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ON LINEAR TRANSFORMATIONS OF WIENER PROCESS

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ABSTRACT. We characterize those linear transformations of a Wiener process $\mathbb{W}$ which result in a time-changed Wiener process. We present a complete discussion of the transformation $\big(\alpha_t \mathbb{W}(\eta_t),\ t \in T\big)$ and of two particular cases of the transformation $\big(\alpha_t \mathbb{W}(\eta_t) + \beta_t \mathbb{W}(\zeta_t),\ t \in T\big)$.

1. Introduction into the problem

Asymptotic investigation of statistical estimators and test statistics often leads to a linear transformation of a Wiener process that turns out to be a Wiener process, a Brownian bridge or an Ornstein-Uhlenbeck process. Let us recall some typical examples of such transformations. Given a Wiener process $\big(\mathbb{W}(t),\ t \geq 0\big)$, the process $\big(t\,\mathbb{W}(\frac{1}{t}),\ t > 0\big)$ is a Wiener processes, $\big(\mathbb{W}(t) - t\,\mathbb{W}(1),\ t \in [0,1]\big)$ and $\big(t\,\mathbb{W}(\frac{1-t}{t}),\ t \in (0,1)\big)$ are Brownian bridges and $\big(e^{-t}\,\mathbb{W}(e^{2t}),\ t \in \mathbb{R}\big)$ is an Ornstein-Uhlenbeck process.

In this paper a more general scheme is considered. We look at a collection of stochastic integrals of non-random real functions w.r.t. a Wiener process $\big(\mathbb{W}(t),\ t \geq 0\big)$, i.e.,

$$\left(\int_0^{+\infty} a_t(\eta)\,\mathrm{d}\mathbb{W}(\eta)\,,\ t \in T\right), \qquad \text{where} \quad a_t \in \mathrm{L}_2(\mathbb{R}_+)\ \forall t \in T. \tag{1}$$

We seek conditions under which the transformed process fulfills

$$\int_0^{+\infty} a_t(\eta)\,\mathrm{d}\mathbb{W}(\eta) = V\big(\xi(t)\big) \quad \text{a.s.} \qquad \forall t \in T\,, \tag{2}$$

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where V is a Wiener process and $\xi\colon T \to \mathbb{R}_+$ is some function.

Two particular cases of (2) are treated by Deheuvels, [1]. Let ϕ be a function for which $\Phi(t) = \int_0^t \phi(v)\,\mathrm{d}v$ exists for each $t \in \mathbb{R}_+$. Theorem 7 in [1] solves (2) for $T = \mathbb{R}_+$, $\xi(t) = t$ and

$$a_t(\eta) = \big(1 - \Phi(t) + \Phi(\eta)\big)\,\mathbb{I}_{[0,t)}(\eta)\,.$$

Also, Theorem 9 in [1] solves (2) with Brownian motion V for $T = [0,1]$, $\xi(t) = t$ and

$$\begin{aligned} a_t(\eta) = {} & \big(1 - \Phi(t) + \Phi(\eta)\big)\,\mathbb{I}_{[0,t)}(\eta) - t\big(1 - \Phi(t) + \Phi(\eta)\big)\,\mathbb{I}_{[0,1)}(\eta) \\ & - (1-t)\int_0^1 \Phi(v)\,\mathrm{d}v\,\mathbb{I}_{[0,1)}(\eta)\,. \end{aligned}$$

2. General observations

The problem posed above was treated in [2] with the following result.

Theorem 1. *The process* (1) *fulfills* (2) *if and only if*

$$\min\left\{ \int_0^{+\infty} a_t^2(\eta)\,\mathrm{d}\eta\,,\ \int_0^{+\infty} a_s^2(\eta)\,\mathrm{d}\eta \right\} = \int_0^{+\infty} a_t(\eta)\,a_s(\eta)\,\mathrm{d}\eta \quad \forall t, s \in T\,. \tag{3}$$

The aim of this paper is to derive solutions of (3) in some particular cases. At first we start with two general observations, the first of which is evident.

