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CLOSURE OPERATORS AND CONCEPT EQUATIONS IN NON-COMMUTATIVE FUZZY LOGIC

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Dedicated to Professor Ján Jakubík on the occasion of his 80th birthday

ABSTRACT. We introduce the notions of closure operator and closure system in a non-commutative fuzzy framework, where the structure of truth values is a generalized residuated lattice, L. We investigate the relationship between L-Galois connections and a weakened form of L-closure operators, which will eventually lead us to three ways of indicating a hierarchy of formal concepts: by means of relations, of Galois connections and of closure operators. So the situation from the classical case (as well as the one from the commutative fuzzy case) is recovered (of course, with some loses, expressed by the notions of strong L-Galois connection and weak L-closure operator, loses generated by the existence of two residua for the conjunction in the structure of truth values). Finally, we study systems of concept equations, which are in fact a form of (sometimes incomplete) specification of a conceptual hierarchy.

1. Introduction

Lately, there has been done a lot of work in the study of fuzzy logic, assuming that the conjunction of the truth values structure is not necessarily commutative ([22], [23], [24]).

On the other hand, we have some well built theories of closure operators ([14]), Galois connections ([14], [25]) and formal concepts ([17]), with significant relationships between them. As shown by Bělohlávek in his papers ([4]–[10]), many of the results also hold, with undiminished glamour, in a very general fuzzy framework, where the truth values come from a residuated lattice.

Thus, there appears naturally the attempt of generalizing the results regarding closure operators, Galois connections and formal concepts to the case where

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the conjunction is not necessarily commutative, the structure of truth values being a generalized residuated lattice. We have two residua of the conjunction, hence two ways of expressing logical implication. This situation, being in some sense beyond our intuition, creates the tendency of splitting everything in two. The unitary approach is sometimes ([18], [19]) still possible, suggesting that there might be a true, "serious" state of affairs involving non-commutative fuzziness. The current paper joins the series of efforts on doing without commutativity in fuzzy logic and set theory.

After some preliminaries mainly regarding the logical framework[1], we introduce in Section 3 the notion of L-closure operator and L-closure system and prove the bijective correspondence between them.

Section 4 deals with relating L-Galois connections to what we call weak L-closure operators (systems), which are actually "primitive", ununified forms of L-closure operators (systems), in which the "non-commutative syndrome" of splitting still exists. We also give a way of defining formal concept hierarchies by means of weak L-closure systems, via a certain type of L-Galois connections. We generalize some results of [8][2].

In the final section, we treat, following [10], concept equations, where there are given triples of fuzzy sets (objects/attributes/attributes) and the unknown part is a relation between objects and attributes which puts the triples concepts into the corresponding context. It turns out that the relation, if it exists, can be recovered. We also consider the degree to which a relation is the solution of a system; thus we get approximative solutions, that will become, with a less precise truth values structure, full solutions.

2. Preliminaries

Definition 2.1. A *generalized residuated lattice* ([26]) is a structure $(L, \vee, \wedge, *, \rightarrow, 0, 1)$ such that the following are true:

(GR1) $(L, \vee, \wedge, 0, 1)$ is a bounded lattice;

(GR2) $(L, *, 1)$ is a monoid;

(GR3) (residuation).

[1] As a matter of fact, we shall not place ourselves inside a non-commutative fuzzy framework, but simulate one.

[2] Actually, we generalize some weaker forms of the results from [8], where the discussion takes place considering another parameter, besides the truth values structure, L; namely, a subset K of L, *the set of relevant values*. We believe that we could have safely taken this parameter too.

For all $a, b, c \in L$ we have the equivalences:

$$a \le b \to c \quad \text{iff} \quad a * b \le c\,,$$
$$a \le b \Rightarrow c \quad \text{iff} \quad b * a \le c\,.$$

Lemma 2.1. *In a complete generalized residuated lattice L, the following properties hold for all $a, b, c \in L$, $(a_i)_{i\in I} \subseteq L$, $(b_i)_{i\in I} \subseteq L$, $(a_{ij})_{(i,j)\in I\times J} \subseteq L$:*

(1) $a \le b$ *iff* $a \to b = 1$ *iff* $a \Rightarrow b = 1$;

(2) $a \to 1 = a \Rightarrow 1 = 1$; $1 \to a = 1 \Rightarrow a = a$;

(3) $\to$ *and* $\Rightarrow$ *are antitone in the first and isotone in the second argument;*

(4) $*$ *is isotone in both arguments;*

(5) $a * b \le a$ *and* $a * b \le b$;

(6) $0 * a = a * 0 = 0$;

(7) $\left(\bigvee_{i\in I} a_i\right) * a = \bigvee_{i\in I}(a_i * a)$ *and* $a * \left(\bigvee_{i\in I} a_i\right) = \bigvee_{i\in I}(a * a_i)$;

(8) $(a \to b) * a \le b$ *and* $a * (a \Rightarrow b) \le b$;

(9) $a \le (a \to b) \Rightarrow b$ *and* $a \le (a \Rightarrow b) \to b$;

(10) $\left(\bigwedge_{i\in I} a_i\right) * a \le \bigwedge_{i\in I}(a_i * a)$ *and* $a * \left(\bigwedge_{i\in I} a_i\right) \le \bigwedge_{i\in I}(a * a_i)$;

(11) $\left(\bigvee_{i\in I} a_i\right) \to a = \bigwedge_{i\in I}(a_i \to a)$ *and* $a \to \left(\bigwedge_{i\in I} a_i\right) = \bigwedge_{i\in I}(a \to a_i)$;

(12) $\left(\bigvee_{i\in I} a_i\right) \Rightarrow a = \bigwedge_{i\in I}(a_i \Rightarrow a)$ *and* $a \Rightarrow \left(\bigwedge_{i\in I} a_i\right) = \bigwedge_{i\in I}(a \Rightarrow a_i)$;

(13) $\bigwedge_{i\in I}\bigwedge_{j\in J} a_{ij} = \bigwedge_{j\in J}\bigwedge_{i\in I} a_{ij} = \bigwedge_{(i,j)\in I\times J} a_{ij}$; $\bigvee_{i\in I}\bigvee_{j\in J} a_{ij} = \bigvee_{j\in J}\bigvee_{i\in I} a_{ij} = \bigvee_{(i,j)\in I\times J} a_{ij}$;

(14) $a \to (b \Rightarrow c) = b \Rightarrow (a \to c)$ *and* $a \Rightarrow (b \to c) = b \to (a \Rightarrow c)$;

(15) $*$ *is commutative iff* $\to\, = \,\Rightarrow$;

(16) $\left(\bigvee_{i\in I} a_i\right) \to \left(\bigvee_{i\in I} b_i\right) \le \bigvee_{i\in I}(a_i \to b_i)$;

(17) $\left(\bigvee_{i\in I} a_i\right) \Rightarrow \left(\bigvee_{i\in I} b_i\right) \le \bigvee_{i\in I}(a_i \Rightarrow b_i)$;

(18) $(b \to a) * (c \to b) \le (c \to a)$; $(a \Rightarrow b) * (b \Rightarrow c) \le (a \Rightarrow c)$.

Let L be a generalized residuated lattice. Define the following binary operations on L. For all $a, b \in L$,

$$a \leftrightarrow b = (a \to b) \wedge (b \to a)\,;$$
$$a \Leftrightarrow b = (a \Rightarrow b) \wedge (b \Rightarrow a)\,;$$
$$d(a, b) = (a \leftrightarrow b) \wedge (a \Leftrightarrow b)\,.$$

For a set X, call an element from L^X (the set of functions from X to L, which is obvious a complete lattice with the component-wise inclusion, where infima and suprema are the component-wise infima and suprema from L) an L-fuzzy-subset (L-subset) of X. We can identify the subsets of X (called crisp subsets, i.e., elements from 2^X) with particular L-subsets of X in the obvious way. For each $a \in L$ and $x \in X$, we consider $\{a/x\} \in L^X$ (called singleton) defined by:

$$\{a/x\}(x') = \begin{cases} 0, & \text{if } x' \neq x, \\ a, & \text{if } x' = x \end{cases}$$

for all $x' \in X$. Given two sets X and Y, an L-relation between X and Y is an element from $L^{X\times Y}$. Given two L-subsets A and B of X, we consider the right (S_1) and the left (S_2) subsethood degree of A in B:

$$S_1(A,B) = \bigwedge_{x\in X} \big(A(x) \to B(x)\big),$$

$$S_2(A,B) = \bigwedge_{x\in X} \big(A(x) \Rightarrow B(x)\big)$$

for any $A, B \in L^X$. Now, $A \subseteq B$ means $A(x) \leq B(x)$ for all $x \in X$, or equivalently, one of the identities $S_1(A,B) = 1$, $S_2(A,B) = 1$ holds (notice that, when full inclusion holds, the right and left subsethood degrees coincide and equal 1). According to Proposition 2.1 from [18], for all $A, B, C \in L^X$,

(1) $A \subseteq B$ implies $S_1(B,C) \leq S_1(A,C)$ and $S_2(B,C) \leq S_2(A,C)$;

(2) $A \subseteq B$ implies $S_1(C,A) \leq S_1(C,B)$ and $S_2(C,A) \leq S_2(C,B)$;

(3) $A \subseteq B$ iff $S_1(A,B) = 1$ iff $S_2(A,B) = 1$;

(4) $S_1(B,C) * S_1(A,B) \leq S_1(A,C)$;

(5) $S_2(A,B) * S_2(B,C) \leq S_2(A,C)$.

