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OPTIMAL 2^K PAIRED COMPARISON DESIGNS FOR PARTIAL PROFILES

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ABSTRACT. In psychological research often paired comparisons are used in which only partial profiles of the alternatives are presented. For this situation we introduce appropriate models and derive optimal designs under both an additive utility structure and in the presence of interactions when all attributes are dichotomous.

1. Introduction

The method of paired comparisons is a standard scaling technique for situations when objects have to be evaluated with respect to a subjective criterion like, for example, preference. Traditionally, objects are presented in pairs and respondents have to indicate which object dominates the other with respect to the chosen criterion, e.g., which of the objects they prefer. That is, only qualitative information is collected. In recent applications of the method (e.g., in marketing), however, rating scales are widely used which permit the judges to indicate the intensity of their response. In this situation data are available from the comparison rather than the direct valuation of a single alternative. The alternatives need not represent existing objects but are mostly subject to hypothetic combinations of different components (so-called attributes) to be judged in view of a potential future market placement. Nevertheless, the paired comparison technique yields valid estimates of the underlying utilities.

The objects to be evaluated are usually described in terms of a large number of different attributes resulting in very demanding comparison tasks. To reduce the burden on the side of the respondents only a subset of the attributes is used in every paired comparison. Therefore, partial profiles are shown to the judges, i.e., within each single paired comparison only a few attributes are specified

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while the others remain undetermined. In fact, the judges have to valuate partially specified alternatives irrespectively of potential (hypothetic) choices of the undetermined attributes. The number of attributes to be presented is bounded by the profile strength which typically does not exceed four, while there may be twenty or more attributes.

The aim of this paper is to carefully introduce an appropriate model for this situation and to derive optimal designs under both an additive utility structure and in the presence of interactions. From the statistical point of view the corresponding analysis coincides with that based on paired observations with fixed pair effects as long as the alternatives are well distinguishable. We treat the particular case when two different levels are available for each attribute independently. This factorial 2^K setting has been thoroughly investigated by van Berkum (1987) and Street *et al.* (2001) in the case of a full profile when all attributes are presented to the judges simultaneously. Detailed proofs and refined results for an arbitrary number of levels are given in Graßhoff *et al.* (2003).

2. General model

We start with a general linear model

$$\tilde{Y}_n(\mathbf{t}) = \mu_n + \mathbf{f}(\mathbf{t})^\top \boldsymbol{\beta} + \tilde{Z}_n(\mathbf{t})$$

for the hypothetical case that the utility $\tilde{Y}_n(\mathbf{t})$ of a single alternative $\mathbf{t}$ could be observed on its own. Here, the index n denotes the nth presentation in which $\mathbf{t}$ is chosen from a set $\mathcal{T}$ of possible realizations for the alternative. In general, each alternative $\mathbf{t} = (t_1, \ldots, t_K)$ consists of a variety of, say, K components which are called attributes. In this setting $\mathbf{f}$ is a vector of known regression functions which describe the form of the functional relationship between the alternative $\mathbf{t}$ and the corresponding mean response $E(\tilde{Y}_n(\mathbf{t}))$, and $\boldsymbol{\beta}$ is the vector of parameters of interest. Each observation is additionally influenced by a presentation effect μ_n which may differ across single presentations due to a variety of unobservable influences. If as in paired comparisons more than one alternative is shown during one presentation, then μ_n can be regarded as a block or pair effect which might be assumed to be fixed or random. Moreover, every observation is subject to a random error $\tilde{Z}_n(\mathbf{t})$.

In contrast to standard design problems, however, the utilities for the alternatives are usually not directly measurable in paired comparisons. Only preferences can be observed for one or the other of two alternatives presented. We assume that the preference is numerically given as the difference between utilities $Y_n(\mathbf{s}, \mathbf{t}) = \tilde{Y}_n(\mathbf{s}) - \tilde{Y}_n(\mathbf{t})$ when a pair of alternatives $\mathbf{s}$ and $\mathbf{t}$ is compared.

In that case the observations are properly described by the linear model

$$Y_n(\mathbf{s},\mathbf{t}) = \big(\mathbf{f}(\mathbf{s}) - \mathbf{f}(\mathbf{t})\big)^\top \boldsymbol{\beta} + Z_n(\mathbf{s},\mathbf{t})$$

with settings $(\mathbf{s},\mathbf{t})$ for the paired comparisons chosen from the design region $\mathcal{T} \times \mathcal{T}$ of possible pairs of alternatives. Here, $\mathbf{f}(\mathbf{s}) - \mathbf{f}(\mathbf{t})$ is the derived regression function. The comparisons are assumed to result in uncorrelated and homoscedastic errors Z_n.