Lemma 1. *Let the functions $a, b\colon \mathbb{R}_+ \to \mathbb{R}$ be integrable. Then a and b satisfy the equation*

$$\int_0^{+\infty} a^2(\eta)\,\mathrm{d}\eta = \int_0^{+\infty} a(\eta)\,b(\eta)\,\mathrm{d}\eta \tag{4}$$

if and only if

$$\int_0^{+\infty} a(\eta)\big(b(\eta) - a(\eta)\big)\,\mathrm{d}\eta = 0\,. \tag{5}$$

Furthermore, both (4) *and* (5) *imply*

$$\int_0^{+\infty} a^2(\eta)\,\mathrm{d}\eta \le \int_0^{+\infty} b^2(\eta)\,\mathrm{d}\eta\,. \tag{6}$$

The problem can be restricted to $T \subset \mathbb{R}_+$ and $\int_0^{+\infty} a_t^2(\eta)\,\mathrm{d}\eta = t$ for each $t \in T$. The argument is given by the following lemma.

Lemma 2. *Let $\varphi\colon \mathbb{R}_+ \to T$ be such that*

$$\int_0^{+\infty} a^2_{\varphi(r)}(\eta)\,\mathrm{d}\eta = r \qquad \forall\, r \in \left\{ \int_0^{+\infty} a_t^2(\eta)\,\mathrm{d}\eta \colon t \in T \right\}. \tag{7}$$

Then the process (1) *satisfies* (3) *if and only if the process*

$$\left(\int_0^{+\infty} a^2_{\varphi(r)}(\eta)\,\mathrm{d}\mathbb{W}(\eta)\,,\ \ r \in \left\{ \int_0^{+\infty} a_t^2(\eta)\,\mathrm{d}\eta \colon t \in T \right\} \setminus \{0\} \right). \tag{8}$$

satisfies condition (3) *and*

$$a_t = 0 \quad \text{a.e.} \quad \textit{whenever} \int_0^{+\infty} a_t^2(\eta)\,\mathrm{d}\eta = 0\,, \tag{9}$$

$$a_t = a_s \quad \text{a.e.} \quad \textit{whenever} \int_0^{+\infty} a_t^2(\eta)\,\mathrm{d}\eta = \int_0^{+\infty} a_s^2(\eta)\,\mathrm{d}\eta\,. \tag{10}$$

Proof. The proof is straightforward. Given (3) for the process (1), we immediately obtain (3) for the process (8), condition (9) is evident, and condition (10) can be shown after a little algebra using Lemma 1. The reverse implication is similarly easy. □

3. Functions with one jump

In this section we restrict our interest to the functions

$$a_t = \alpha_t \mathbb{I}_{[0,\eta_t)}\,, \quad t \in T, \qquad \text{where } \alpha_t, \eta_t \in \mathbb{R}\,,\ \eta_t > 0\,. \tag{11}$$

First we state an auxiliary lemma.

Lemma 3. *Assume that the process* (1) *satisfies* (3) *and* (11). *Let $t, s \in T$ be such that $0 < \int_0^{+\infty} a_t^2(\eta)\,\mathrm{d}\eta < \int_0^{+\infty} a_s^2(\eta)\,\mathrm{d}\eta$.*

(i) *If $0 < \eta_t \le \eta_s$ then $\alpha_t = \alpha_s$.*

(ii) *If $\eta_t > \eta_s > 0$ then $\alpha_t \eta_t = \alpha_s \eta_s$.*

P r o o f . Using Lemma 1 we see the following.

1. If $0 < \eta_t \le \eta_s$ then (5) turns into

$$\int_0^{+\infty} a_t(\eta)\big(a_s(\eta) - a_t(\eta)\big)\,\mathrm{d}\eta = \alpha_t(\alpha_s - \alpha_t)\eta_t = 0\,.$$

   Thus $\alpha_t = \alpha_s$.