In what follows, we prefer to call universes the sets from where the fuzzy subsets are taken, while the fuzzy subsets shall be fuzzy sets inside the respective universes. L shall denote a fixed generalized residuated lattice, taken as the structure of truth values.

3. Closure operators

During this section, X will be a fixed non-empty universe.

Definition 3.1. A function $C\colon L^X \longrightarrow L^X$ is called an *L-closure operator* if the following properties hold for each $A, A_1, A_2 \in L^X$:

1) $A \subseteq C(A)$;

2) $S_1(A_1, A_2) \leq S_1\big(C(A_1), C(A_2)\big)$; $S_2(A_1, A_2) \leq S_2\big(C(A_1), C(A_2)\big)$;
3) $C(A) = CC(A)$.

Proposition 3.1. *The following are equivalent for a function* $C\colon L^X \longrightarrow L^X$:

a) *C is an L-closure operator;*
b) *For all $A, A_1, A_2 \in L^X$, we have $A \subseteq C(A)$,*
$S_1\big(A_1, C(A_2)\big) \leq S_1\big(C(A_1), C(A_2)\big)$ *and* $S_2(A_1, A_2) \leq S_2\big(C(A_1), C(A_2)\big)$;
c) *For all $A, A_1, A_2 \in L^X$, we have*
$S_1\big(A_1, C(A_2)\big) = S_1\big(C(A_1), C(A_2)\big)$ *and* $S_2\big(A_1, C(A_2)\big) = S_2\big(C(A_1), C(A_2)\big)$.

Proof. c) implies b): All we need to prove is $A \subseteq C(A)$, which follows from $1 = S_1\big(C(A), C(A)\big) = S_1\big(A, C(A)\big)$.
a) implies c): Since $A_1 \subseteq C(A_1)$ and S_1 is decreasing on the first argument, we have $S_1\big(A_1, C(A_2)\big) \geq S_1\big(C(A_1), C(A_2)\big)$. Now, by 2) and 3) from Definition 3.1,

$$S_1\big(A_1, C(A_2)\big) \leq S_1\big(C(A_1), CC(A_2)\big) = S_1\big(C(A_1), C(A_2)\big).$$

The other equality from c) follows analogously.
b) implies a): We need to show 2) and 3) from Definition 3.1. First, we prove 2).

$$S_1(A_1, A_2) \leq S_1\big(A_1, C(A_2)\big) \leq S_1\big(C(A_1), C(A_2)\big)$$

(we applied the increasingness of S_1 on the second argument and $A_2 \subseteq C(A_2)$). The fact that $S_2(A_1, A_2) \leq S_2\big(C(A_1), C(A_2)\big)$ follows similarly.
We now prove 3): $1 = S_1\big(C(A), C(A)\big) \leq S_1\big(CC(A), C(A)\big)$, hence $CC(A) \subseteq C(A)$. But we know that $C(A) \subseteq C\big(C(A)\big)$. So $CC(A) = C(A)$. □

Definition 3.2. A nonempty set $\mathcal{S} = (A_i)_{i\in I} \subseteq \mathcal{P}(L^X)$ is called an *L-closure system* if, for any $A \in L^X$,

$$\bigwedge_{i\in I}\big(S_1(A, A_i) \Rightarrow A_i\big) \in \mathcal{S},$$

$$\bigwedge_{i\in I}\big(S_2(A, A_i) \rightarrow A_i\big) \in \mathcal{S}.$$

Remark 3.1. Notice that $S_1(A, A_i) \Rightarrow A_i$ and $S_2(A, A_i) \rightarrow A_i$ are functions from L^X, $S_1(A, A_i)$ and $S_2(A, A_i)$ being treated here as constant functions. The same happens for a_i from the next proposition. As mentioned before, L^X is a complete lattice, where suprema and infima are taken pointwise. Moreover, since generalized residuated lattices form a semi-variety, L^X, with pointwise operations, is a generalized residuated lattice.

PROPOSITION 3.2. *A nonempty $\mathcal{S} = (A_i)_{i\in I} \subseteq \mathcal{P}(L^X)$ is an L-closure system iff, for all $(a_i)_{i\in I} \subseteq L$,*

$$\bigwedge_{i\in I}(a_i \to A_i) \in \mathcal{S} \quad \text{and} \quad \bigwedge_{i\in I}(a_i \Rightarrow A_i) \in \mathcal{S}\,.$$

P r o o f . Suppose first that the two above conditions hold. Let $A \in L^X$. We first define $a_i = S_2(A, A_i)$ and then $a_i = S_1(A, A_i)$ for all $i \in I$, getting the desired properties from Definition 3.2.

Conversely, suppose S is an L-closure system and let $(a_i)_{i\in I} \subseteq L$. We want to show that $A = \bigwedge_{i\in I}(a_i \to A_i) \in \mathcal{S}$. It is sufficient to prove that $\bigwedge_{i\in I}\big(S_2(A, A_i) \to A_i\big) = A$. Let $x \in X$.

$$A(x) \le \bigwedge_{i\in I}\big(S_2(A, A_i) \to A_i(x)\big)$$

means

$$\forall\, i \in I, \qquad A(x) \le S_2(A, A_i) \to A_i(x)\,,$$

that is

$$\forall\, i \in I, \qquad A(x) * S_2(A, A_i) \le A_i(x)\,,$$

which is true since

$$A(x) * S_2(A, A_i) \le A(x) * \big(A(x) \Rightarrow A_i(x)\big) \le A_i(x)\,.$$

For the converse inequality, we need to show

$$\bigwedge_{i\in I}\big(S_2(A, A_i) \to A_i(x)\big) \le \bigwedge_{i\in I}\big(a_i \to A_i(x)\big)\,.$$

It is sufficient to show that

$$\forall\, i \in I, \qquad S_2(A, A_i) \to A_i(x) \le a_i \to A_i(x)\,.$$

Further, it suffices to show

$$\forall\, i \in I, \qquad a_i \le S_2(A, A_i)\,,$$

which means, consecutively,

$$\forall\, i \in I, \qquad a_i \le \bigwedge_{y\in X}\big(A(y) \Rightarrow A_i(y)\big)\,,$$

$$\forall\, i \in I,\ \forall\, y \in X, \qquad a_i \le A(y) \Rightarrow A_i(y)\,,$$

$$\forall\, i \in I,\ \forall\, y \in X, \qquad A(y) * a_i \le A_i(y)\,,$$

$$\forall\, i \in I,\ \forall\, y \in X, \qquad \left(\bigwedge_{j \in I} (a_j \to A_j(y)) \right) * a_i \le A_i(y)\,,$$

$$\forall\, i \in I,\ \forall\, y \in X, \qquad \left(\bigwedge_{j \in I} (a_j \to A_j(y)) \right) \le a_i \to A_i(y)\,,$$

the last being obviously true. The fact that $\bigwedge\limits_{i\in I} (a_i \Rightarrow A_i) \in \mathcal{S}$ can be proved similarly. □

Remark 3.2. Notice that if $\mathcal{S} \subseteq \mathcal{P}(L^X)$ is closed to arbitrary infima, then $\mathcal{S}$ is an L-closure system iff, for all $a \in L$ and $A \in \mathcal{S}$, $a \to A \in \mathcal{S}$.

Proposition 3.3. *Let $\mathcal{S} = (A_i)_{i\in I} \subseteq \mathcal{P}(L^X)$ be an L-closure system. Then, for each $A \in L^X$, we have*

$$\bigwedge_{i \in I} (S_2(A, A_i) \to A_i) = \bigwedge_{i \in I} = (S_1(A, A_i) \Rightarrow A_i) = \bigwedge_{i\in I,\, A \subseteq A_i} A_i\,.$$

P r o o f. We want to prove $\bigwedge\limits_{i \in I} (S_1(A, A_i) \Rightarrow A_i) = \bigwedge\limits_{i\in I,\, A \subseteq A_i} A_i$. First,

$$\bigwedge_{i \in I} (S_1(A, A_i) \Rightarrow A_i(x)) \le \bigwedge_{i \in I,\, S_1(A, A_i) = 1} (1 \Rightarrow A_i(x)) = \bigwedge_{i\in I,\, A \subseteq A_i} A_i(x)\,.$$

For the converse inequality, notice that

$$\forall\, i \in I, \qquad A(x) \le S_1(A, A_i) \Rightarrow A_i(x)\,,$$

because

$$S_1(A, A_i) * A(x) \le \big(A(x) \to A_i(x)\big) * A(x) \le A_i(x)\,.$$

Now, since $\bigwedge\limits_{i \in I} (S_1(A, A_i) \Rightarrow A_i) \in \mathcal{S}$, we have $\bigwedge\limits_{i\in I,\, A \subseteq A_i} A_i \le \bigwedge\limits_{i \in I} (a_i \Rightarrow A_i)$.

