and $\varphi(u)$). We have $i, j \leqq k \leqq m$, thus

$$f^{(m-j)+i}(\varphi(p)) = f^{m-j}(f^i(\varphi(p))) = f^{m-j}(f^j(v)) = f^m(v) = p\,.$$

If $m - j + i > 0$, then put $n = m - j + i$ and $t = p$; we obtain the required elements t and n.

Let $m - j + i = 0$. We have $\varphi(p) = p$, $\varphi \notin \operatorname{Aut}(A, f)$, thus by 1.13d) there are $l \in \mathbb{N}$ and $q \in A$ such that

$$q = f^l(p)\,, \quad \varphi(q) \neq q\,, \quad \varphi(f^\alpha(p)) = f^\alpha(p) \text{ for each } \alpha \in \mathbb{N} \cup \{0\}\,,\ \alpha < l. \quad (1)$$

Then $\{f^{l-1}(p), q\} \in E$ implies $\{f^{l-1}(p), \varphi(q)\} = \{\varphi(f^{l-1}(p)), \varphi(q)\} \in E$, i.e., either $\varphi(q) = f^l(p)$ or $f(\varphi(q)) = f^{l-1}(p)$. Thus either $\varphi(q) = q$ or $f^2(\varphi(q)) = f^l(p) = q$. The first relation yields a contradiction; in the second case we can put $t = q$, $n = 2$.

2) Now suppose that $m < k$. Denote $\psi = \varphi^{-1}$. Then $\psi \in \operatorname{Aut} \mathcal{G}(A, f) - \operatorname{Aut}(A, f)$. We have

$$f^m(v) = f^k(\psi(v))$$

and we apply the previous part of the proof with the distinction that we take v, u, ψ, m, k instead of u, v, φ, k, m. Then there exist $t' \in A$ and $n' \in \mathbb{N}$ such that either (a) $f^{n'}\bigl(\psi(t')\bigr) = t'$ or (b) $f^{n'}(t') = \psi(t')$. Denote $t = \psi(t')$. If (a) is valid, then

$$f^{n'}(t) = t' = \psi^{-1}(t) = \varphi(t)\,.$$

Let (b) hold. Then

$$f^{n'}\bigl(\varphi(t)\bigr) = f^{n'}\bigl(\varphi\bigl(\psi(t')\bigr)\bigr) = f^{n'}(t') = \psi(t') = t\,,$$

which completes the proof. □

In the remaining part of this section we suppose that t and n are as in 3.1. Next, let $q \in A$ be such that if $f^n\bigl(\varphi(t)\bigr) = t$, then $q = f^{n-1}\bigl(\varphi(t)\bigr)$, and if $f^n(t) = \varphi(t)$, then $q = f(t)$. This implies that one of the following conditions is satisfied:

(α) $f^n\bigl(\varphi(t)\bigr) = t$, $q = f^{n-1}\bigl(\varphi(t)\bigr)$, $f\bigl(\varphi(q)\bigr) = \varphi(t)$,

(β) $f^n(t) = \varphi(t)$, $q = f(t)$, $f\bigl(\varphi(t)\bigr) = \varphi(q)$,

(γ) $f^n\bigl(\varphi(t)\bigr) = t$, $q = f^{n-1}\bigl(\varphi(t)\bigr)$, $f\bigl(\varphi(t)\bigr) = \varphi(q)$,

(δ) $f^n(t) = \varphi(t)$, $q = f(t)$, $f\bigl(\varphi(q)\bigr) = \varphi(t)$.

The cases (α) and (β) are similar, because we could consider φ^{-1} instead of φ, thus we will deal only with the case (α). Analogously we will omit the case (δ), because it is similar to (γ).

4.2. Lemma. *Let* (α) *hold. Then* (A, f) *is periodic along a broken path.*

P r o o f . We are going to define a path $\{a_i\}_{i\in\mathbb{Z}}$. Let us denote

$$b_0 = t\,, \quad b_1 = q = f^{n-1}\bigl(\varphi(t)\bigr)\,, \quad b_2 = f^{n-2}\bigl(\varphi(t)\bigr), \dots, b_n = \varphi(t) \tag{1}$$

and then, by induction, if $m \in \mathbb{N}$, $m > n$ and b_l is defined for each $l \in \mathbb{N}$, $l < m$, then we set

$$b_l = \varphi(b_{l-n})\,. \tag{2}$$

Further we put

$$b_{-j} = \varphi^{-1}(b_{n-j}) \qquad \text{for each } j \in \mathbb{N}\,. \tag{3}$$