2. If $\eta_t > \eta_s > 0$ then (5) turns into

$$\int_0^{+\infty} a_t(\eta)\big(a_s(\eta) - a_t(\eta)\big)\,\mathrm{d}\eta = \alpha_t(\alpha_s\eta_s - \alpha_t\eta_t) = 0\,.$$

   Thus $\alpha_s\eta_s = \alpha_t\eta_t$.

□

THEOREM 2. *Assuming* (11), *the process* (1) *satisfies* (3) *if and only if one of the following two cases is attained:*

(i) *There is $C \in \mathbb{R}$ such that for all $t \in T$ either $\alpha_t = 0$ or $\alpha_t = C$. Thus*

$$\textit{either}\quad \int_0^{+\infty} a_t(\eta)\,\mathrm{d}\mathbb{W}(\eta) = \mathbb{W}(0) \quad\textit{or}\quad \int_0^{+\infty} a_t(\eta)\,\mathrm{d}\mathbb{W}(\eta) = C\,\mathbb{W}(\eta_t)\,.$$

(ii) *There is $C \in \mathbb{R}$ such that for all $t \in T$ either $\alpha_t = 0$ or $\alpha_t\eta_t = C$. Thus*

$$\textit{either}\quad \int_0^{+\infty} a_t(\eta)\,\mathrm{d}\mathbb{W}(\eta) = \mathbb{W}(0) \quad\textit{or}\quad \int_0^{+\infty} a_t(\eta)\,\mathrm{d}\mathbb{W}(\eta) = \frac{C}{\eta_t}\,\mathbb{W}(\eta_t)\,.$$

P r o o f . If either the case (i) or (ii) is satisfied then one can easily verify condition (3) for the process (1). Thus, we only have to show the reverse implication.

As shown in Lemma 2, we may suppose $T \subset \mathbb{R}_+ \setminus \{0\}$ and $\int_0^{+\infty} a_t^2(\eta)\,\mathrm{d}\eta = \alpha_t^2\eta_t = t$ without any loss of generality. Let us fix $t_o \in T$ and investigate the following six exhaustive cases.

1. Let for all $t, s \in T$, $t < t_o < s$ us have $\eta_t \le \eta_{t_o} \le \eta_s$. According to Lemma 3, $\alpha_t = \alpha_{t_o}$ for all $t \in T$. Hence we are in the case (i).
2. Let for all $t, s \in T$, $t < t_o < s$ us have $\eta_t > \eta_{t_o} > \eta_s$. According to Lemma 3, $\alpha_t\eta_t = \alpha_{t_o}\eta_{t_o}$ for all $t \in T$. Hence we are in the case (ii).

3. Assume $t, s \in T$ such that $t < t_o < s$ and $\eta_t \le \eta_s < \eta_{t_o}$. According to Lemma 3, $\alpha_t = \alpha_s$, $\alpha_t = \alpha_{t_o}$ and $\alpha_s \eta_s = \alpha_{t_o} \eta_{t_o}$. Consequently, $\eta_s = \eta_{t_o}$ and this is a contradiction.
4. Assume $t, s \in T$ such that $t < t_o < s$ and $\eta_s < \eta_t \le \eta_{t_o}$. According to Lemma 3, $\alpha_t = \alpha_{t_o}$ and $\alpha_t \eta_t = \alpha_s \eta_s = \alpha_{t_o} \eta_{t_o}$. Consequently, $\eta_s = \eta_{t_o}$ and this is a contradiction.
5. Assume $t, s \in T$ such that $t < t_o < s$ and $\eta_{t_o} < \eta_t \le \eta_s$. According to Lemma 3, $\alpha_{t_o} = \alpha_s$, $\alpha_t = \alpha_s$ and $\alpha_t \eta_t = \alpha_{t_o} \eta_{t_o}$. Consequently, $\eta_t = \eta_{t_o}$ contradicts again.
6. Assume $t, s \in T$ such that $t < t_o < s$ and $\eta_{t_o} \le \eta_s < \eta_t$. According to Lemma 3, $\alpha_s = \alpha_{t_o}$ and $\alpha_t \eta_t = \alpha_s \eta_s = \alpha_{t_o} \eta_{t_o}$. Thus, $\eta_s = \eta_{t_o}$ is a final contradiction.