The fact that $\bigwedge\limits_{i \in I} (S_2(A, A_i) \to A_i) = \bigwedge\limits_{i\in I,\, A \subseteq A_i} A_i$ follows analogously. □

Remark 3.3. The previous proposition gives us another sample of unification against the "non-commutative splitting". Provided they are in $\mathcal{S}$, the two possible fuzzy closures of a set A coincide and they are the same as, say, the classical closure, given by the intersection of all the upper bounds of A from the given closure system.

We now state the bijective correspondence between L-closure operators and L-closure systems.

Proposition 3.4.

1) Let $\mathcal{S} \subseteq \mathcal{P}(L^X)$ be an L-closure system. Define $C_{\mathcal{S}}\colon L^X \longrightarrow L^X$ by

$$C_{\mathcal{S}}(A) = \bigwedge_{B\in\mathcal{S},\, A\subseteq B} B, \qquad \forall\, A \in L^X .$$

Then $C_{\mathcal{S}}$ is an L-closure operator.

2) Let $C\colon L^X \longrightarrow L^X$ be an L-closure operator. Define

$$\mathcal{S}_C = \{A \in L^X \colon A = C(A)\} .$$

Then $\mathcal{S}_C$ is an L-closure system.

3) The mappings $\mathcal{S} \mapsto C_{\mathcal{S}}$ and $C \mapsto \mathcal{S}_C$ are inverse to each other.

Proof.

1): The fact that $A \subseteq C_{\mathcal{S}}(A)$ is obvious. Also, since, by Proposition 3.3, $C_{\mathcal{S}}(A) \in \mathcal{S}$, we have

$$C_{\mathcal{S}}(A) = \min\{B \in \mathcal{S}, C_{\mathcal{S}}(A) \subseteq B\} = \bigwedge_{B\in\mathcal{S},\, C_{\mathcal{S}}(A)\subseteq B} B = C_{\mathcal{S}}(C_{\mathcal{S}}(A)) .$$

We now show that $S_1(A_1, A_2) \le S_1(C_{\mathcal{S}}(A_1), C_{\mathcal{S}}(A_2))$ for each $A_1, A_2 \in L^X$. We need to prove

$$S_1(A_1, A_2) \le \bigwedge_{x\in X} \left(C_{\mathcal{S}}(A_1)(x) \to C_{\mathcal{S}}(A_2)(x)\right) .$$

This condition reduces, successively, to:

$$\begin{aligned} &\forall\, x \in X, \qquad S_1(A_1, A_2) \le C_{\mathcal{S}}(A_1)(x) \to C_{\mathcal{S}}(A_2)(x) , \\ &\forall\, x \in X, \qquad S_1(A_1, A_2) * C_{\mathcal{S}}(A_1)(x) \le C_{\mathcal{S}}(A_2)(x) \end{aligned}$$

and further, using Proposition 3.3, to

$$\begin{aligned} &\forall\, x \in X,\, \forall\, B \in \mathcal{S}, \qquad S_1(A_1, A_2) * C_{\mathcal{S}}(A_1)(x) \le S_1(A_2, B) \Rightarrow B(x) , \\ &\forall\, x \in X,\, \forall\, B \in \mathcal{S}, \qquad S_1(A_2, B) * S_1(A_1, A_2) * C_{\mathcal{S}}(A_1)(x) \le B(x) . \end{aligned}$$

Since $S_1(A_2, B) * S_1(A_1, A_2) \le S_1(A_1, B)$, it suffices to show

$$\forall\, x \in X,\, \forall\, B \in \mathcal{S}, \qquad S_1(A_1, B) * C_{\mathcal{S}}(A_1)(x) \le B(x) ,$$

that is

$$\forall\, x \in X,\, \forall\, B \in \mathcal{S}, \qquad C_{\mathcal{S}}(A_1)(x) \le S_1(A_1, B) \Rightarrow B(x) ,$$

which is true since, by Proposition 3.3,

$$C_{\mathcal{S}}(A_1)(x) = \bigwedge_{D\in\mathcal{S}} \left(S_1(A_1, D) \Rightarrow D(x)\right) .$$

The fact that $S_2(A_1, A_2) \le S_2\big(C_{\mathcal{S}}(A_1), C_{\mathcal{S}}(A_2)\big)$ has a similar proof.

2): We want to prove that, for each $A \in L^X$, $D = \bigwedge_{B \in \mathcal{S}_C} \big(S_1(A, B) \Rightarrow B\big) \in \mathcal{S}_C$. It suffices to show that $D = C(A)$, because then we would get $C(D) = CC(A) = C(A) = D$, hence $D \in \mathcal{S}_C$. First, we have

$$\bigwedge_{B \in \mathcal{S}_C} (S_1(A, B) \Rightarrow B(x)) \le S_1\big(A, C(A)\big) \Rightarrow C(A)(x) = 1 \Rightarrow C(A)(x) = C(A)(x).$$

For the converse inequality, take $x \in X$. We need to show that

$$\forall\, B \in \mathcal{S}_C, \qquad C(A)(x) \le S_1(A, B) \Rightarrow B(x),$$

that is

$$\forall\, B \in \mathcal{S}_C, \qquad S_1(A, B) * C(A)(x) \le B(x),$$

which is true, since

$$\begin{aligned} S_1(A, B) * C(A)(x) &\le S_1\big(C(A), C(B)\big) * C(A)(x) \\ &\le \big(C(A)(x) \to C(B)(x)\big) * C(A)(x) \le C(B)(x) = B(x). \end{aligned}$$

Similarly, one can prove that $\bigwedge_{B \in \mathcal{S}_C} \big(S_2(A, B) \to B\big) \in \mathcal{S}_C$ for each $A \in L^X$.

3): We have already shown, at 2), that, for an L-closure operator C, for each $A \in L^X$, $\bigwedge_{B \in \mathcal{S}_C} (S_1(A, B) \Rightarrow B) = C(A)$. Applying Proposition 3.3, we get $C = C_{\mathcal{S}_C}$.

Now, let $\mathcal{S}$ be an L-closure system. $\mathcal{S} = \mathcal{S}_{C_{\mathcal{S}}}$ means that, for each $A \in L^X$, $A = C_{\mathcal{S}}(A)$ iff $A \in \mathcal{S}$. Obviously, $A \in \mathcal{S}$ implies $A = C_{\mathcal{S}}(A)$. Conversely, if $A = C_{\mathcal{S}}(A) = \bigwedge_{B \in \mathcal{S}, A \subseteq B} B$, then $A \in \mathcal{S}$ because, by Proposition 3.3, $\bigwedge_{B \in \mathcal{S}, A \subseteq B} B \in \mathcal{S}$.

□

4. Galois connections and concept lattices

For the whole section, we fix two nonempty universes, X and Y.

Definition 4.1. ([18]) An *L-Galois connection* between X and Y is a quadruple $(\uparrow, \Uparrow, \downarrow, \Downarrow)$ of functions

$$\uparrow\colon L^X \longrightarrow L^Y, \quad \Uparrow\colon L^X \longrightarrow L^Y, \quad \downarrow\colon L^Y \longrightarrow L^X, \quad \Downarrow\colon L^Y \longrightarrow L^X$$

such that the following conditions are fulfilled:

(G1) $S_1(A_1, A_2) \le S_2\big(A_2^{\uparrow}, A_1^{\uparrow}\big)$; $S_2(A_1, A_2) \le S_1\big(A_2^{\Uparrow}, A_1^{\Uparrow}\big)$;

(G2) $S_1(B_1, B_2) \le S_2\big(B_2^{\downarrow}, B_1^{\downarrow}\big)$; $S_2(B_1, B_2) \le S_1\big(B_2^{\Downarrow}, B_1^{\Downarrow}\big)$;

(G3) $A \subseteq A^{\uparrow\Downarrow}$; $A \subseteq A^{\Uparrow\downarrow}$;

(G4) $B \subseteq B^{\downarrow\Uparrow}$; $B \subseteq B^{\Downarrow\uparrow}$.

The following two propositions, the first giving an alternative definition for L-Galois connections, are also proved in [18].

Proposition 4.1. *A quadruple* $(\uparrow, \Uparrow, \downarrow, \Downarrow)$ *of functions*

$$\uparrow, \Uparrow \colon L^X \longrightarrow L^Y, \qquad \downarrow, \Downarrow \colon L^Y \longrightarrow L^X$$

forms an L*-Galois connection iff* $S_1(A, B^{\Downarrow}) = S_2(B, A^{\uparrow})$, $S_2(A, B^{\downarrow}) = S_1(B, A^{\Uparrow})$ *for all* $A \in L^X$, $B \in L^Y$.

