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IDEALS OF PSEUDO-EFFECT ALGEBRAS AND THEIR APPLICATIONS

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Dedicated to Professor Ján Jakubík on the occasion of his 80th birthday

ABSTRACT. We give elements of theory of ideals of pseudo-effect algebras. Such algebras are in important cases intervals in unital po-groups with a special kind of the Riesz decomposition property. We define ideals, normal, prime and their counterparts in the corresponding unital po-groups. We characterize prime ideals and antilattice pseudo-effect algebras, and we apply them to the characterization of antilattice unital po-groups.

1. Introduction

In the beginning of nineties, there appeared a new family of quantum structures which are called D-posets [KoCh] with a primary notion of the difference of comparable elements in the poset. They were reformulated as effect algebras [FoBe] with a primary notion of a partial addition. These structures, namely the set $E(H)$ of all effect operators A of a Hilbert space, H, which are between the zero and identity operator, play an important role in mathematical formulation of quantum mechanical measurement. For a basic source on effect algebras, see, e.g., [DvPu].

At the end of the last decade, some non-commutative structures, called pseudo MV-algebras, have appeared in [GeIo]. They are a generalization of MV-algebras. As it was shown in [Dvu 1], they are always intervals in unital ℓ-groups. The author with his student Vetterlein introduced in [DvVe 1, 2] a non-commutative generalization of effect algebras called pseudo-effect algebras. Under some natural conditions, like the Riesz decomposition property, it was shown in [DvVe 1, 2] that they are also intervals but now in unital po-groups with analogical Riesz decomposition property.

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These non-commutative algebras have nowadays many applications even in non-commutative logic [Haj] and a programming language [Bau].

In the present paper, we study the basic properties of ideals with accent to characterization prime ideals and antilattice pseudo-effect algebras, and to the characterization of ideals and corresponding subgroups in the representing unital po-group. The paper is organized as follows. In Section 2, we introduce pseudo-effect algebras and we mention their basic properties. Section 3 is devoted to theory of ideals. We characterize different types of ideals, prime ideals, and values. In Section 4, we study quotients of pseudo-effect algebras through normal ideals. Using that, we can characterize prime normal ideals and antilattice pseudo-effect algebras. Finally, in Section 5, we characterize prime subgroups and antilattice unital po-groups.

2. Pseudo-effect algebras

According to [DvVe 1, 2], a partial algebra $(E;+,0,1)$, where $+$ is a partial binary operation and 0 and 1 are constants, is called a *pseudo-effect algebra* if, for all $a,b,c\in E$, the following holds

(i) $a+b$ and $(a+b)+c$ exist if and only if $b+c$ and $a+(b+c)$ exist, and in this case $(a+b)+c=a+(b+c)$;

(ii) there is exactly one $d\in E$ and exactly one $e\in E$ such that $a+d=e+a=1$;

(iii) if $a+b$ exists, there are elements $d,e\in E$ such that $a+b=d+a=b+e$;

(iv) if $1+a$ or $a+1$ exists, then $a=0$.

If we define $a\le b$ if and only if there exists an element $c\in E$ such that $a+c=b$, then $\le$ is a partial ordering on E such that $0\le a\le 1$ for any $a\in E$. It is possible to show that $a\le b$ if and only if $b=a+c=d+a$ for some $c,d\in E$. We write $c=a\,/\,b$ and $d=b\setminus a$. Then

$$(b\setminus a)+a=a+(a\,/\,b)=b,$$

and we write $a^-=1\setminus a$ and $a^\sim=a\,/\,1$ for any $a\in E$.

For basic properties of pseudo-effect algebras see [DvVe 1] and [DvVe 2]. We recall that if $+$ is commutative, E is said to be an *effect algebra*; for effect algebras see, e.g., [DvPu].

For example, if (G,u) is a unital (not necessary Abelian) po-group with strong unit u, and

$$\Gamma(G,u):=\{g\in G\colon 0\le g\le u\}, \tag{2.1}$$

then $(\Gamma(G,u);+,0,u)$ is a pseudo-effect algebra if we restrict the group addition $+$ to $\Gamma(G,u)$.