Since all possible cases are checked, the theorem is proved. □

4. Functions with two jumps

In this section we consider the functions

$$a_t = \alpha_t \mathbb{I}_{[0,\eta_t)} + \beta_t \mathbb{I}_{[\eta_t,\zeta_t)}, \quad t \in T,\ \alpha_t, \beta_t \in \mathbb{R},\ 0 < \eta_t < \zeta_t . \tag{12}$$

Again, first we state a lemma.

Lemma 4. *Assume that the process* (1) *satisfies* (3) *and* (12). *Let* $t, s \in T$ *be such that* $0 < \int\limits_0^{+\infty} a_t^2(\eta)\,\mathrm{d}\eta \le \int\limits_0^{+\infty} a_s^2(\eta)\,\mathrm{d}\eta$. *Then*

$$\alpha_s \int\limits_0^{\zeta_s \wedge \zeta_t} a_t(\eta)\,\mathrm{d}\eta + (\beta_s - \alpha_s) \int\limits_{\eta_s \wedge \zeta_t}^{\zeta_s \wedge \zeta_t} a_t(\eta)\,\mathrm{d}\eta = \int\limits_0^{+\infty} a_t^2(\eta)\,\mathrm{d}\eta . \tag{13}$$

P r o o f. We only have to evaluate the covariance, which is

$$\begin{aligned} \int\limits_0^{+\infty} a_t(\eta)\, a_s(\eta)\,\mathrm{d}\eta &= \int\limits_0^{\zeta_s \wedge \zeta_t} a_t(\eta)\, a_s(\eta)\,\mathrm{d}\eta \\ &= \alpha_s \int\limits_0^{\zeta_s \wedge \zeta_t} a_t(\eta)\,\mathrm{d}\eta + (\beta_s - \alpha_s) \int\limits_{\eta_s \wedge \zeta_t}^{\zeta_s \wedge \zeta_t} a_t(\eta)\,\mathrm{d}\eta , \end{aligned}$$

proving the lemma. □

Unfortunately, we have no exhaustive discussion of all cases yet. Therefore, we merely present two particular results.

THEOREM 3. *Let* $T = \{0, 1, \dots, N\}$, $\zeta_n \le \eta_{n+1}$ *and*

$$0 < \int_0^{+\infty} a_n^2(\eta)\,\mathrm{d}\eta \le \int_0^{+\infty} a_{n+1}^2(\eta)\,\mathrm{d}\eta \qquad \text{for all } n = 0, 1, \dots, N-1\,. \tag{14}$$

Let the considered functions be of the shape (11). *Then the process* (1) *satisfies* (3) *if and only if* $\int_0^{+\infty} a_o(\eta)\,\mathrm{d}\eta \ne 0$ *and*

$$\alpha_1 = \dots = \alpha_N = \beta_1 = \dots = \beta_{N-1} = \frac{\int_0^{+\infty} a_o^2(\eta)\,\mathrm{d}\eta}{\int_0^{+\infty} a_o(\eta)\,\mathrm{d}\eta}\,. \tag{15}$$

Hence,

$$\int_0^{+\infty} a_n(\eta)\,\mathrm{d}\mathbb{W}(\eta) = \frac{\int_0^{+\infty} a_o^2(\eta)\,\mathrm{d}(\eta)}{\int_0^{+\infty} a_o(\eta)\,\mathrm{d}(\eta)}\,\mathbb{W}(\zeta_n)\,. \tag{16}$$