Note that the pair effects μ_n are immaterial in this setting. In fact, for fixed pair effects the analysis based on paired differences $Y_n(\mathbf{s},\mathbf{t}) = \tilde{Y}_n(\mathbf{s}) - \tilde{Y}_n(\mathbf{t})$ coincides with that based on directly observed pairs $\big(\tilde{Y}_n(\mathbf{s}), \tilde{Y}_n(\mathbf{t})\big)$, in general (see Graßhoff *et al.*, 2002). Moreover, in the case of a large variance component compared to the residual variance this approach is also efficient for random effects (see, e.g., Goos and Vandebroek, 2001).

The quality of the statistical analysis based on a paired comparison experiment depends on the pairs of alternatives presented. The choice of such pairs $(\mathbf{s}_1,\mathbf{t}_1),\ldots,(\mathbf{s}_N,\mathbf{t}_N)$ is called a design ξ. Its performance is given by the standardized information matrix

$$\mathbf{M}(\xi) = \frac{1}{N}\sum_{n=1}^{N}\big(\mathbf{f}(\mathbf{s}_n) - \mathbf{f}(\mathbf{t}_n)\big)\big(\mathbf{f}(\mathbf{s}_n) - \mathbf{f}(\mathbf{t}_n)\big)^\top$$

which is proportional to the inverse of the covariance matrix for the best linear unbiased estimator of the parameter vector $\boldsymbol{\beta}$. Commonly used performance measures include the D-, A- and E-criteria which aim at minimizing the determinant, the trace and the largest eigenvalue of the covariance matrix, respectively. The standardization $\frac{1}{N}$ has been introduced here to compare designs of different sample sizes and is related to the concept of generalized designs introduced by Kiefer (1959).

It is worthwhile mentioning that the difference model considered here can be interpreted as a linearization of the binary response model by Bradley and Terry (1952) in the particular case of equal utilities (see Großmann *et al.*, 2002).

3. Main effects model

Usually, in paired comparisons the alternatives are specified by a couple of, say, K attributes $k = 1,\ldots,K$ which may vary over a finite number of levels. In the present note we confine ourselves to the case of two levels for each attribute. For notational ease we code these two levels by $+1$ and -1, respectively.

For the main effects model without interactions the direct response is described by

$$E\big(\tilde{Y}_n(\mathbf{t})\big) = \mu_n + \sum_{k=1}^{K} \beta_k t_k ,$$

where β_k is the standardized effect of attribute k and the regression functions are given by $f_k(t_k) = t_k$, $t_k = \pm 1$. For the response in the paired comparisons we, hence, obtain

$$E\big(Y_n(\mathbf{s},\mathbf{t})\big) = \sum_{k=1}^{K} \beta_k (s_k - t_k) .$$

Due to the limited information processing capacity in psychological applications it is usually too difficult for a judge to express his preference for two alternatives which differ with respect to a too large number of attributes. Therefore, in this setting only *partial profiles* are presented within a single paired comparison. That is, every alternative is described by only a predefined maximal number of attributes, the *profile strength* S, while the other attributes remain undetermined. The selection which of the attributes are presented varies between the single comparisons. Of course, for these partial profiles no direct valuations of the reduced alternatives are possible while paired comparisons still lead to meaningful results.

To describe the implications of the profile strength we introduce an additional, artificial level, say "0", for each attribute which characterizes the absence of the particular attribute in a comparison: If $s_k = t_k = 0$ then no alternatives are shown for the kth attribute. Moreover, it is assumed that parameter values for the utilities are not affected by the choice which of the attributes are presented: The corresponding regression functions are consequently extended to $f_k(0) = 0$.

The variety of all comparisons which are allowed in this restricted setting is given by those pairs for which exactly S attributes are presented,

$$\mathcal{X}^{(S)} = \big\{(\mathbf{s},\mathbf{t})\colon s_k \neq 0\,,\ t_k \neq 0 \text{ for } S \text{ components and} \\ s_k = t_k = 0 \text{ for } K - S \text{ components}\big\} .$$

The optimality criteria under consideration are all invariant with respect to both permutations of the levels ± 1 for each attribute and to permutations of the attributes themselves. Hence, optimal designs can be found within the class of invariant designs which are uniform on the orbits, i.e., on the sets

$$\mathcal{X}_d^{(S)} = \big\{(\mathbf{s},\mathbf{t}) \in \mathcal{X}^{(S)}\colon s_k \neq t_k \text{ for exactly } d \text{ components}\big\}$$

of *comparison depth* $d \leq S$. For the uniform design ξ_d on $\mathcal{X}_d^{(S)}$ the corresponding information matrix equals

$$\mathbf{M}(\xi_d) = h_1(d)\ \mathbf{I}_K ,$$

where $h_1(d) = 4d/K$ and $\mathbf{I}_K$ denotes the $K \times K$ identity matrix. Now, $h_1(d)$ attains its maximum for the largest possible comparison depth $d = S$, when all attributes are distinct. Further improvements are not possible by mixtures of different comparison depths:

Theorem 1. *The uniform design ξ_S on $\mathcal{X}_S^{(S)}$ is universally optimal, i.e., optimal for every design criterion which is invariant with respect to both permutations of the levels and permutations of the attributes, in the main effects model.*

The concept of universal optimality is used here in the spirit of Kiefer (1975, see also the chapter on Kiefer optimality in Pukelsheim, 1993).