In important cases, this is a prototypical example of pseudo-effect algebras when they satisfy a kind of the Riesz decomposition property as it was proved in [DvVe 1, 2].

According to [DvVe 1], we introduce for pseudo-effect algebras the following forms of the Riesz decomposition properties:

(a) For $a, b \in E$, we write $a \,\mathbf{com}\, b$ to mean that for all $a_1 \leq a$ and $b_1 \leq b$, a_1 and b_1 commute.

(b) We say that E fulfils the *Riesz interpolation property*, (RIP) for short, if for any $a_1, a_2, b_1, b_2 \in E$ such that $a_1, a_2 \leq b_1, b_2$ there is a $c \in E$ such that $a_1, a_2 \leq c \leq b_1, b_2$.

(c) We say that E fulfils the *weak Riesz decomposition property*, (RDP_0) for short, if for any $a, b_1, b_2 \in E$ such that $a \leq b_1 + b_2$ there are $d_1, d_2 \in E$ such that $d_1 \leq b_1$, $d_2 \leq b_2$ and $a = d_1 + d_2$.

(d) We say that E fulfils the *Riesz decomposition property*, (RDP) for short, if for any $a_1, a_2, b_1, b_2 \in E$ such that $a_1 + a_2 = b_1 + b_2$ there are $d_1, d_2, d_3, d_4 \in E$ such that $d_1 + d_2 = a_1$, $d_3 + d_4 = a_2$, $d_1 + d_3 = b_1$, $d_2 + d_4 = b_2$.

(e) We say that E fulfils the *commutational Riesz decomposition property*, (RDP_1) for short, if for any $a_1, a_2, b_1, b_2 \in E$ such that $a_1 + a_2 = b_1 + b_2$ there are $d_1, d_2, d_3, d_4 \in E$ such that (i) $d_1 + d_2 = a_1$, $d_3 + d_4 = a_2$, $d_1 + d_3 = b_1$, $d_2 + d_4 = b_2$, and (ii) $d_2 \,\mathbf{com}\, d_3$.

(f) We say that E fulfils the *strong Riesz decomposition property*, (RDP_2) for short, if for any $a_1, a_2, b_1, b_2 \in E$ such that $a_1 + a_2 = b_1 + b_2$ there are $d_1, d_2, d_3, d_4 \in E$ such that (i) $d_1 + d_2 = a_1$, $d_3 + d_4 = a_2$, $d_1 + d_3 = b_1$, $d_2 + d_4 = b_2$, and (ii) $d_2 \wedge d_3 = 0$.

We have the implications

$$(\mathrm{RDP}_2) \Rightarrow (\mathrm{RDP}_1) \Rightarrow (\mathrm{RDP}) \Rightarrow (\mathrm{RDP}_0) \Rightarrow (\mathrm{RIP}).$$

The converse of any of these implications does not hold. For commutative effect algebras we have

$$(\mathrm{RDP}_2) \Rightarrow (\mathrm{RDP}_1) \Leftrightarrow (\mathrm{RDP}) \Leftrightarrow (\mathrm{RDP}_0) \Rightarrow (\mathrm{RIP}).$$

Proposition 2.1. *If a pseudo-effect algebra E satisfies (RDP_1) and if $a_1 + \cdots + a_m = b_1 + \cdots + b_n$, then there is a system of elements $\{c_{ij} : i = 1, \dots, m, j = 1, \dots, n\}$ in E such that $a_i = c_{i1} + \cdots + c_{in}$ and $b_j = c_{1j} + \cdots + c_{mj}$ for all i and j, and*

$$(c_{i,j+1} + \cdots + c_{in}) \,\mathbf{com}\, (c_{i+1,j} + \cdots + c_{mj}) \tag{2.2}$$

for all $1 \leq i < m$ and all $1 \leq j < n$. Similarly, if E satisfies (RDP_2), then in (2.2) we can have $\wedge$ instead of **com**, *and in such a case all c_{ij}'s are uniquely determined.*