P r o o f . According to Lemma 4 we have

$$\alpha_k \int_0^{+\infty} a_n(\eta)\,\mathrm{d}\eta = \int_0^{+\infty} a_n^2(\eta)\,\mathrm{d}\eta \qquad \forall n, k \in T,\ n < k\,.$$

Hence $\int_0^{+\infty} a_o(\eta)\,\mathrm{d}\eta \ne 0$ and $\alpha_1 = \dots = \alpha_N = \frac{\int_0^{+\infty} a_o^2(\eta)\,\mathrm{d}\eta}{\int_0^{+\infty} a_o(\eta)\,\mathrm{d}\eta}$. Moreover,

$$\alpha_1 = \frac{\int_0^{+\infty} a_k^2(\eta)\,\mathrm{d}\eta}{\int_0^{+\infty} a_k(\eta)\,\mathrm{d}\eta} = \frac{\alpha_1^2\eta_k + \beta_k^2(\zeta_k - \eta_k)}{\alpha_1\eta_k + \beta_k(\zeta_k - \eta_k)} \qquad \forall k = 1, 2, \dots, N-1\,,$$

and so $\beta_1 = \dots = \beta_{N-1} = \alpha_1$. □

THEOREM 4. *Let* $T \subset [0,1]$ *and* $0, 1 \in T$. *Let* $\eta_t \le \eta_s$ *and* $\zeta_t \le \zeta_s$ *for all* $t, s \in T$, $t < s$. *Let the considered functions be of the shape* (12) *and*

$$\int_0^{+\infty} a_t(\eta)\,\mathrm{d}\eta = 0 \qquad \text{for all} \quad t \in T\,, \tag{17}$$

$$0 < \int_0^{+\infty} a_t^2(\eta)\,\mathrm{d}\eta \le \int_0^{+\infty} a_s^2(\eta)\,\mathrm{d}\eta \qquad \text{for all } t, s \in T,\ t < s\,. \tag{18}$$

Then the process (1) *satisfies* (3) *if and only if* $\eta_1 < \zeta_0$, $\beta_t \neq 0$ *for all* $t \in T$ *and there is a pair of sets* T_1 *and* T_2 *such that*

(i) $T_1 \cup T_2 = T$;

(ii) *If* $t_1 \in T_1$ *and* $t_2 \in T_2$ *then* $t_1 < t_2$;

(iii) *If* $t \in T_1$ *then* $\zeta_t = \zeta_0$,

$$\alpha_t = -\frac{\int\limits_0^{+\infty} a_o^2(\eta)\,\mathrm{d}\eta}{\beta_o \zeta_o}, \quad \beta_t = \frac{\int\limits_0^{+\infty} a_o^2(\eta)\,\mathrm{d}\eta}{\beta_o \zeta_o} \frac{\eta_t}{\zeta_o - \eta_t},$$

and so

$$\int\limits_0^{+\infty} a_t(\eta)\,\mathrm{d}\mathbb{W}(\eta) = \frac{\int\limits_0^{+\infty} a_o^2(\eta)\,\mathrm{d}\eta}{\beta_o \zeta_o (\zeta_o - \eta_t)} \big(\eta_t \mathbb{W}(\zeta_o) - \zeta_o \mathbb{W}(\eta_t)\big);$$

(iv) *If* $t \in T_2$ *then* $\eta_t = \eta_1$ *and* $\zeta_t > \zeta_0$,

$$\alpha_t = -\frac{\int\limits_0^{+\infty} a_o^2(\eta)\,\mathrm{d}\eta}{\beta_o} \frac{\zeta_t - \eta_1}{\zeta_t(\zeta_o - \eta_1)}, \quad \beta_t = \frac{\int\limits_0^{+\infty} a_o^2(\eta)\,\mathrm{d}\eta}{\beta_o} \frac{\eta_1}{\zeta_t(\zeta_o - \eta_1)},$$

and so

$$\int\limits_0^{+\infty} a_t(\eta)\,\mathrm{d}\mathbb{W}(\eta) = \frac{\int\limits_0^{+\infty} a_o^2(\eta)\,\mathrm{d}\eta}{\beta_o \zeta_t (\zeta_o - \eta_1)} \big(\eta_1 \mathbb{W}(\zeta_t) - \zeta_t \mathbb{W}(\eta_1)\big).$$