Since $\varphi \in \operatorname{Aut}\mathcal{G}(A,f)$, $\varphi^{-1} \in \operatorname{Aut}\mathcal{G}(A,f)$, the definition of $\{b_i\}_{i\in\mathbb{Z}}$ implies that the elements are distinct and that $\{b_i, b_{i+1}\} \in E$ for each $i \in \mathbb{Z}$, i.e., either $f(b_i) = b_{i+1}$ or $f(b_{i+1}) = b_i$ for each $i \in \mathbb{Z}$. We have $f(b_1) = b_0$, thus either $f(b_i) = b_{i-1}$ for each $i \in \mathbb{Z}$ or there is $m \in \mathbb{Z}$ such that

$$f(b_i) = b_{i-1} \qquad \text{for each } i \in \mathbb{Z}\,,\ i > m \text{ and } f(b_m) \neq b_{m-1}\,. \tag{4}$$

In the first case, the system $\{b_i\}_{i\in\mathbb{Z}}$ is a path of (A, f). (Notice that the path is not broken). Let us consider the second case. We have $f(b_{m+1}) = b_m$ and

$f(b_{m-1}) = b_m$. Next, in view of (4) the fact that either $f(b_{m-2}) = b_{m-1}$ or $f(b_{m-1}) = b_{m-2}$ implies that $f(b_{m-2}) = b_{m-1}$. By induction, $f(b_j) = b_{j+1}$ for each $j < m$. If $i \in \mathbb{Z}$, then denote $a_i = b_{i-m}$. We obtain

$$f(a_i) = \begin{cases} b_{i-m-1} = a_{i-1} & \text{for each } i \in \mathbb{N}, \\ b_{i-m+1} = a_{i+1} & \text{for each } i \in \mathbb{N},\ i < 0. \end{cases}$$

Therefore $\{a_i\}_{i\in\mathbb{Z}}$ is a path.

In both cases we have constructed a path $\{a_i\}_{i\in\mathbb{Z}}$ such that $\varphi(a_i) = a_{i+n}$ for each $i \in \mathbb{Z}$. In this situation φ maps the sets $[a_i]$ onto $[a_{i+n}]$ for $i \in \mathbb{Z}$ and it is obvious that the partial algebras $([a_i], p_R)$ and $([a_{i+n}], p_R)$, where $R = \{a_j\}_{j\in\mathbb{N}\cup\{0\}}$, are then isomorphic. In the first case it follows from 1.13d) that $\varphi \in \mathrm{Aut}(A, f)$, which is a contradiction. In the second case we obtain the required assertion, since we have shown that (A, f) is n-periodic along the path $\{a_i\}_{i\in\mathbb{Z}}$ and that the path is broken. □

4.3. Lemma. *Let (γ) hold and suppose that (A, f) is not periodic along a broken path. Denote $p = \varphi(t)$. Then*

$$\varphi\big(f^{n-i}(p)\big) = f^i(p) \qquad \text{for each } i \in \mathbb{N}\cup\{0\},\ i \leqq n/2. \tag{1}$$

Proof. By (γ), $\varphi\big(f^n(p)\big) = p$, $\varphi\big(f^{n-1}(p)\big) = f(p)$. We prove (1) by induction with respect to i. If $i = 0$ or $i = 1$, then the assertion is valid, thus the assertion holds for $n \in \{1, 2, 3\}$.

Let $n > 3$. Take $i \in \mathbb{N}$, $1 < i \leqq n/2$, and assume that (1) is valid for each $m \in \mathbb{N}\cup\{0\}$ such that $m < i$. Let $u = f^{n-i}(p)$, $v = f(u)$. We have $\{u, v\} \in E$, thus $\big\{\varphi(u), \varphi(v)\big\} \in E$ and either $f\big(\varphi(v)\big) = \varphi(u)$ or $f\big(\varphi(u)\big) = \varphi(v)$. By induction hypothesis, $\varphi(v) = \varphi\big(f(u)\big) = \varphi\big(f^{n-(i-1)}(p)\big) = f^{i-1}(p)$. In the first case we obtain

$$\varphi(u) = f\big(\varphi(v)\big) = f^i(p)\,,$$

as we wanted to show. Suppose that $f\big(\varphi(u)\big) = \varphi(v)$, i.e., $f\big(\varphi(u)\big) = f^{i-1}(p)$. This yields

$$u = f^{n-i}(p) = f^{n-2i+1}\big(f^{i-1}(p)\big) = f^{n-2i+1}\big(\varphi(v)\big) = f^{(n-2i+2)-1}\big(\varphi(v)\big)\,,$$
$$f^{n-2i+2}\big(\varphi(v)\big) = f\big(f^{n-2i+1}\big(\varphi(v)\big)\big) = f(u) = v\,,$$

i.e., (α) is valid (with the distinction that we take $v, u, n-2i+2$ instead of t, q, n). By 3.2, (A, f) is periodic along a broken path, which is a contradiction. Therefore (1) holds. □

4.4. Lemma. *Let (γ) hold and suppose that (A,f) is not periodic along a broken path. Then there is $a \in A$ such that L_a is symmetric, or $f(a) \neq a$ and D_a is symmetric.*

P r o o f. Let $p = \varphi(t)$. Then (1) of 3.3 is valid.