Proof. We will follow ideas of Fuchs [Fuc, p. 68]. The case $m = 2 = n$ follows from the definition of (RDP_1). Suppose thus that the theorem is true for integers m' and n' with $m' \le m$, $n' < n$ $(n \ge 3)$, and write the equality from the Proposition in the form

$$a_1 + \cdots + a_m = b_1 + \cdots + b_{n-2} + (b_{n-1} + b_n).$$

By the induction hypothesis, there exist elements c_{ij} $(1 \le i \le m,\ 1 \le j \le n-2)$ and elements d_i $(1 \le i \le m)$ in E such that

$$\begin{aligned} a_i &= c_{i1} + \cdots + c_{i,n-2} + d_i\,, \\ b_j &= c_{1j} + \cdots + c_{mj} \end{aligned}$$

for $1 \le j \le n-2$, and $b_{n-1} + b_n = d_1 + \cdots + d_m$.

We again apply the induction hypothesis, now to the last equation, to conclude the existence of elements c_{ij} in E, $1 \le i \le m$, $j = n-1, n$, with the property

$$d_i = c_{i,n-1} + c_{in}\,, \quad b_j = c_{1j} + \cdots + c_{mj}\,.$$

We now prove (2.2). The induction hypothesis yields

$$(c_{i+1,j} + \cdots + c_{mj}) \ \mathbf{com}\ (c_{i,j+1} + \cdots + c_{i,n-2} + d_i)$$

for $i < m$ and $j \le n-2$ and

$$(c_{i+1,n-1} + \cdots + c_{m,n-1}) \ \mathbf{com}\ c_{in}\,,$$

respectively, which completes the proof of (2.2).

The rest of the lattice case is same as that in [Fuc]. $\square$

We recall, see [DvVe 1, Lem 3.2], that if a pseudo-effect algebra E satisfies (RDP_0) and if $a \wedge b = 0$, then $a+b$, $b+a$ and $a \vee b$ are defined in E, and

$$a + b = a \vee b = b + a\,. \tag{2.3}$$

In addition, in any pseudo-effect algebra E we have, if $a \wedge b \in E$, then

$$\big(a \setminus (a \wedge b)\big) \wedge \big(b \setminus (a \wedge b)\big) = 0 = \big((a \wedge b) / a\big) \wedge \big((a \wedge b) / b\big)\,. \tag{2.4}$$

It is easy to prove that if $a \le b \le c$, then

$$c \setminus (a / b) = (c \setminus b) + a\,, \quad (b \setminus a) / c = a + (b / c)\,. \tag{2.5}$$

Analogical notions we introduce for po-groups. Let G be a po-group and for $a, b \in G^+$, we write $a \ \mathbf{com}\ b$ iff, for all $a_1, b_1 \in G^+$ such that $a_1 \le a$ and $b_1 \le b$, we have $a_1 + b_1 = b_1 + a_1$. According to [DvVe 1, 2], we introduce different types of the Riesz interpolation properties.

Let $(G;+,0,\le)$ be a directed po-group.

(a) We say that G fulfils the *Riesz interpolation property*, (RIP) for short, if for any $a_1,a_2,b_1,b_2 \ge 0$ such that $a_1,a_2 \le b_1,b_2$, there is a $c \in G$ such that $a_1,a_2 \le c \le b_1,b_2$.

(b) We say that G fulfils the *weak Riesz decomposition property*, (RDP_0) for short, if for any $a,b_1,b_2 \ge 0$ such that $a \le b_1+b_2$, there are $d_1,d_2 \in G$ such that $0 \le d_1 \le b_1$, $0 \le d_2 \le b_2$ and $a = d_1+d_2$.

(c) We say that G fulfils the *Riesz decomposition property*, (RDP) for short, if for any $a_1,a_2,b_1,b_2 \ge 0$ such that $a_1+a_2 = b_1+b_2$, there are $d_1,d_2,d_3,d_4 \ge 0$ such that $d_1+d_2 = a_1$, $d_3+d_4 = a_2$, $d_1+d_3 = b_1$, $d_2+d_4 = b_2$.