P r o o f. Necessarily, $\eta_s < \zeta_t$ for all $t, s \in T$, $t < s$, since the opposite case would contradict Theorem 3. According to Lemma 4 we have

$$(\beta_s - \alpha_s) \int\limits_{\eta_s}^{\zeta_t} a_t(\eta)\,\mathrm{d}\eta = (\beta_s - \alpha_s)\beta_t(\zeta_t - \eta_s) = \int\limits_0^{+\infty} a_t^2(\eta)\,\mathrm{d}\eta$$

for all $t, s \in T$, $t < s$.

Let $t, s \in T$ and $t < s$. Then,

$$\beta_s - \alpha_s = \frac{\int\limits_0^{+\infty} a_t^2(\eta)\,\mathrm{d}\eta}{\beta_t(\zeta_t - \eta_s)}.$$

Considering the assumption $\int\limits_0^{+\infty} a_t(\eta)\,\mathrm{d}\eta = 0$, we have

$$\alpha_s = -\frac{\int\limits_0^{+\infty} a_t^2(\eta)\,\mathrm{d}\eta}{\beta_t}\,\frac{\zeta_s - \eta_s}{\zeta_s(\zeta_t - \eta_s)} \quad \text{and} \quad \beta_s = \frac{\int\limits_0^{+\infty} a_t^2(\eta)\,\mathrm{d}\eta}{\beta_t}\,\frac{\eta_s}{\zeta_s(\zeta_t - \eta_s)}\,.$$

Hence,

$$\int\limits_0^{+\infty} a_s^2(\eta)\,\mathrm{d}\eta = \frac{\left(\int\limits_0^{+\infty} a_t^2(\eta)\,\mathrm{d}\eta\right)^2}{\beta_t^2}\,\frac{(\zeta_s - \eta_s)\eta_s}{\zeta_s(\zeta_t - \eta_s)^2}$$

and

$$\frac{\int\limits_0^{+\infty} a_s^2(\eta)\,\mathrm{d}\eta}{\beta_s} = \frac{\int\limits_0^{+\infty} a_t^2(\eta)\,\mathrm{d}\eta}{\beta_t}\,\frac{\zeta_s - \eta_s}{\zeta_t - \eta_s}\,.$$

Now, let $t \in T$ and $0 < t < 1$. Then we have

$$\begin{aligned}\beta_1 - \alpha_1 &= \frac{\int\limits_0^{+\infty} a_0^2(\eta)\,\mathrm{d}\eta}{\beta_0}\,\frac{1}{\zeta_0 - \eta_1} = \frac{\int\limits_0^{+\infty} a_t^2(\eta)\,\mathrm{d}\eta}{\beta_t}\,\frac{1}{\zeta_t - \eta_1} \\ &= \frac{\int\limits_0^{+\infty} a_0^2(\eta)\,\mathrm{d}\eta}{\beta_0}\,\frac{\zeta_t - \eta_t}{\zeta_0 - \eta_t}\,\frac{1}{\zeta_t - \eta_1}\,.\end{aligned}$$

Thus either $\eta_t = \eta_1$ or $\zeta_t = \zeta_0$. Therefore, the index set T is divided into $T_1 = \{t \in T \colon \zeta_t = \zeta_0\}$ and $T_2 = \{t \in T \colon \zeta_t > \zeta_0\}$. Evidently, $T = T_1 \cup T_2$ and $t < s$ whenever $t \in T_1$ and $s \in T_2$. $\square$

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