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RELATIVELY COMPACT PSEUDO-COMPLEMENTS IN ALGEBRAIC, DISTRIBUTIVE LATTICES

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Dedicated to Professor Ján Jakubík on the occasion of his 80th birthday

ABSTRACT. The annihilators and relative annihilators were studied in the theory of normal autometrized ℓ-algebras as members of the lattice of ideals. In this paper, these notions are generalized to any algebraic, distributive lattice as generalized compact and relatively compact elements. The results can be applied, e.g., in the theories of ℓ-groups, MV-algebras, GMV-(or pseudo MV-)algebras, and DR ℓ-monoids.

1. Introduction

In this paper we will deal with algebraic, distributive lattices. If L is such a lattice then 0 and 1 will be the least and greatest element of L, respectively. The join-subsemilattice of compact elements of L will be denoted by $\mathrm{Com}(L)$. It is known that any algebraic, distributive lattice L satisfies the join-infinite distributive law (JID), i.e.,

$$a \wedge \bigvee_{\alpha\in\Gamma} b_\alpha = \bigvee_{\alpha\in\Gamma} (a \wedge b_\alpha)$$

for any $a, b_\alpha \in L$, $\alpha \in \Gamma$, and hence it is a Heyting lattice.

Recall that a *Heyting* (or *Brouwer*) *lattice* L (see, e.g., [1]) is a lattice with 0 in which for all elements a and b, the set $\{x \in L : a \wedge x \leq b\}$ has a largest element. This largest element is called the *relative pseudo-complement of a with*

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respect to b and is denoted by $a \to b$. The element $a^* = a \to 0$ is called *the pseudo-complement* of a in L.

Many of important lattices which are in connection with lattice ordered groups (in short: ℓ-groups) or with algebras of some logics are Heyting lattices.

EXAMPLES.

1. Let G be an ℓ-group. Then the lattice $\mathcal{C}(G)$ of convex ℓ-subgroups of G is an algebraic, distributive lattice. (See, e.g., [2], [10], [12].)

2. If M is a GMV-algebra (see [15]), or equivalently a pseudo MV-algebra (see [8], [9]), then ideals of M form an algebraic, distributive lattice $\mathcal{C}(M)$. (See also [6].) Particularly for M an MV-algebra (see [4]).

3. More generally, if R is a dually residuated lattice ordered monoid (in short: DRℓ-monoid) (see [15]), then the complete lattices of ideals of R is algebraic and distributive. (See [13].)

4. If A is a normal autometrized lattice ordered algebra (in short: NAℓ-algebra (see [20], [21]) and $\mathcal{I}(A)$ is the lattice of ideals of A, then $\mathcal{I}(A)$ is an algebraic, distributive lattice.

In all above instances, the compact elements of the lattices of convex ℓ-subgroups and ideals, respectively, coincide with the principal ones. Further, it is always possible to consider the polars of subsets. We can recall this notions, e.g., for NAℓ-algebras. Let $A = (A; +, 0, \vee, \wedge, *)$ be an NAℓ-algebra and $B \subseteq A$. Then $B^{\perp} = \{x \in A\colon (x * 0) \wedge (b * 0) = 0, \text{ for each } b \in B\}$ is called *the polar of* B. For $B = \{b\}$ put $b^{\perp} = \{b\}^{\perp}$. A subset C of A is called a *polar in* A if $C = B^{\perp}$ for some $B \subseteq A$. Every polar in A belongs to $\mathcal{I}(A)$ and it is an intersection of polars in the form $b^{\perp}$. Moreover, the polars are exactly those of ideals of A, hence it is possible to consider only the polars of ideals. If $I \in \mathcal{I}(A)$ then we can recognize $I^{\perp}$ in terms of the operations in $\mathcal{I}(A)$ because $I^{\perp} = I^* = I \to \{0\}$.

In [3] the notions of an annihilator and relative annihilator of an NAℓ-algebra, which generalize those of a polar, are introduced. If a and b are elements in an NAℓ-algebra A then the subset $\langle a, b\rangle$ of A such that

$$\langle a, b\rangle = \{x \in A\colon (a * 0) \wedge (x * 0) \le n(b * 0) \text{ for some } n \in \mathbb{N}\}$$

is called *the relative annihilator of* a *with respect to* b. A subset B of A is called a *relative annihilator in* A if $B = \langle a, b\rangle$ for some elements $a, b \in A$. A subset $C \subseteq A$ is called an *annihilator of* A if C is the intersection of a system of relative annihilators in A. (The intersection of relative annihilators in A need not be a relative annihilator in A, see [3].) The annihilators in A are ideals of A and, analogously as the polars, they can be described in the lattice $\mathcal{I}(A)$. Namely, if $\langle a, b\rangle$ is the relative annihilator of a with respect to b, then $\langle a, b\rangle = I(a) \to I(b)$ in $\mathcal{I}(A)$, where $I(a)$ and $I(b)$ are the principal ideals of A generated by a and b, respectively. Moreover, for the following purpose it is important that the principal ideals coincide with the compact elements in $\mathcal{I}(A)$.