(d) We say that G fulfils the *commutational Riesz decomposition property*, (RDP_1) for short, if for any $a_1,a_2,b_1,b_2 \ge 0$ such that $a_1+a_2 = b_1+b_2$, there are $d_1,d_2,d_3,d_4 \ge 0$ such that (i) $d_1+d_2 = a_1$, $d_3+d_4 = a_2$, $d_1+d_3 = b_1$, $d_2+d_4 = b_2$, and (ii) $d_2\ \mathbf{com}\ d_3$.

(e) We say that G fulfils the *strong Riesz decomposition property*, (RDP_2) for short, if for any $a_1,a_2,b_1,b_2 \ge 0$ such that $a_1+a_2 = b_1+b_2$, there are $d_1,d_2,d_3,d_4 \ge 0$ such that (i) $d_1+d_2 = a_1$, $d_3+d_4 = a_2$, $d_1+d_3 = b_1$, $d_2+d_4 = b_2$, and (ii) $d_2 \wedge d_3 = 0$.

Then

$$(\mathrm{RDP}_2) \Rightarrow (\mathrm{RDP}_1) \Rightarrow (\mathrm{RDP}) \Rightarrow (\mathrm{RDP}_0) \Leftrightarrow (\mathrm{RIP}),$$

and if G is Abelian, then $(\mathrm{RDP}_0) \Leftrightarrow (\mathrm{RDP}_1)$; if G is not Abelian, the converse implications do not hold, in general, [DvVe 1, 2]. In addition, (RDP_2) holds in G iff G is an ℓ-group.

Let $\mathcal{PEA}_{\mathrm{RDP}_1}$ be the category of pseudo-effect algebras whose objects are pseudo-effect algebras with (RDP_1) and morphisms are homomorphisms of pseudo-effect algebras[1]. Let $\mathcal{UG}_{\mathrm{RDP}_1}$ be the category of unital po-groups with (RDP_1) whose objects are unital po-groups (G,u) with fixed strong unit u satisfying (RDP_1) and morphisms are homomorphisms of unital po-groups[2]. We define a functor $\Gamma\colon \mathcal{UG}_{\mathrm{RDP}_1} \to \mathcal{PEA}_{\mathrm{RDP}_1}$ such that, for any unital po-group (G,u) from $\mathcal{UG}_{\mathrm{RDP}_1}$, we assign $\Gamma(G,u)$ via (2.1), and for any morphism $f\colon (G,u) \to (H,v)$ we set a homomorphism $\Gamma(f) := f|\Gamma(G,u)$ from $\Gamma(G,u)$ into $\Gamma(H,v)$.

[1]A mapping $h\colon E \to F$, where E and F are pseudo-effect algebras, is said to be a *homomorphism* if (i) $h(0)=0$ and $h(1)=1$, and (ii) $h(a+b) = h(a)+h(b)$ whenever $a+b$ is defined in E. If h is injective and surjective such that also h^{-1} is a homomorphism, then h is said to be an *isomorphism*, and E and F are *isomorphic*.

[2]Let (G,u) and (H,v) be unital po-groups. A mapping $f\colon G \to H$ such that (i) $f(u)=v$, (ii) $f(g_1+g_2) = f(g_1)+f(g_2)$, $g_1,g_2 \in G$, (iii) $f(g) \ge 0$ whenever $g \in G^+$ is said to be a *homomorphism* of unital po-groups.

The following fundamental result was proved in [DvVe 2].

Theorem 2.2. *The functor Γ defines a categorical equivalence of the category $\mathcal{PEA}_{\mathrm{RDP}_1}$ of pseudo-effect algebras satisfying* (RDP_1) *and the category $\mathcal{UG}_{\mathrm{RDP}_1}$ of unital po-groups satisfying* (RDP_1).

Moreover, if $E=\Gamma(G,u)$, then all finite meets and joins from E are preserved in (G,u), see [DvVe 2, Prop. 6.3].

As a corollary we have the following statement.