First suppose that n is even. Denote $a = f^{n/2}(p)$. By (1), $\varphi(a) = a$. Let $u_1 = f(a)$, $u_2 = \varphi(u_1)$ and let U_1, U_2 be the partial subalgebras of the algebra $L_a = (A, l_a)$ such that

$$U_j = \bigcup_{k \in \mathbb{N} \cup \{0\}} l_a^{-k}(u_j) \cup \{a\}$$

(for $j = 1,2$). Then $U_1 \cap U_2 = \{a\}$, U_1, U_2 are connected partial subalgebras of L_a. We have $\varphi \in \operatorname{Aut} \mathcal{G}(A,f)$ and $\mathcal{G}(A,f) = \mathcal{G}(L_a)$ (by [3]). Since $\varphi(u_1) = u_2$, this implies that the partial algebras (U_1, l_a) and (U_2, l_a) are isomorphic. Thus L_a is symmetric in view of 1.8.

Now let n be odd, $a = f^{(n-1)/2}(p)$. Then (1) of 3.3 yields

$$\varphi\big(f(a)\big) = \varphi\big(f^{(n+1)/2}(p)\big) = \varphi\big(f^{n-(n-1)/2}(p)\big) = f^{(n-1)/2}(p) = a\,.$$

Then $f(a) \neq a$. Consider the algebra D_a. We have $\mathcal{G}(A,f) = \mathcal{G}(D_a)$ according to [2]. In D_a we have a cycle $\{a,b\}$, where $b = f(a)$; there exist connected partial subalgebras U_1, U_2 of D_a such that $U_1 \cup U_2 = A$, $U_1 \cap U_2 = \emptyset$. Since $\varphi \in \operatorname{Aut} \mathcal{G}(D_a)$ and $\varphi(b) = a$, we obtain that partial subalgebras U_1 and U_2 of D_a are isomorphic. Therefore D_a is symmetric. □

5. Characterization of the class $\mathcal{M}$

In Section 3 connected monounary algebras with a large cycle belonging to the class $\mathcal{M}$ were described. Now we will first describe the remaining connected elements of $\mathcal{M}$. Next we will use these descriptions to give the characterization of all monounary algebras belonging to the class $\mathcal{M}$.

5.1. Lemma. *Suppose that (A,f) is a connected monounary algebra possessing no large cycle and such that (A,f) is periodic along a broken path. Then there is $\varphi \in \operatorname{Aut} \mathcal{G}(A,f) - \operatorname{Aut}(A,f)$.*

P r o o f. Let $\{a_i\}_{i \in \mathbb{Z}}$ be a broken path, $k \in \mathbb{N}$ be such that (A,f) is periodic according to $\{a_i\}_{i \in \mathbb{Z}}$, with a period k. Thus

$$f(a_i) = \begin{cases} a_{i+1} & \text{if } i \in \mathbb{Z}\,,\ i < 0\,, \\ a_{i-1} & \text{if } i \in \mathbb{N}\,. \end{cases}$$

Further, for each $i \in \mathbb{Z}$ there is an isomorphism φ_i of partial algebras $\big([a_i], p_R\big)$ onto $\big([a_{i+k}], p_R\big)$, where $R = \{a_i\}_{i\in\mathbb{N}\cup\{0\}}$. Denote

$$\varphi(x) = \begin{cases} \varphi_i(x) & \text{if } x \in [a_i]\,, \\ x & \text{otherwise}. \end{cases}$$

Then φ is an automorphism of P_R and $\varphi \in \operatorname{Aut}\mathcal{G}(P_R)$. By [2], $\varphi \in \operatorname{Aut}\mathcal{G}(P_R) = \operatorname{Aut}\mathcal{G}(A,f)$. Further, if we suppose that $\varphi \in \operatorname{Aut}(A,f)$, then

$$a_{k-1} = f(a_k) = f\big(\varphi(a_0)\big) = \varphi\big(f(a_0)\big) \in \varphi\big([a_0]\big) = [a_k]\,,$$

which is a contradiction. Therefore $\varphi \in \operatorname{Aut}\mathcal{G}(A,f) - \operatorname{Aut}(A,f)$. □