2. Compact pseudo-complements

Now, we will study analogous elements in a more general situation.

Definition. Let L be an algebraic, distributive lattice and $\mathrm{Com}(L)$ be the set of compact elements of L. Then by a *relatively compact pseudo-complement in* L (in short: *rcpc*) will be meant any relative pseudo-complement $a \to b$, where $a, b \in \mathrm{Com}(L)$.

Remark. a) Recall that structure properties of, e.g., ℓ-groups were also studied in the language of algebraic lattices in [7], [11], [14], [19].

b) Throughout the paper unless stated otherwise, L will denote an algebraic, distributive lattice.

c) For any $c \in \mathrm{Com}(L)$, its relative complement $c^* = c \to 0$ is rcpc. Particularly, $1 = 0^*$ is rcpc.

d) The meet of a set of elements which all are rcpc need not be rcpc, therefore we will introduce the following notion.

Definition. If $b \in L$ then b is called a *generalized compact pseudo-complement in* L (in short: *gcpc*) if it is the meet of a set of rcpc. The set of all gcpc will be denoted by $\mathrm{CPC}(L)$.

Remark. If A is, e.g., an NAℓ-algebra and $I \in \mathcal{I}(A)$ then I is a relative annihilator in A if and only if I is rcpc in the lattice $\mathcal{I}(A)$, and I is an annihilator in A if and only if I is gcpc in $\mathcal{I}(A)$.

Proposition 1. *If $a \in L$ and $c \in \mathrm{Com}(L)$ then $a \to c \in \mathrm{CPC}(L)$.*

Proof. Let $a \in L$ and $c \in \mathrm{Com}(L)$. Then there are $y_\alpha \in \mathrm{Com}(L)$, $\alpha \in \Gamma$, such that $a = \bigvee\limits_{\alpha\in\Gamma} y_\alpha$. Hence by Lemma 5 in [18], if z, x_α ($\alpha \in \Gamma$) are elements in Heyting lattice then

$$\left(\bigvee_{\alpha\in\Gamma} x_\alpha\right) \to z = \bigwedge_{\alpha\in\Gamma} (x_\alpha \to z).$$

Hence in our case we have

$$a \to c = \left(\bigvee_{\alpha\in\Gamma} y_\alpha\right) \to c = \bigwedge_{\alpha\in\Gamma} (y_\alpha \to c).$$

$\square$

Corollary 2. $\mathrm{CPC}(L)$ *is with respect to induced order a complete lattice which is a complete meet-subsemilattice of* L.

PROPOSITION 3. $\mathrm{Com}(L) \subseteq \mathrm{CPC}(L)$.

Proof. Let $c \in \mathrm{Com}(L)$ and let $\bar{c}$ be the least element in $\mathrm{CPC}(L)$ that exceeds c. If $a \in L$ and $b \in \mathrm{Com}(L)$ are such that $c \leq a \to b$, then $\bar{c} \leq a \to b$. Since $\bar{c} \wedge c \leq c$, we have $c \leq \bar{c} \to c$, and thus also $\bar{c} \leq \bar{c} \to c$. But this means that $\bar{c} \wedge \bar{c} \leq c$, therefore $\bar{c} = c$. □

PROPOSITION 4. *Let $b \in \mathrm{Com}(L)$. Let us denote by $\mathrm{CPC}(b)$ the set of all elements in L which are meets of sets of rcpc in the form $c \to b$, where $c \in \mathrm{Com}(L)$. Then the elements in $\mathrm{CPC}(b)$ are exactly the elements in the form $a \to b$, where $a \in L$.*

Proof. Let $x \in L$, $x = \bigwedge\limits_{\alpha\in\Gamma} (c_\alpha \to b)$, $c_\alpha \in \mathrm{Com}(L)$, $\alpha \in \Gamma$. Then

$$x = \bigwedge_{\alpha\in\Gamma} (c_\alpha \to b) = \left(\bigvee_{\alpha\in\Gamma} c_\alpha \right) \to b\,.$$

Since L is algebraic, the converse implication can be verified analogously. □

COROLLARY 5. *If $b \in \mathrm{Com}(L)$ then $\mathrm{CPC}(b)$ is under the induced order a complete lattice which is a complete meet-subsemilattice of L.*

Remark. In particular, $\mathrm{CPC}(0)$ is the complete Boolean algebra of all pseudocomplements of elements in L.

If $b \in \mathrm{Com}(L)$, set $P(b) = \{a \to b \colon b \leq a \in L\}$.