Theorem 2.3. *Suppose that $h\colon \Gamma(G,u)\to\Gamma(H,v)$ is a homomorphism of pseudo-effect algebras with* (RDP_1), *then there is a unique homomorphism $f\colon (G,u)\to(H,v)$ of unital po-groups such that $h=\Gamma(f)$, and*

(i) *if h is surjective, so is f;*

(ii) *if h is an isomorphism, so is f.*

We recall that a pseudo-effect algebra E with (RDP_1) is a lattice iff E satisfies (RDP_2), or equivalently, if it is a pseudo MV-algebra (see, [DvVe 2]). Theorem 2.1 holds also for the category pseudo-effect algebras satisfying (RDP_2) and the category of unital ℓ-groups (see also [Dvu 1]).

3. Ideals of pseudo-effect algebras

A non-empty subset I of a pseudo-effect algebra E is said to be an *ideal* of E if (i) $x+y\in I$ whenever $x,y\in I$ and if $x+y$ is defined in E, and (ii) if $x\le y$ for $x\in E$ and $y\in I$, then $x\in I$. Then E as well as $\{0\}$ are ideals of E. We denote by $\mathcal{I}(E)$ the set of all ideals of E. It is evident that the intersection of any system of ideals is again an ideal, so we can define an ideal $I_0(X)$ for any subset $X\subseteq E$ which is generated by X. In particular, if $\{I_t\}_t$ is a system, then the least upper bound and the greatest lower bound for $\{I_t\}_t$ are denoted by $\bigvee\limits_t I_t$ and $\bigwedge\limits_t I_t$, respectively, and then $\bigwedge\limits_t I_t=\bigcap\limits_t I_t$. In particular, if $a\in E$, the ideal $I_0(a)$ is an ideal of E generated by a.

An ideal I of E is (i) *normal* if $a+I=I+a$[3], (ii) *maximal* if I is a proper subset of E and it is not included in any proper ideal of E as a proper subset, and (iii) *prime* if $I_0(a)\cap I_0(b)\subseteq I$ implies $a\in I$ or $b\in I$. We denote by $\mathcal{N}(E)$, $\mathcal{M}(E)$, and $\mathcal{P}(E)$ the set of all normal ideals, maximal ideals, and prime ideals, respectively, of E. Using Zorn's lemma, we see that $\mathcal{M}(E)$ is non-void.

[3] If A is a non-empty subset of E, then $a+A:=\{a+x : x\in A$ and $a+x$ is defined in $E\}$. In a similar way we define $A+a$.

We recall that $\{0\}, E \in \mathcal{N}(E)$ and if f is a homomorphism from a pseudo-effect E into another one F, then

$$\operatorname{Ker}(f) := \{x \in E \colon f(x) = 0\}$$

is a normal ideal of E.

Proposition 3.1. *If I is an ideal of a pseudo-effect algebra E with (RDP_0) and a is an element of E, then the ideal $I_0(I,a)$ generated by I and a is given by*

$$I_0(I,a) = \{x \in E \colon x = x_1+a_1+\cdots+x_n+a_n,\ x_i \in I,\ a_i \le a,\ 1 \le i \le n,\ n \ge 1\}. \tag{3.1}$$

If I is a normal ideal, then

$$I_0(I,a) = \{x \in E \colon x = y + a_1 + \cdots + a_n,\ y \in I,\ a_i \le a,\ i = 1,\dots,n,\ n \ge 1\}. \tag{3.2}$$

Moreover,

$$I_0(a) = \{x \in E \colon x = a_1 + \cdots + a_n,\ a_i \le a,\ i = 1,\dots,n,\ n \ge 1\}. \tag{3.3}$$

If $A \subseteq E$, then

$$I_0(A) = \{x \in E \colon x = x_1 + \cdots + x_n,\ \exists\, a_i \in A,\ x_i \le a_i,\ i = 1,\dots,n,\ n \ge 1\}. \tag{3.4}$$

If I is a normal ideal and J an ideal of E, then

$$I \vee J = \{x \in E \colon x = x_1 + y_1,\ x_1 \in I,\ y_1 \in J\}. \tag{3.5}$$

In particular, if I and J are two normal ideals of E, then $I \vee J$ is a normal ideal of E.