5.2. Lemma. *Suppose that (A,f) is a connected monounary algebra possessing no large cycle and $a \in A$ is such that L_a is symmetric. Then there is $\varphi \in \operatorname{Aut}\mathcal{G}(A,f) - \operatorname{Aut}(A,f)$.*

Proof. By the assumption, there are connected partial subalgebras U_1, U_2 of L_a with at least 2 elements such that $(U_1, l_a) \cong (U_2, l_a)$ and $U_1 \cup U_2 = \{a\}$. Let ψ be the isomorphism of (U_1, l_a) onto (U_2, l_a). Further we define a mapping of A into A by putting

$$\varphi(x) = \begin{cases} \psi(x) & \text{if } x \in U_1\,, \\ \psi^{-1}(x) & \text{if } x \in U_2\,, \\ x & \text{otherwise}. \end{cases}$$

Then φ is an automorphism of L_a and $\varphi \in \operatorname{Aut}\mathcal{G}(L_a)$. By [2], $\varphi \in \operatorname{Aut}\mathcal{G}(L_a) = \operatorname{Aut}\mathcal{G}(A,f)$. We can suppose that $f(a) \in U_1$. Denote $\varphi\big(f(a)\big) = b$. Then $\varphi(b) = f(a)$, $f(b) = a$. If $\varphi \in \operatorname{Aut}(A,f)$, then

$$a = \varphi(a) = \varphi\big(f(b)\big) = f\big(\varphi(b)\big) = f\big(f(a)\big) \in U_1 - \{a\}\,,$$

which is a contradiction. □

5.3. Lemma. *Suppose that (A,f) is a connected monounary algebra possessing no large cycle and $a \in A$ with $f(a) \neq a$ is such that D_a is symmetric. Then there is $\varphi \in \operatorname{Aut}\mathcal{G}(A,f) - \operatorname{Aut}(A,f)$.*

Proof. The assumption yields that that there are connected partial subalgebras (U_1, d_a), (U_2, d_a) of D_a with at least 2 elements and such that $(U_1, d_a) \cong (U_2, d_a)$, $U_1 \cup U_2 = A$ and $U_1 \cap U_2 = \emptyset$. Let ψ be the isomorphism of (U_1, l_a) onto (U_2, l_a). Further we define a mapping of A into A by putting

$$\varphi(x) = \begin{cases} \psi(x) & \text{if } x \in U_1\,, \\ \psi^{-1}(x) & \text{if } x \in U_2\,, \\ x & \text{otherwise}. \end{cases}$$

Then φ is an automorphism of L_a and $\varphi \in \operatorname{Aut} \mathcal{G}(L_a)$. By [2], $\varphi \in \operatorname{Aut} \mathcal{G}(L_a) = \operatorname{Aut} \mathcal{G}(A, f)$.

Let $\{a, b\}$ be the cycle of D_a. This yields that $\varphi(a) = b, \varphi(b) = a$. Then either $f(a) = b$ or $f(b) = a$; we can suppose that $f(a) = b$. Without loss of generality, $a \in U_1$, $b \in U_2$. Then $f(b) = U_2$ and the definition of φ implies that $\varphi(f(b)) = \psi^{-1}(f(b)) \in U_1$. If $\varphi \in \operatorname{Aut}(A, f)$, then

$$\varphi(f(b)) = f(\varphi(b)) = f(a) = b \in U_2 ,$$

which is a contradiction. □

5.4. Proposition. *Let (A, f) be a connected monounary algebra possessing no large cycle. Then $(A, f) \in \mathcal{M}$ if and only if*

(a) *for any broken path S of (A, f), (A, f) fails to be periodic along S,*

(b) *for any $a \in A$, neither L_a nor D_a (if $f(a) \neq a$) is symmetric.*

P r o o f. Let $(A, f) \notin \mathcal{M}$, i.e., there is $\varphi \in \operatorname{Aut} \mathcal{G}(A, f) - \operatorname{Aut}(A, f)$. Applying results of Section 4, either (α) or (γ) is valid. If (α) holds, then 4.2 implies that (a) is not satisfied. Suppose that (γ) holds. Then either (A, f) is periodic along some broken path, i.e., (a) is not satisfied, or we obtain according to 4.4 that (b) is not satisfied.