LEMMA 6. *If $b \in \mathrm{Com}(L)$ then $\mathrm{CPC}(b) = P(b)$.*

Proof. Let $x \in L$.

a) If $x \leq b$ then $x \to b = 1 = b \to b$.

b) If $x \parallel b$ then

$$x \to b = (x \to b) \wedge 1 = (x \to b) \wedge (b \to b) = (x \vee b) \to b\,.$$

Hence for any $x \in L$ there is $a_x \geq b$ such that $x \to b = a_x \to b$. □

In the following theorem we will complete the properties of the lattices $\mathrm{CPC}(b)$ from Corollary 5.

THEOREM 7. *If $b \in \mathrm{Com}(L)$ then $\mathrm{CPC}(b)$ is a complete Boolean algebra with the greatest element 1 and the least element b.*

Proof. Since L is a Heyting lattice, every of its intervals $[u, v]$ is also a Heyting lattice, and for any elements $x, y \in [u, v]$, the relative pseudocomplement of x with respect to y in $[u, v]$ is equal to $(x \to y) \wedge v$. (See, e.g., [1], Theorem IX.2.8.)

If $b \in \mathrm{Com}(L)$ then, by Lemma 6, the elements in $\mathrm{CPC}(b)$ are just the pseudo-complements of elements of the Heyting lattice $[b, 1]$, and hence $\mathrm{CPC}(b)$ is a Boolean algebra. $\square$

The next theorem is now an immediate consequence.

THEOREM 8. *Let L be any algebraic, distributive lattice. Then there exists a system $\mathcal{B}_L$ of Boolean algebras such that every $B \in \mathcal{B}_L$ is a complete meet-subsemilattice of L, all have the common greatest element 1, and $\bigcup \mathcal{B}_L$ meet-generates* $\mathrm{CPC}(L)$.

3. Closed elements

If G is an ℓ-group, then its (order) closed convex ℓ-subgroups, i.e., the members of $\mathcal{C}(G)$ which contain all existing joins of their subsets in the underlying lattice $(G; \vee, \wedge)$, have an important role; for example, they make it possible to characterize the completely distributive ℓ-groups. (See [2], [10], [12].)

Analogously, if M is an MV- or GMV-algebra then the complete distributivity of M can be characterized by means of closed ideals from the lattice $\mathcal{C}(M)$. (See [5], [16], [17]).

These notions were generalized in [18] for the situation where general algebraic, distributive lattices are considered instead of the lattices $\mathcal{C}(G)$ and $\mathcal{C}(M)$.

DEFINITION. Let L be an algebraic lattice, $a \in L$ and $(a] = \{x \in L : x \leq a\}$. Then a is called a *closed element* of L if it satisfies the following condition: If $B \subseteq (a] \cap \mathrm{Com}(L)$ and $\sup\limits_{\mathrm{Com}(L)} B$ exists, then $\sup\limits_{\mathrm{Com}(L)} B \in (a]$.

Recall that if L is an algebraic, distributive lattice then L satisfies the join-infinite distributive law (JID). We can generalize the condition (JID) to any lattice, which need not be complete. (See also [18]).

DEFINITION. We say that a lattice L *satisfies the conditional join-infinite distributive law* (CJID) if the following condition holds: If $a, b_\alpha \in L$, $\alpha \in \Gamma$, and if $\bigvee\limits_{\alpha \in \Gamma} b_\alpha$ exists, then $\bigvee\limits_{\alpha \in \Gamma} (a \wedge b_\alpha)$ also exists and

$$a \wedge \bigvee_{\alpha \in \Gamma} b_\alpha = \bigvee_{\alpha \in \Gamma} (a \wedge b_\alpha).$$

THEOREM 9. *Let L be an algebraic, distributive lattice, let* $\mathrm{Com}(L)$ *be a sublattice of L and let the lattice* $\mathrm{Com}(L)$ *satisfy* (CJID). *Then every element in* $\mathrm{CPC}(L)$ *is closed.*

Proof. Let the assumptions of the theorem be satisfied and let $b \in \mathrm{CPC}(L)$. Hence

$$b = \bigwedge_{\beta \in \Delta} (a_\beta \to b_\beta),$$

where $a_\beta, b_\beta \in \mathrm{Com}(L)$. In [18], Proposition 6, it is proved that if L satisfies the assumptions of the theorem and $a \in L$, $c \in \mathrm{Com}(L)$, then the element $a \to c$ is closed. Hence in our case, since the meet of any set of closed elements is also closed, b is a closed element in L. □

Remark. By the last proposition, every annihilator, and especially every polar, of an NAℓ-algebra is closed. Analogously for ℓ-groups, MV- and GMV-algebras, and, in general, for DRℓ-monoids (which need not be commutative, in general).

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