The normal ideal of E generated by a, $N_0(a)$, is the set

$$N_0(a) = \{x \in E \colon x = x_1 / (a_1 + x_1) + \cdots + x_n / (a_n + x_n) = (y_1 + a_1') \setminus y_1 + \cdots + (y_m + a_m') \setminus y_m,\ x_i, y_j \in E,\ a_i, a_j' \le a\}.$$

If $a_1, \dots, a_n \in E$, and if E satisfies (RDP), then there exists an element $a \in E$ with $a_i \le a$, $i = 1,\dots,n$, such that $\bigvee_{i=1}^{n} I_0(a_i) = I_0(a)$.

Proof. It is clear that if $x, y \in I_0(I,a)$ and $x + y$ is defined in E, then $x + y \in I_0(I,a)$. If $x \le y \in I_0(I,a)$, then (RDP_0) gives the necessary property $x \in I_0(I,a)$. Other is now clear.

For the last equality, since $a = (0+a) \setminus 0 = 0 / (a+0)$, we see that $a \in N_0(a)$. It is clear that if $x, y \in N_0(a)$ and $x + y \in E$, then $x + y \in N_0(a)$. Assume $z \le x \in N_0(a)$; by (RDP_0), it is enough to assume $z \le x / (a' + x)$, where

$a' \le a$ and $x \in E$. Then $a_0 := (x+z) \setminus x \le a' \le a$, so that $z = x \,/\, (a_0 + x)$. By symmetry, if $z \le (x' + a'') \setminus x'$, then $z \in N_0(a)$ proving $N_0(a)$ is an ideal of E.

To prove that $N_0(a)$ is normal, check: $x \,/\, (a' + x) + b = b + t$ for some $t \in E$, $a' \le a$. Then $a' + x + b = x + b + t$, i.e., $t = (x+b) \,/\, \big(a' + (x+b)\big)$. By the induction, we have the general case $x_1 \,/\, (a_1 + x_1) + \cdots + x_n \,/\, (a_n + x_n) + b = b + (x_1 + b) \,/\, (a_1 + x_1 + b) + \cdots + (x_n + b) \,/\, (a_n + x_n + b)$.

Analogously we have $b + (y_1 + a_1') \setminus y_1 + \cdots + (y_m + a_m') \setminus y_m = (b + y_1 + a_1') \setminus (b + y_1) + \cdots + (b + y_m + a_m') \setminus (b + y_m) + b$.

Hence, let $x_1 \,/\, (a_1 + x_1) + \cdots + x_n \,/\, (a_n + x_n) = (y_1 + a_1') \setminus y_1 + \cdots + (y_m + a_m') \setminus y_m \in N_0(a)$, and let $x_1 \,/\, (a_1 + x_1) + \cdots + x_n \,/\, (a_n + x_n) + b = (y_1 + a_1') \setminus y_1 + \cdots + (y_m + a_m') \setminus y_m + b \in E$, then $b + (x_1 + b) \,/\, (a_1 + x_1 + b) + \cdots + (x_n + b) \,/\, (a_n + x_n + b) = b' + (y_1 + a_1') \setminus y_1 + \cdots + (y_m + a_m') \setminus y_m = (b' + y_1 + a_1') \setminus (b' + y_1) + \cdots + (b' + y_m + a_m') \setminus (b' + y_m) + b' = b'' + (b' + y_1 + a_1') \setminus (b' + y_1) + \cdots + (b' + y_m + a_m') \setminus (b' + y_m)$. Since $b = b''$, we have $(x_1 + b) \,/\, (a_1 + x_1 + b) + \cdots + (x_n + b) \,/\, (a_n + x_n + b) = (b' + y_1 + a_1') \setminus (b' + y_1) + \cdots + (b' + y_m + a_m') \setminus (b' + y_m) \in N_0(a)$.

Let a_1, a_2 be elements of E. There are three elements $a_1', a_2', c \in E$ such that $a_1 = a_1' + c$, $a_2 = a_2' + c$ and $a := a_1' + a_2' + c = a_2' + a_1' + c \in E$. By (3.5), we have $I_0(a_1) \vee I_0(a_2) = I_0(a)$. The general case follows by mathematical induction. □

For the lattice $\mathcal{I}(E)$ with respect to the set-theoretical inclusion, we have the following statement.