The converse implication follows from 5.1–5.3. □

5.5. Notation. Let (B_1, f), (B_2, f) be connected monounary algebras. Let us introduce a relation ρ by putting

$$\big((B_1, f), (B_2, f)\big) \in \rho$$

if and only if one of the following conditions is satisfied:

(i) (B_1, f) contains a cycle $C_1, (B_2, f)$ contains a cycle C_2, $|C_1| = |C_2| = n > 2$ and there exist $c_1 \in C_1$, $c_2 \in C_2$ such that $\big([f^i(c_1)], f\big) \cong \big([f^{n-i}(c_2)], f\big)$ for each $i \in \mathbb{Z}$;

(ii) (B_1, f), (B_2, f) contain no large cycles and either there is $a \in B_2$ with $(B_1, f) \cong L_a$ or $f(a) \neq a$, $(B_1, f) \cong D_a$, or there is a chain $R \subseteq B_2$ with $(B_1, f) \cong P_R$.

5.6. Lemma. *The relation ρ is symmetric. Further, if (B_1, f), (B_2, f) are connected monounary algebras with $\big((B_1, f), (B_2, f)\big) \in \rho$, then $\mathcal{G}(B_1, f) \cong \mathcal{G}(B_2, f)$.*

P r o o f. It is a consequence of [2], Thm 3.5 and 3.6. □

Propositions 3.3 and 5.4 characterize connected members of $\mathcal{M}$; we will characterize all (A, f) of $\mathcal{M}$ as follows:

5.7. THEOREM. *Let (A,f) be a monounary algebra. Then (A,f) belongs to the class $\mathcal{M}$ if and only if*

(i) *whenever B is a connected component of (A,f), then $(B,f)\in\mathcal{M}$;*

(ii) *if B_1, B_2 are connected components of (A,f) such that $(B_1,f)\not\cong(B_2,f)$, then $\big((B_1,f),(B_2,f)\big)\notin\rho$.*

Proof. Suppose that $(A,f)\in\mathcal{M}$. Let B be a connected component of (A,f), $\varphi\in\operatorname{Aut}\mathcal{G}(B,f)$. Put

$$\psi(x)=\begin{cases}\varphi(x) & \text{if } x\in B,\\ x & \text{if } x\in A-B.\end{cases}$$

Obviously, $\psi\in\operatorname{Aut}\mathcal{G}(A,f)$, thus the relation $(A,f)\in\mathcal{M}$ implies that $\psi\notin\operatorname{Aut}(A,f)$. By the definition of ψ we get $\varphi\notin\operatorname{Aut}(B,f)$ and $(B,f)\in\mathcal{M}$.

Let B_1, B_2 be connected components of (A,f) such that (B_1,f), (B_2,f) are nonisomorphic and that $\big((B_1,f),(B_2,f)\big)\in\rho$. In view of 5.6, $\mathcal{G}(B_1,f)\cong\mathcal{G}(B_2,f)$; let φ be the corresponding graph isomorphism. We define a mapping $\psi\colon A\to A$ as follows:

$$\psi(x)=\begin{cases}\varphi(x) & \text{if } x\in B_1,\\ \varphi^{-1}(x) & \text{if } x\in B_2,\\ x & \text{otherwise}.\end{cases}$$

Then $\psi\in\operatorname{Aut}\mathcal{G}(A,f)$ and, since φ is not an isomorphism of algebras, $\psi\notin\operatorname{Aut}(A,f)$. Hence $(A,f)\notin\mathcal{M}$, a contradiction.

Conversely, assume that $(A,f)\neq\mathcal{M}$ and that (i), (ii) hold. Then there is $\varphi\in\operatorname{Aut}\mathcal{G}(A,f)-\operatorname{Aut}(A,f)$. If B is a connected component of (A,f), then by (i) there exists a connected component $B'\neq B$ with $\varphi(B)=B'$, since in the case $\varphi(B)=B$ we obtain $\varphi\restriction B\in\operatorname{Aut}\mathcal{G}(B,f)-\operatorname{Aut}(B,f)$ which is a contradiction to (i). Hence for each connected component B we get $\varphi(B)\neq B$ and the relation $\varphi\in\operatorname{Aut}\mathcal{G}(A,f)$ implies that $\mathcal{G}(B,f)\cong\mathcal{G}\big(\varphi(B),f\big)$. We have $\varphi\notin\operatorname{Aut}(A,f)$, thus there exists a connected component B of (A,f) such that $(B,f)\not\cong\big(\varphi(B),f\big)$. Then [2], 3.5 and 3.6 imply that $\big((B,f),\big(\varphi(B),f\big)\big)\in\rho$, a contradiction to (ii). $\square$

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Received November 27, 2002

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