Proposition 4.2. *Let* $(\uparrow, \Uparrow, \downarrow, \Downarrow)$ *be an* L*-Galois connection. Then, for any* $a \in L$, $x \in X$, $y \in Y$, $(A_j)_{j \in J} \subseteq L^X$ *and* $(B_j)_{j \in J} \subseteq L^Y$ *we have:*

(1) $\{1|x\}^{\uparrow}(y) = \{1|y\}^{\Downarrow}(x)$;

(2) $\{1|x\}^{\S}(y) = \{1|y\}^{\downarrow}(x)$;

(3) $a \Rightarrow \{1|x\}^{\Uparrow}(y) = \{a|x\}^{\Uparrow}(y)$;

(4) $a \to \{1|x\}^{\uparrow}(y) = \{a|x\}^{\uparrow}(y)$;

(5) $a \Rightarrow \{1|y\}^{\Downarrow}(x) = \{a|y\}^{\Downarrow}(x)$;

(6) $a \to \{1|y\}^{\downarrow}(x) = \{a|y\}^{\downarrow}(x)$;

(7) $\left(\bigvee_{j \in J} A_j\right)^{\uparrow} = \bigwedge_{j \in J} A_j^{\uparrow}$;

(8) $\left(\bigvee_{j \in J} A_j\right)^{\Uparrow} = \bigwedge_{j \in J} A_j^{\Uparrow}$;

(9) $\left(\bigvee_{j \in J} B_j\right)^{\downarrow} = \bigwedge_{j \in J} B_j^{\downarrow}$;

(10) $\left(\bigvee_{j \in J} B_j\right)^{\Downarrow} = \bigwedge_{j \in J} B_j^{\Downarrow}$.

In the classical case, as well as in the fuzzy commutative case ([8]), there exists a natural relation between Galois connections and closure operators. This relation also holds for the non-commutative case, just that we need to split the notion of closure operator into two weaker forms, corresponding to the two implications (the right one, $\to$ and the left one, $\Rightarrow$).

Definition 4.2. A function $C \colon L^X \longrightarrow L^X$ is called a *right* L*-closure operator* (a *left* L*-closure operator*) iff the following properties hold for each $A, A_1, A_2 \in L^X$:

1) $A \subseteq C(A)$;

2) $S_1(A_1, A_2) \leq S_1(C(A_1), C(A_2))$
(respectively, $S_2(A_1, A_2) \leq S_2(C(A_1), C(A_2))$);

3) $C(A) = CC(A)$.

Notice that C is an L-closure operator iff it is both a right and a left L-closure operator. Similarly, we define right and left L-closure systems.

Definition 4.3. A nonempty set $\mathcal{S} = (A_i)_{i\in I} \subseteq \mathcal{P}(L^X)$ is called a right L-closure system (a left L-closure system) iff, for any $A \in L^X$,

$$\bigwedge_{i\in I} \big(S_1(A, A_i) \Rightarrow A_i\big) \in \mathcal{S}\,,$$

$$\bigwedge_{i\in I} \big(S_2(A, A_i) \rightarrow A_i\big) \in \mathcal{S}\,.$$

An easy adaptation of the proofs from the previous section gives us:

Proposition 4.3.

1) *The following statements equivalent for a function $C\colon L^X \longrightarrow L^X$:*
   a) *C is a right (left) L-closure operator;*
   b) *For all $A, A_1, A_2 \in L^X$, we have $A \subseteq C(A)$, $S_1\big(A_1, C(A_2)\big) \leq S_1\big(C(A_1), C(A_2)\big)$, $S_2(A_1, A_2) \leq S_2\big(C(A_1), C(A_2)\big)$;*
   c) *For all $A, A_1, A_2 \in L^X$, we have $S_1\big(A_1, C(A_2)\big) = S_1\big(C(A_1), C(A_2)\big)$, $S_2(A_1, A_2) = S_2\big(C(A_1), C(A_2)\big)$;*
2) *A nonempty $\mathcal{S} = (A_i)_{i\in I} \subseteq \mathcal{P}(L^X)$ is a right L-closure system (a left L-closure system) iff, for all $(a_i)_{i\in I} \subseteq L$,*

$$\bigwedge_{i\in I} (a_i \Rightarrow A_i) \in \mathcal{S}\,,$$

$$\bigwedge_{i\in I} (a_i \rightarrow A_i) \in \mathcal{S}\,.$$

3) *Let $\mathcal{S} = (A_i)_{i\in I} \subseteq \mathcal{P}(L^X)$ be a right (left) L-closure system. Then, for each $A \in L^X$, we have*

$$\bigwedge_{i\in I} (a_i \Rightarrow A_i) = \bigwedge_{i\in I,\, A\subseteq A_i} A_i\,,$$

$$\bigwedge_{i\in I} (a_i \rightarrow A_i) = \bigwedge_{i\in I,\, A\subseteq A_i} A_i\,.$$

4) *There exists a bijective correspondence between right (left) L-closure operators and right (left) L-closure systems. This correspondence is given in the same way as in Proposition 3.4.*

Notice that considering a left or right L-closure system (operator) implies making a choice between the two possible implications, hence between the two possible ways of taking the subsethood degree. It is thus natural to consider the right L-closure systems equipped with the right subsethood degree and the left ones with the left subsethood degree.

Definition 4.4. Any left or right L-closure system (operator) shall be called *weak L-closure system* (operator).

Let $\mathcal{U}_1$ and $\mathcal{U}_2$ be two weak L-closure systems and a bijective function $\alpha\colon \mathcal{U}_1 \longrightarrow \mathcal{U}_2$. α is said to be an *anti-isomorphism* iff, for all $A_1, A_2 \in \mathcal{U}_1$, we have $D_1(A_1, A_2) = D_2\big(\alpha(A_2), \alpha(A_1)\big)$, where, for $i \in \{1,2\}$, $D_i = S_1$ if $\mathcal{U}_i$ is a right L-closure operator and $D_i = S_2$ if $\mathcal{U}_i$ is a left one.

For each weak L-closure system $\mathcal{S}$, we denote by $C_{\mathcal{S}}$ the associated weak L-closure operator; also, if C is a weak L-closure operator, $\mathcal{S}_C$ will denote its corresponding weak L-closure system.

An L-Galois connection $(\uparrow, \Uparrow, \downarrow, \Downarrow)$ between X and Y induces two right L-closure operators ($\uparrow\Downarrow$ on X and $\downarrow\Uparrow$ on Y) and two left L-closure operators ($\Uparrow\downarrow$ on X and $\downarrow\Uparrow$ on Y), such that $\mathcal{S}_{\uparrow\Downarrow}$ and $\mathcal{S}_{\Downarrow\uparrow}$ are anti-isomorphic, and so are $\mathcal{S}_{\Uparrow\uparrow}$ and $\mathcal{S}_{\downarrow\Uparrow}$. In fact, if we consider the four above mentioned weak L-closure operators together with the two anti-isomorphisms, we entirely determine the L-Galois connection we started with.

Proposition 4.4.

1) Let $\mathcal{G} = (\uparrow, \Uparrow, \downarrow, \Downarrow)$ be an L-Galois connection between X and Y. Let

$$\mathcal{M}_{\mathcal{G}} = (\mathcal{U}_1, \alpha, \mathcal{T}_1, \mathcal{U}_2, \beta, \mathcal{T}_2),$$

where $\mathcal{U}_1 = \mathcal{S}_{\uparrow\Downarrow}$, $\mathcal{T}_1 = \mathcal{S}_{\Downarrow\uparrow}$, $\mathcal{U}_2 = \mathcal{S}j$, $\mathcal{T}_2 = \mathcal{S}_{\downarrow\Downarrow}$, while $\alpha\colon \mathcal{U}_1 \longrightarrow \mathcal{T}_1$ and $\beta\colon \mathcal{U}_2 \longrightarrow \mathcal{T}_2$ are defined by: $\alpha(A) = A^{\uparrow}$ for all $A \in \mathcal{U}_1$ and $\beta(A) = A^{\Uparrow}$ for all $A \in \mathcal{U}_2$.

Then $\uparrow\Downarrow$ and $\downarrow\Uparrow$ are right L-closure operators, $\Uparrow\downarrow$ and $\Downarrow\uparrow$ are left L-closure operators (hence $\mathcal{U}_1$ and $\mathcal{T}_2$ are right and $\mathcal{U}_2$ and $\mathcal{T}_1$ are left L-closure systems) and α and β are weak L-closure system anti-isomorphisms.

2) Let $\mathcal{M} = (\mathcal{U}_1, \alpha, \mathcal{T}_1, \mathcal{U}_2, \beta, \mathcal{T}_2)$, where $\mathcal{U}_1, \mathcal{U}_2 \subseteq \mathcal{P}(L^X)$, $\mathcal{T}_1, \mathcal{T}_2 \subseteq \mathcal{P}(L^Y)$ such that $\mathcal{U}_1$ and $\mathcal{T}_2$ are right L-closure systems, $\mathcal{U}_2$ and $\mathcal{T}_1$ are left L-closure systems and $\alpha\colon \mathcal{U}_1 \longrightarrow \mathcal{T}_1$ and $\beta\colon \mathcal{U}_2 \longrightarrow \mathcal{T}_2$ are weak L-closure system anti-isomorphisms.