4. First order interactions

In paired comparisons it is common practice to assume an additive structure as exhibited in the previous section. However, in applications one is often faced with some possible interactions between the components. Certain level combinations may result in higher (or lower) utilities than would be expected when only single attributes were evaluated.

It should be noted that in the present setting interactions can obviously occur only between those attributes which are presented. Potential interactions which would include undetermined attributes have to be set to zero. We thus consider the first order interaction model

$$E(\tilde{Y}_n(\mathbf{t})) = \mu_n + \sum_{k=1}^{K} \beta_k t_k + \sum_{k<\ell} \beta_{k,\ell} t_k t_\ell$$

for the direct response. The corresponding paired comparison model is given by

$$E(Y_n(\mathbf{s},\mathbf{t})) = \sum_{k=1}^{K} \beta_k (s_k - t_k) + \sum_{k<\ell} \beta_{k,\ell} (s_k s_\ell - t_k t_\ell)\,.$$

Again, all reasonable design criteria are invariant with respect to permutations of levels and attributes. It is thus sufficient to look for optimality in the class of invariant designs which are uniform on their orbits of fixed comparison depth. In particular, for designs concentrated on a single comparison depth d the information matrix $\mathbf{M}(\xi_d)$ is diagonal,

$$\mathbf{M}(\xi_d) = \begin{pmatrix} h_1(d)\mathbf{I}_K & 0 \\ 0 & h_2(d)\mathbf{I}_{K(K-1)/2} \end{pmatrix},$$

where $h_2(d) = 8\frac{d(S-d)}{K(K-1)}$ is the information associated with each interaction term.

We first consider optimal designs for the main effects and the interaction terms separately which can be obtained by maximizing the corresponding entries $h_1(d)$ and $h_2(d)$, respectively, in the information matrix:

Theorem 2. *The uniform design ξ_S is universally optimal for the main effects $(\boldsymbol{\beta}_1, \dots, \boldsymbol{\beta}_K)$ with respect to both permutations of the levels and permutations of the attributes in the first order interactions model.*

For interactions the design ξ_S does not contain any information since $h_2(S) = 0$. We consider instead the intermediate comparison depth $d^* = S/2$ if S is even and $d^* = (S+1)/2$ if S is odd.

Theorem 3. *The uniform design ξ_{d^*} is universally optimal for the interaction effects $(\boldsymbol{\beta}_{1,2}, \boldsymbol{\beta}_{1,3}, \dots, \boldsymbol{\beta}_{K-1,K})$ with respect to both permutations of the levels and permutations of the attributes in the first order interactions model.*

Also in this situation mixtures of different comparison depths do not provide improvements. In view of Theorems 2 and 3 no universally optimal design exists for the whole parameter vector. We therefore restrict our attention to the commonly used D-, A- and E-criteria:

Theorem 4. *a) If S is odd or $S < K$, then the uniform design ξ_{d^*} is D-, A- and E-optimal in the first order interactions model.*

b) In the case $S = K$, even, the design $\xi_{d^*,\, d^*+1}$ which is uniform on the union of the two adjacent comparison depths $\mathcal{X}_{d^*}^{(S)} \cup \mathcal{X}_{d^*+1}^{(S)}$ is D-, A- and E-optimal in the first order interactions model.

It is straightforward to see that ξ_{d^*} (or ξ_{d^*+1}) is optimal among those designs supported by a single comparison depth. A rigorous proof of the more general result in Theorem 4 can be based on the behaviour of the directional derivatives occurring in the corresponding equivalence theorems (see Graßhoff *et al.*, 2003, for some details) or some particular properties of the eigenvalues h_1 and h_2 of the information matrix. The case $S = K$ of full profiles can be derived from the results in van Berkum (1987).

5. Discussion

For paired comparisons in a linear model without interactions optimal designs require that the alternatives presented show distinct levels in each attribute. A universally optimal design is constituted by the set of all these pairs. In particular, this design is optimal for the commonly used D-, A- and E-criteria.

For first order interaction models we have to consider pairs in which approximately half of the attributes are distinct and half of the attributes coincide. The set of all those pairs turns out to be simultaneously D-, A- and E-optimal. It might be expected that this design is optimal for all reasonable criteria based on the whole parameter vector. However, it is obviously not universally optimal.

Due to the large number of comparisons in the above mentioned optimal designs it is essential to construct fractions which share the property of optimality and which can be realized with a reasonable number of comparisons. To reach this aim combinatorial reduction methods have been recently exhibited by Mukerjee *et al.* (2002) and Street and Burgess (2001) for full profiles and by Graßhoff *et al.* (2002) for partial profiles in main effects models.

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