Define $\mathcal{G}_{\mathcal{M}} = (\uparrow, \Uparrow, \downarrow, \Downarrow)$, where

$$\uparrow\colon L^X \longrightarrow L^Y, \quad \Uparrow\colon L^X \longrightarrow L^Y, \quad \downarrow\colon L^Y \longrightarrow L^X, \quad \Downarrow\colon L^Y \longrightarrow L^X$$

are given by:

$$A^{\uparrow} = \alpha\big(C_{\mathcal{U}_1}(A)\big), \qquad A^{\Uparrow} = \beta\big(C_{\mathcal{U}_2}(A)\big),$$
$$B^{\downarrow} = \beta^{-1}\big(C_{\mathcal{T}_2}(B)\big), \quad B^{\Downarrow} = \alpha^{-1}\big(C_{\mathcal{T}_1}(B)\big),$$

for all $A \in L^X$, $B \in L^Y$. *Then* $\mathcal{G}_{\mathcal{M}}$ *is an* L*-Galois connection.*

3) The correspondences $\mathcal{G} \mapsto \mathcal{M}_{\mathcal{G}}$ *and* $\mathcal{M} \mapsto \mathcal{G}_{\mathcal{M}}$ *are bijective and inverse to each other.*

Proof. 1): The fact that $\uparrow\Downarrow$ and $\downarrow\Uparrow$ are right and $\Uparrow\downarrow$ and $\Downarrow\uparrow$ are left L-closure operators follows immediately from the definition of L-Galois connection and the properties of the subsethood degrees. Now, α is surjective because, for each $B \in \mathcal{T}_1$, we have $B = B^{\Downarrow\uparrow}$, so $(B^{\Downarrow})^{\uparrow\Downarrow} = B^{\Downarrow}$, which means $B^{\Downarrow} \in \mathcal{U}_1$ and $\alpha(B^{\Downarrow}) = B^{\Downarrow\uparrow} = B$. Further, α is injective because, for all $A_1, A_2 \in \mathcal{U}_1$, $\alpha(A_1) = \alpha(A_2)$ means $A_1^{\uparrow} = A_2^{\uparrow}$, which implies $A_1 = A_1^{\uparrow\Downarrow} = A_2^{\uparrow\Downarrow} = A_2$. Moreover, by the L-Galois connection definition, for all $A_1, A_2 \in \mathcal{U}_1$, we have

$$\mathcal{S}_1(A_1, A_2) \le \mathcal{S}_2(A_2^{\uparrow}, A_1^{\uparrow}) \le \mathcal{S}_1(A_1^{\uparrow\Downarrow}, A_2^{\uparrow\Downarrow}) = \mathcal{S}_1(A_1, A_2)\,.$$

Hence α is an anti-isomorphism. Similarly, β is an anti-isomorphism.

2): In order to prove that $\mathcal{G}_{\mathcal{M}}$ is an L-Galois connection, we apply the criterion from Proposition 4.1. Let $A \in L^X$ and $B \in L^Y$. We want to prove $\mathcal{S}_1(A, B^{\Downarrow}) = \mathcal{S}_2(B, A^{\uparrow})$. We have

$$\begin{aligned}\mathcal{S}_1(A, B^{\Downarrow}) &= \mathcal{S}_1\big(A, \alpha^{-1}\big(C_{\mathcal{T}_1}(B)\big)\big) = \mathcal{S}_1\big(C_{\mathcal{U}_1}(A), \alpha^{-1}\big(C_{\mathcal{T}_1}(B)\big)\big)\\ &= \mathcal{S}_2\Big(\alpha\big(\alpha^{-1}\big(C_{\mathcal{T}_1}(B)\big)\big), \alpha\big(C_{\mathcal{U}_1}(A)\big)\Big) = \mathcal{S}_2\big(C_{\mathcal{T}_1}(B), \alpha\big(C_{\mathcal{U}_1}(A)\big)\big)\\ &= \mathcal{S}_2\big(B, \alpha\big(C_{\mathcal{U}_1}(A)\big)\big) = \mathcal{S}_2(B, A^{\uparrow})\end{aligned}$$

(we used Proposition 3.1, first for $C_{\mathcal{U}_1}$, then for $C_{\mathcal{T}_1}$).

Similarly, we get $\mathcal{S}_2(A, B^{\downarrow}) = \mathcal{S}_1(B, A^{\Uparrow})$.

3): Let $\mathcal{G}$ be an L-Galois connection and $\mathcal{M}$ be a six-tuple like at point 2).

The fact that $\mathcal{G}_{\mathcal{M}_{\mathcal{G}}} = \mathcal{G}$ follows from

$$A^{\uparrow} = A^{\uparrow\Downarrow\uparrow} = \big(C_{\mathcal{U}_1}(A)\big)^{\uparrow} = \alpha\big(C_{\mathcal{U}_1}(A)\big)\,, \qquad \forall\, A \in L^X$$

and the other three similar statements for $\Uparrow$, $\downarrow$ and $\Downarrow$.

Now, $\mathcal{M}_{\mathcal{G}_{\mathcal{M}}} = \mathcal{M}$ follows easily. For example, let $\mathcal{U}_1$ be the first component of the six-tuple $\mathcal{M}$ and $\mathcal{U}_1'$ that of the six-tuple $\mathcal{M}_{\mathcal{G}_{\mathcal{M}}}$ and let $A \in L^X$. Also, let $\mathcal{G}_{\mathcal{M}} = (\uparrow, \Uparrow, \downarrow, \Downarrow)$. We have

$$\begin{aligned}C_{\mathcal{U}_1'}(A) = A^{\uparrow\Downarrow} &= \big(\alpha\big(C_{\mathcal{U}_1}(A)\big)\big)^{\Downarrow} = \alpha^{-1}\Big(C_{\mathcal{T}_1}\big(\alpha\big(C_{\mathcal{U}_1}(A)\big)\big)\Big)\\ &= \alpha^{-1}\big(\alpha\big(C_{\mathcal{U}_1}(A)\big)\big) = C_{\mathcal{U}_1}(A)\end{aligned}$$

(we used the fact that, since $C_{\mathcal{U}_1}(A) \in \mathcal{U}_1$, we have $\alpha\big(C_{\mathcal{U}_1}(A)\big) \in \mathcal{T}_1$, hence $C_{\mathcal{T}_1}\big(\alpha\big(C_{\mathcal{U}_1}(A)\big)\big) = \alpha\big(C_{\mathcal{U}_1}(A)\big)$). $\square$

Any binary L-relation $I \in L^{X\times Y}$ naturally induces an L-Galois connection. Define the four functions

$$\uparrow_I\colon L^X \longrightarrow L^Y, \quad \Uparrow_I\colon L^X \longrightarrow L^Y, \quad \downarrow_I\colon L^Y \longrightarrow L^X, \quad \Downarrow_I\colon L^Y \longrightarrow L^X$$

by

$$A^{\uparrow_I}(y) = \bigwedge_{x\in X} \big(A(x) \to I(x,y)\big)\,,$$
$$A^{\Uparrow_I}(y) = \bigwedge_{x\in X} \big(A(x) \Rightarrow I(x,y)\big)\,,$$
$$B^{\downarrow_I}(x) = \bigwedge_{y\in Y} \big(B(y) \to I(x,y)\big)\,,$$
$$B^{\Downarrow_I}(x) = \bigwedge_{y\in Y} \big(B(y) \Rightarrow I(x,y)\big)\,,$$

for any $A \in L^X$, $B \in L^Y$ and $x \in X$, $y \in Y$.

The following facts are proved in [18]:

Proposition 4.5.

1) *For any binary L-relation $I \in L^{X\times Y}$, the quadruple $(\uparrow_I, \Uparrow_I, \downarrow_I, \Downarrow_I)$ is an L-Galois connection.*
2) *For an L-Galois connection $(\uparrow, \Uparrow, \downarrow, \Downarrow)$, the following conditions are equivalent:*
   (a) $\{1|x\}^{\uparrow} = \{1|x\}^{\Uparrow}$ *for all* $x \in X$*;*
   (b) $\{1|y\}^{\downarrow} = \{1|y\}^{\Downarrow}$ *for all* $y \in Y$*;*
   (c) *L is induced by some L-relation $I \in L^{X\times Y}$ (call* **strong** *any L-Galois connection that satisfies* (a))
   (d) *$I \mapsto (\uparrow_I, \Uparrow_I, \downarrow_I, \Downarrow_I)$ is a bijective correspondence from the set of L-relations to the set of strong L-Galois connections between X and Y given by $I \mapsto (\uparrow_I, \Uparrow_I, \downarrow_I, \Downarrow_I)$, its inverse being $(\uparrow, \Uparrow, \downarrow, \Downarrow) \mapsto I$, where, for all $x \in X$, $y \in Y$, $I(x,y) = \{1/x\}^{\uparrow}(y)$.*

Remark 4.1. Thus, in order for an L-Galois connection to be induced by an L-relation, it is necessary and sufficient that there exists some agreement between the component functions with respect to singletons.

L-Galois connections coming from L-relations are important because, considering the triples of fixed points, we get to the formal concepts in a given context.

Definition 4.5. ([19]) Call *context* any triple (X, Y, I), where $I \in L^{X\times Y}$. Let (X, Y, I) be a context. The triple $(A, B_1, B_2) \in L^{X\times Y\times Y}$ is said to be a *concept* (in this context) if:

$$A^{\uparrow_I} = B_1\,; \quad B_1^{\Downarrow_I} = A; \quad A^{\Uparrow_I} = B_2\,; \quad B_2^{\downarrow_I} = A\,.$$

Remark 4.2. As argued in [19], because of the two implication in our non-commutative fuzzy framework, any time we want to generalize a notion from the classical, crisp mathematics, we face a centrifuge tendency of splitting in two. Trying to treat unitarily the problem of formal concepts, we give priority to objects, considering properties just as tools helping to get to objects. Hence, a formal concept (or, simply, a concept) consists of one extent, but two intents, that "define" the extent in the two possible ways; we thus have, for each concept, the right ($\to$) intent and the left ($\Rightarrow$) intent.

The set of concepts in a given context,

$$\mathcal{L}(X, Y, I) = \left\{(A, B_1, B_2) \in L^{X\times Y\times Y} : A^{\uparrow} = B_1,\ B_1^{\Downarrow} = A,\ A^{\Uparrow} = B_2,\ B_2^{\downarrow} = A\right\},$$

equipped with the partial order $\leq$ defined as follows:

$$(A, B_1, B_2) \leq (A', B_1', B_2') \quad \text{iff} \quad A \subseteq A'$$
$$(\text{or, same thing},\ B_1' \subseteq B_1,\ \text{ or, same thing, } B_2' \subseteq B_2),$$

becomes a complete lattice, as proved in [19], where it is also given an abstract characterization of the lattice $\mathcal{L}(X, Y, I)$.

As we have seen, if the universe X of objects and the universe Y of attributes are provided, in order to have a concept structure (hierarchy), we can either establish a fuzzy binary relation between objects and attributes, or define a strong L-Galois connection between X and Y. A third possibility is using weak L-closure operators or systems.

Definition 4.6. Let $\mathcal{U}_1$ and $\mathcal{U}_2$ be two weak L-closure operators on a universe Z. They are said to *have the same starting point* if $C_{\mathcal{U}_1}$ and $C_{\mathcal{U}_2}$ coincide on crisp singletons, i.e., $C_{\mathcal{U}_1}(\{1/x\}) = C_{\mathcal{U}_2}(\{1/x\})$ for all $x \in Z$.

Using Proposition 4.5(d) and adapting Proposition 4.4, we immediately get:

Proposition 4.6. *The following sets are isomorphic:*

1) *The set of binary L-relations between X and Y;*
2) *The set of strong L-Galois connections between X and Y;*
3) *The set of six-tuples $(\mathcal{U}_1, \alpha, \mathcal{T}_1, \mathcal{U}_2, \beta, \mathcal{T}_2)$, where $\mathcal{U}_1, \mathcal{U}_2 \subseteq \mathcal{P}(L^X)$, $\mathcal{T}_1, \mathcal{T}_2 \subseteq \mathcal{P}(L^Y)$ such that $\mathcal{U}_1$ and $\mathcal{T}_2$ are right L-closure systems, $\mathcal{U}_2$ and $\mathcal{T}_1$ are left L-closure systems, $\mathcal{U}_1$ and $\mathcal{U}_2$ have the same starting point, $\mathcal{T}_1$ and $\mathcal{T}_2$ have the same starting point and $\alpha\colon \mathcal{U}_1 \longrightarrow \mathcal{T}_1$ and $\beta\colon \mathcal{U}_2 \longrightarrow \mathcal{T}_2$ are weak L-closure systems anti-isomorphisms.*

5. Concept equations

In this section, we shall consider the problem (partially treated in [19]) of solving concept equations inside a non-commutative fuzzy framework. Following quite closely [10], we shall generalize the results from there obtaining, in some cases, a non-commutative flavor and specificity.

Definition 5.1. Let X be a universe. Define $E_{\rightarrow}, E_{\Rightarrow} \in L^{L^X \times L^X}$ by

$$E_{\rightarrow}(A_1, A_2) = \bigwedge_{x \in X} \left(A_1(x) \leftrightarrow A_2(x)\right),$$
$$E_{\Rightarrow}(A_1, A_2) = \bigwedge_{x \in X} \left(A_1(x) \Leftrightarrow A_2(x)\right),$$
$$E(A_1, A_2) = E_{\rightarrow}(A_1, A_2) \wedge E_{\Rightarrow}(A_1, A_2).$$

for all $A_1, A_2 \in L^X$.

Remark 5.1. While E is an L-equality (see [19]), measuring the degree to which two L-sets are equal, $E_{\rightarrow}$ and $E_{\Rightarrow}$ are not, because each of them only considers half of the implication apparatus. However, they become useful in the cases where we are only interested in the respective half, as in Definition 5.2.

In what follows, no matter what universe we consider (be it X or Y) we shall use the same notation, $E_{\rightarrow}, E_{\Rightarrow}$, for the corresponding above defined fuzzy relations, the difference being made, each time, by the context.

Definition 5.2. Let X and Y be two universes. A *system of concept equations* will be any $T \subseteq L^X \times L^Y \times L^Y$. For each $I \in L^{X \times Y}$,

$$\mathrm{Sol}(T, I) = \bigwedge_{(A, B_1, B_2) \in T} E_{\rightarrow}(A^{\uparrow_I}, B_1) \wedge E_{\Rightarrow}(A^{\Uparrow_I}, B_2) \wedge E_{\Rightarrow}(A, B_1^{\Downarrow_I}) \wedge E_{\rightarrow}(A, B_2^{\downarrow_I})$$

is said to be the *degree* to which I is a solution of T. I is a full solution of T if $\mathrm{Sol}(T, I) = 1$, so we define

$$\mathrm{Sol}(T) = \left\{ I \in L^{X \times Y} :\ \mathrm{Sol}(T, I) = 1 \right\},$$

the set of full solutions of the system T. T is called *consistent* if $\mathrm{Sol}(T) \neq \emptyset$.

Remark 5.2.

a) $\mathrm{Sol}(T, I)$ is in fact the degree to which all the triples from T constitute concepts in the context of I. And a triple (A, B_1, B_2) constitutes a concept to the same degree to which B_1 is the right intent of A (i.e., B_1 is the fuzzy set of attributes of all the objects from A with respect to the right implication, $\rightarrow$), B_2 is the left intent of A and A is the left extent of B_1 and the right extent of B_2.

b) One can easily see that

$$\mathrm{Sol}(T) = \{I \in L^{X\times Y} : T \subseteq \mathcal{L}(X,Y,I)\}.$$

So I is a solution of T iff T forms a set of concepts in the context (X,Y,I).

For the rest of this section, we fix two universes X and Y and a system $T \subseteq L^{X\times Y\times Y}$ of concept equations.

Lemma 5.1. *Let $(I_j)_{j\in J} \subseteq L^{X\times Y}$ be a nonempty family of fuzzy relations. Denote $I = \bigwedge_{j\in J} I_j$. Then, for all $A \in L^X$, $B \in L^Y$, we have:*

$$A^{\uparrow_I} = \bigwedge_{j\in J} A^{\uparrow_{I_j}}, \quad A^{\Uparrow_I} = \bigwedge_{j\in J} A^{\Uparrow_{I_j}}, \quad B^{\downarrow_I} = \bigwedge_{j\in J} B^{\downarrow_{I_j}}, \quad B^{\Downarrow_I} = \bigwedge_{j\in J} B^{\Downarrow_{I_j}}.$$

P r o o f. We only prove $A^{\uparrow_I} = \bigwedge_{j\in J} A^{\uparrow_{I_j}}$, the other three equalities following similarly. Let $y \in Y$.

$$\begin{aligned}
A^{\uparrow_I}(y) &= \bigwedge_{x\in X}\big(A(x) \to I(x,y)\big) = \bigwedge_{x\in X}\Big(A(x) \to \Big(\bigwedge_{j\in J} I_j(x,y)\Big)\Big)\\
&= \bigwedge_{x\in X}\bigwedge_{j\in J}\big(A(x) \to I_j(x,y)\big) = \bigwedge_{j\in J}\bigwedge_{x\in X}\big(A(x) \to I_j(x,y)\big) = \bigwedge_{j\in J} A^{\uparrow_{I_j}}(y).
\end{aligned}$$

We used Lemma 2.1 (11, 13). □

Proposition 5.1. *Let $(I_j)_{j\in J} \subseteq L^{X\times Y}$ be a nonempty family of fuzzy relations. Then*

$$\bigwedge_{j\in J} \mathrm{Sol}(T, I_j) \le \mathrm{Sol}\Big(T, \bigwedge_{j\in J} I_j\Big).$$

P r o o f. Notice first that, for any $(a_j)_{j\in J} \subseteq L$ and $b \in L$,

$$\bigwedge_{j\in J}(a_j \to b) \le \Big(\bigwedge_{j\in J} a_j\Big) \to b \qquad \text{and} \qquad \bigwedge_{j\in J}(a_j \Rightarrow b) \le \Big(\bigwedge_{j\in J} a_j\Big) \Rightarrow b \quad (\#)$$

(because J is nonempty and $\to$, $\Rightarrow$ are decreasing on the first component).

Now, denote $I = \bigwedge_{j\in J} I_j$. We have that

$$\begin{aligned}
&\bigwedge_{j\in J} \mathrm{Sol}(T, I_j)\\
&= \bigwedge_{j\in J}\ \bigwedge_{(A,B_1,B_2)\in T} E_{\to}(A^{\Uparrow_{I_j}}, B_1) \wedge E_{\Rightarrow}(A^{\Uparrow_{I_j}}, B_2) \wedge E_{\Rightarrow}(A, B_1^{\Downarrow_{I_j}}) \wedge E_{\to}(A, B_2^{\downarrow_{I_j}})\\
&= \bigwedge_{(A,B_1,B_2)\in T}\ \bigwedge_{j\in J} E_{\to}(A^{\uparrow_{I_j}}, B_1) \wedge E_{\Rightarrow}(A^{\Uparrow_{I_j}}, B_2) \wedge E_{\Rightarrow}(A, B_1^{\Downarrow_{I_j}}) \wedge E_{\to}(A, B_2^{\downarrow_{I_j}}).
\end{aligned}$$

By the definition of $\mathrm{Sol}(T,I)$, it is sufficient to take $(A,B_1,B_2)\in T$ and prove:

$$\bigwedge_{j\in J} E_{\to}(A^{\uparrow I_j},B_1)\le E_{\to}(A^{\uparrow I},B_1);\quad \bigwedge_{j\in J} E_{\Rightarrow}(A^{\Uparrow I_j},B_2)\le E_{\Rightarrow}(A^{\Uparrow I},B_2);$$

$$\bigwedge_{j\in J} E_{\Rightarrow}(A,B_1^{\Downarrow I_j})\le E_{\Rightarrow}(A,B_1^{\Downarrow I});\quad \bigwedge_{j\in J} E_{\to}(A,B_2^{\downarrow I_j})\le E_{\to}(A,B_2^{\downarrow I}).$$

We only prove the first inequality, the others following similarly.

$$\begin{aligned}\bigwedge_{j\in J} E_{\to}(A^{\uparrow I_j},B_1) &= \bigwedge_{j\in J}\bigwedge_{y\in Y} A^{\uparrow I_j}(y)\leftrightarrow B_1(y)\\ &= \bigwedge_{y\in Y}\bigwedge_{j\in J}(A^{\uparrow I_j}(y)\to B_1(y))\wedge(B_1(y)\to A^{\uparrow I_j}(y))\\ &\le \bigwedge_{y\in Y}\left(\left(\bigwedge_{j\in J}A^{\uparrow I_j}(y)\right)\to B_1(y)\right)\wedge\left(B_1(y)\to\left(\bigwedge_{j\in J}A^{\uparrow I_j}(y)\right)\right)\\ &= \bigwedge_{y\in Y}(A^{\uparrow I}(y)\to B_1(y))\wedge(B_1(y)\to A^{\uparrow I}(y)) = E_{\to}(A^{\uparrow I},B_1).\end{aligned}$$

We used Lemma 2.1(11), (#) and Lemma 5.1. □

As a corollary we obtain the property that was proved directly in [19].

Corollary 5.1. $\mathrm{Sol}(T)$ *is a meet-semilattice inside which there always exist infima of nonempty families.*

Remark 5.3. Since $\mathrm{Sol}(T)$ satisfies the above property, it also has family suprema whenever those families have upper bounds.

The following two propositions are proved in [19]:

Proposition 5.2. *Let* $I\in L^{X\times Y}$ *and denote* $\mathcal{L}=\mathcal{L}(X,Y,I)$. *Then, for any* $x\in X$, $y\in Y$, *we have*

$$I(x,y)=\bigvee_{(A,B_1,B_2)\in\mathcal{L}}(B_1(y)*A(x))=\bigvee_{(A,B_1,B_2)\in\mathcal{L}}(A(x)*B_2(y)),$$

for all $x\in X$, $y\in Y$.

Proposition 5.3. *Define* $I_T\in L^{X\times Y}$ *by*

$$I_T(x,y)=\bigvee_{(A,B_1,B_2)\in T}(B_1(y)*A(x))\vee(A(x)*B_2(y)).$$

We have that:

1) T *is consistent iff* $I_T\in\mathrm{Sol}(T)$;
2) *If* T *is consistent, then* I_T *is the least element from* $\mathrm{Sol}(T)$.

As already mentioned, $\mathcal{L}(X,Y,I)$ is a complete lattice. In [19], it is proved that infima in $\mathcal{L}(X,Y,I)$ have a certain form:

PROPOSITION 5.4.

$$\begin{aligned}\bigwedge_{j\in J}(A_j,B_j,C_j) &= \left(\bigwedge_{j\in J}A_j,\ \left(\bigwedge_{j\in J}A_j\right)^{\uparrow},\ \left(\bigwedge_{j\in J}A_j\right)^{\Uparrow}\right)\\ &= \left(\bigwedge_{j\in J}A_j,\ \left(\bigvee_{j\in J}B_j\right)^{\Downarrow\uparrow},\ \left(\bigvee_{j\in J}C_j\right)^{\downarrow\Uparrow}\right),\end{aligned}$$

for any $((A_j,B_j,C_j))_{j\in J}\subseteq\mathcal{L}(X,Y,I)$.

As a consequence of Propositions 5.3 and 5.4, we have the following:

COROLLARY 5.2. *Define* I_T *like in Proposition 5.3 and consider that* $T=\{(A_p,A_p,A_p)\colon p\in P\}$. *Then* T *is consistent iff*

$$T\cup\left\{\left(\bigwedge_{p\in P'}A_p,\ \left(\bigwedge_{p\in P'}A_p\right)^{\uparrow_{I_T}},\ \left(\bigwedge_{p\in P'}A_p\right)^{\Uparrow_{I_T}}\right):\emptyset\neq P'\subseteq P\right\}$$

is consistent.

Proof. One implication is trivial. For the other, assume that T is consistent. Then, according to Proposition 5.3, $T\subseteq\mathcal{L}(X,Y,I_T)$. Now, the desired property follows from Proposition 5.4. $\square$

LEMMA 5.2. *Let* $I_1,I_2\in L^{X\times Y}$, $A\in L^X$, $B\in L^Y$. *Then, for all* $x\in X$, $y\in Y$, *we have:*

1) $E(I_1,I_2)\leq A^{\uparrow_{I_1}}(y)\to A^{\uparrow_{I_2}}(y)$;
2) $E(I_1,I_2)\leq A^{\Uparrow_{I_1}}(y)\Rightarrow A^{\Uparrow_{I_2}}(y)$;
3) $E(I_1,I_2)\leq B^{\Downarrow_{I_1}}(x)\Rightarrow B^{\Downarrow_{I_2}}(x)$;
4) $E(I_1,I_2)\leq B^{\downarrow_{I_1}}(x)\to B^{\downarrow_{I_2}}(x)$.

Proof. We only prove 1), 2)–4) following analogously. Let $y\in Y$. We have to show that

$$E(I_1,I_2)*A^{\uparrow_{I_1}}(y)\leq A^{\uparrow_{I_2}}(y),$$

that is

$$\forall\,x\in X,\qquad E(I_1,I_2)*A^{\uparrow_{I_1}}(y)\leq A(x)\to I_2(x,y).$$

Furthermore, this means

$$\forall\,x\in X,\qquad E(I_1,I_2)*A^{\uparrow_{I_1}}(y)*A(x)\leq I_2(x,y).$$

But this is true, since

$$\begin{aligned} E(I_1,I_2) * A^{\uparrow I_1}(y) * A(x) &\le \big(I_1(x,y) \to I_2(x,y)\big) * \big(A(x) \to I_1(x,y)\big) * A(x) \\ &\le \big(I_1(x,y) \to I_2(x,y)\big) * I_1(x,y) \le I_2(x,y) \end{aligned}$$

(we applied Lemma 2.1 (8, 4)). □

The next proposition gives a non-commutative fuzzy non-trivial refinement to the following trivial crisp assertion: "If I_1 is a solution of T and $I_2 = I_1$, then I_2 is also a solution of T".

Proposition 5.5. *Let $I_1, I_2 \in L^{X\times Y}$. Then*

$$\big(\mathrm{Sol}(T,I_1) * E(I_1,I_2)\big) \wedge \big(E(I_1,I_2) * \mathrm{Sol}(T,I_1)\big) \le \mathrm{Sol}(T,I_2)\,.$$

P r o o f. Let $(A,B_1,B_2) \in T$, $x \in X$, $y \in Y$. It is sufficient to show that:

(I) $E(I_1,I_2) * \mathrm{Sol}(T,I_1) \le B_1(y) \to A^{\uparrow I_2}(y)$;

(II) $\mathrm{Sol}(T,I_1) * E(I_1,I_2) \le A^{\uparrow I_2}(y) \to B_1(y)$;

(III) $\mathrm{Sol}(T,I_1) * E(I_1,I_2) \le B_2(y) \Rightarrow A^{\Uparrow I_2}(y)$;

(IV) $E(I_1,I_2) * \mathrm{Sol}(T,I_1) \le A^{\Uparrow I_2}(y) \Rightarrow B_2(y)$;

(V) $E(I_1,I_2) * \mathrm{Sol}(T,I_1) \le A(x) \to B_2^{\downarrow I_2}(x)$;

(VI) $\mathrm{Sol}(T,I_1) * E(I_1,I_2) \le B_2^{\downarrow I_2}(x) \to A(x)$;

(VII) $\mathrm{Sol}(T,I_1) * E(I_1,I_2) \le A(x) \Rightarrow B_1^{\Downarrow I_2}(x)$;

(VIII) $E(I_1,I_2) * \mathrm{Sol}(T,I_1) \le B_1^{\Downarrow I_2}(x) \Rightarrow A(x)$.

We shall prove (I) and (II). (I) is equivalent to

$$E(I_1,I_2) * \mathrm{Sol}(T,I_1) * B_1(y) \le A^{\uparrow I_2}(y)\,,$$

which is true because

$$\begin{aligned} E(I_1,I_2) * \mathrm{Sol}(T,I_1) * B_1(y) &\le E(I_1,I_2) * \big(B_1(y) \to A^{\uparrow I_1}(y)\big) * B_1(y) \\ &\le E(I_1,I_2) * A^{\uparrow I_1}(y) \le \big(A^{\uparrow I_1}(y) \to A^{\uparrow I_2}(y)\big) * A^{\uparrow I_1}(y) \le A^{\uparrow I_2}(y) \end{aligned}$$

(we used Lemma 2.1 (8) and Lemma 5.2 (1)).

(II) is equivalent to

$$\mathrm{Sol}(T,I_1) * E(I_1,I_2) * A^{\uparrow I_2}(y) \le B_1(y)\,.$$

The last is true, since

$$\begin{aligned} \mathrm{Sol}(T,I_1) * E(I_1,I_2) * A^{\uparrow I_2}(y) &\le \mathrm{Sol}(T,I_1) * \big(A^{\uparrow I_2}(y) \to A^{\uparrow I_1}(y)\big) * A^{\uparrow I_2}(y) \\ &\le \mathrm{Sol}(T,I_1) * A^{\uparrow I_1}(y) \le \big(A^{\uparrow I_1}(y) \to B_1(y)\big) * A^{\uparrow I_1}(y) \le B_1(y)\,. \end{aligned}$$

The inequalities (III)–(VIII) follow in the same fashion.

Now, (I)–(VIII) imply

$$\begin{aligned}\big(\mathrm{Sol}(T,I_1)*E(I_1,I_2)\big)\wedge\big(E(I_1,I_2)*\mathrm{Sol}(T,I_1)\big)&\le\big(B_1(y)\leftrightarrow A^{\uparrow_{I_2}}(y)\big)\\ \wedge\big(B_2(y)\Leftrightarrow A^{\Uparrow_{I_2}}(y)\big)\wedge\big(A(x)&\Leftrightarrow B_1^{\Downarrow_{I_2}}(x)\big)\wedge\big(A(x)\leftrightarrow B_2^{\downarrow_{I_2}}(x)\big)\end{aligned}$$

for all $x\in X$, $y\in Y$, $(A,B_1,B_2)\in T$.

This means

$$\begin{aligned}&\big(\mathrm{Sol}(T,I_1)*E(I_1,I_2)\big)\wedge\big(E(I_1,I_2)*\mathrm{Sol}(T,I_1)\big)\\ &\quad\le E_{\to}\big(A^{\uparrow_{I_2}},B_1\big)\wedge E_{\Rightarrow}\big(A^{\Uparrow_{I_2}},B_2\big)\wedge E_{\Rightarrow}\big(A,B_1^{\Downarrow_{I_2}}\big)\wedge E_{\to}\big(A,B_2^{\downarrow_{I_2}}\big)\end{aligned}$$

for all $(A,B_1,B_2)\in T$.

It follows that

$$\big(\mathrm{Sol}(T,I_1)*E(I_1,I_2)\big)\wedge\big(E(I_1,I_2)*\mathrm{Sol}(T,I_1)\big)\le\mathrm{Sol}(T,I_2)\,. \qquad\square$$

Still following [10], we are now interested in loosing some of the logical precision in order to accept approximate solutions to concept equations. The decrease of logical precision can be obtained by considering an onto morphism between two generalized residuated lattices, $h\colon L_1\longrightarrow L_2$. For each $b\in L_2$, any truth value from $h^{-1}(b)$ shall be flattened into b. For the sake of quantification, we deal by complete generalized residuated lattices and complete onto morphisms. For any $A_1\in L_1^X$, we shall denote by $h(A_1)$ the fuzzy set from L_2^X given by $h(A_1)(x)=h\big(A_1(x)\big)$ for all $x\in X$. Similarly, for $I_1\in L_1^{X\times Y}$ and $T_1\subseteq L_1^X\times L_1^Y\times L_1^Y$, we define $h(I_1)\in L_2^{X\times Y}$ and $h(T_1)\subseteq L_2^X\times L_1^Y\times L_1^Y$ by $h(I_1)(x,y)=h\big(I_1(x,y)\big)$ for all $(x,y)\in X\times Y$ and $h(T_1)=\big\{\big(h(A),h(B_1),h(B_2)\big)\colon(A,B_1,B_2)\in T_1\big\}$.

When the truth values structure is not understood, we denote Sol_L instead of Sol etc., to point out that we consider it to be L.

Remember that we have X,Y,T,L fixed.

Proposition 5.6. *Let $I\in L^{X\times Y}$ and denote by θ the smallest complete congruence on L that contains the pair $(\mathrm{Sol}_L(T,I),1)$ and by $h_\theta =\colon L\longrightarrow L_{/\theta}$ the canonical complete onto morphism. Then $h_\theta(I)\in\mathrm{Sol}_{L_{/\theta}}\big(h_\theta(T)\big)$. Moreover, for any $f\colon L\longrightarrow L'$ complete onto morphism such that $f(I)\in\mathrm{Sol}_{L'}\big(f(T)\big)$, there exists a unique complete onto morphism $g\colon L_{/\theta}\longrightarrow L'$ such that $g\circ h_\theta=f$.*

P r o o f . We have that

$$\begin{aligned}
1 &= h_\theta(\mathrm{Sol}_L(T,I)) \\
&= h_\theta\left(\bigwedge_{(A,B_1,B_2)\in T} E_{\rightarrow}(A^{\uparrow_I},B_1)\wedge E_{\Rightarrow}(A^{\Uparrow_I},B_2)\wedge E_{\Rightarrow}(A,B_1^{\Downarrow_I})\wedge E_{\rightarrow}(A,B_2^{\downarrow_I})\right) \\
&= \bigwedge_{(A,B_1,B_2)\in T} h_\theta(E_{\rightarrow}(A^{\uparrow_I},B_1))\wedge h_\theta(E_{\Rightarrow}(A^{\Uparrow_I},B_2))\wedge h_\theta(E_{\Rightarrow}(A,B_1^{\Downarrow_I}))\wedge \\
&\qquad \wedge h_\theta(E_{\rightarrow}(A,B_2^{\downarrow_I})) = \bigwedge_{(A,B_1,B_2)\in T} h_\theta(\bigwedge_{y\in Y} A^{\uparrow_I}(y)\leftrightarrow B_1(y)) \\
&\wedge h_\theta(\bigwedge_{y\in Y} A^{\Uparrow_I}(y)\Leftrightarrow B_2(y))\wedge h_\theta(\bigwedge_{x\in X} A(x)\Leftrightarrow B_1^{\Downarrow_I}(x))\wedge h_\theta(\bigwedge_{x\in X} A(x)\leftrightarrow B_2^{\downarrow_I}(x)) \\
&= \bigwedge_{(A,B_1,B_2)\in T}\bigwedge_{y\in Y}\big(h_\theta(A^{\uparrow_I}(y))\leftrightarrow h_\theta(B_1(y))\big)\wedge\bigwedge_{y\in Y}\big(h_\theta(A^{\Uparrow_I}(y))\Leftrightarrow h_\theta(B_2(y))\big) \\
&\qquad \wedge\bigwedge_{x\in X}\big(h_\theta(A(x))\Leftrightarrow h_\theta(B_1^{\Downarrow_I}(x))\big)\wedge\bigwedge_{x\in X}\big(h_\theta(A(x))\leftrightarrow h_\theta(B_2^{\downarrow_I}(x))\big) \\
&= \bigwedge_{(h_\theta(A),h_\theta(B_1),h_\theta(B_2))\in h_\theta(T)} E_{\rightarrow}(h_\theta(A^{\uparrow_I}),h_\theta(B_1))\wedge E_{\Rightarrow}(h_\theta(A^{\Uparrow_I}),h_\theta(B_2)) \\
&\qquad \wedge E_{\Rightarrow}(h_\theta(A),h_\theta(B_1^{\Downarrow_I})) = \mathrm{Sol}_{L_{/\theta}}(h_\theta(T),h_\theta(I)).
\end{aligned}$$

Thus $h_\theta(I)\in \mathrm{Sol}_{L_{/\theta}}(h_\theta(T))$.

Consider now $f\colon L\longrightarrow L'$ a complete onto morphism such that $f(I)\in \mathrm{Sol}_{L'}(f(T))$. We have that $\mathrm{Sol}_{L'}(f(T),f(I))=1$, hence $f(\mathrm{Sol}_L(T,I))=1$, the last being justified in the same way as above for h_θ. We obtain $(\mathrm{Sol}_L(T,I),1)\in \mathrm{Ker}(f)$. The existence and unicity of a morphism $g\colon L_{/\theta}\longrightarrow L'$ such that $g\circ h_\theta=f$ follows now from the universal property of the quotient algebra $L_{/\theta}$. Moreover, g is onto because f is onto and g is complete because h_θ is onto and complete and f is complete. ☐

Remark 5.4. The last proposition says that, given an approximate solution I, we can obtain the softest flattening of the truth values structure such that I becomes a full solution.

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