PROPOSITION 3.2. *Let E be a pseudo-effect algebra satisfying* (RDP). *For all $I, I_s \in \mathcal{I}(E)$ $(s \in S)$, $I \cap \bigvee_s I_s = \bigvee_s (I \cap I_s)$. Moreover, $I \vee (I_1 \cap I_2) = (I \vee I_1) \cap (I \vee I_2)$ for all $I, I_1, I_2 \in \mathcal{I}(E)$.*

P r o o f. Let $\{I_s\}_{s \in S}$ be a family of ideals of E and let I_0 be the ideal of E generated by $\bigcup_s I_s$. Clearly, $\bigvee_s (I \cap I_s) \subseteq I \cap \bigvee_s I_s$. Let $x \in I \cap \bigvee_s I_s$, then by Proposition 3.1, $x = x_1 + \cdots + x_n$, with $x_i \in I_{s_i}$, $i = 1, \dots, n$, and while simultaneously $x \in I$, we have $x_i \in I$ for $i = 1, \dots, n$ which gives $x \in \bigvee_s (I \cap I_s)$.

It is clear that $I \vee (I_1 \cap I_2) \subseteq (I \vee I_1) \cap (I \vee I_2)$. Assume $x \in (I \vee I_1) \cap (I \vee I_2)$. Then by Proposition 3.1, $x = x_1 + i_1 + \cdots + x_n + i_n = y_1 + j_1 + \cdots + y_m + j_m$, where $x_i, y_j \in I$ and $i_s \in I_1$, $j_t \in I_2$. Due to (RDP), there exists a matrix $\{c_{st}\}$ of elements of E such that the corresponding equalities hold. Hence, $x = x_1 + c_{21} + c_{22} + c_{23} + \cdots + x_2 + c_{41} + c_{42} + c_{43} + \cdots$. Since $x_i \in I$ and $c_{21}, c_{23}, \ldots \in I$ and $c_{22}, c_{24}, \ldots \in I_1 \cap I_2$, which implies $x \in I \vee (I_1 \cap I_2)$. □

Let a be a non-zero element of E. By Zorn's lemma, there exists an ideal V not containing a and which is maximal with respect to this property. Such an ideal is said to be a *value* of a in E. We denote by $\Gamma(a)$ the set of all values of the element a. For example, every maximal ideal of E is a value of 1.

We say that an ideal J *covers* I if $I \subset J$ and, for any ideal K of E, $I \subseteq K \subseteq J$ implies either $I = K$ or $J = K$; and we say that J is a *cover* of I.

PROPOSITION 3.3. *Every value in a pseudo-effect algebra E has a cover.*

Proof. Let $a > 0$ and V be a value of a in E. Let $V(a)$ be the intersection of all ideals of E that contain V and a. Then $V(a)$ covers V: For it if J is an ideal of E and $J \supset V$, then $a \in J$ by the maximality of V, hence $V(a) \subseteq J$. $\square$

We say that a value V is normal in its cover V^* if $x + V = V + x$ for $x \in V^*$. E is said to be *normal-valued* if any value is normal in its cover. In [Dvu], we proved that pseudo MV-algebras (= pseudo-effect algebras with (RDP_2)) which are normal-valued form a variety.

PROPOSITION 3.4. *Every value V of a pseudo-effect algebra E with* (RDP) *is prime.*

Proof. Let V be a value of an element $a \in E \setminus \{0\}$, and let $I_0(x) \cap I_0(y) \subseteq V$. Assume $x, y \notin V$. The maximality of V implies $a \in \big(V \vee I_0(x)\big) \cap \big(V \vee I_0(y)\big)$. By Proposition 3.1, $a = a_1 + x_1 + \cdots + a_n + x_n = b_1 + y_1 + \cdots + b_m + y_m$, where $a_i, b_j \in V$ and $x_i \in I_0(x)$ and $y_j \in I_0(y)$. By (RDP), there exist elements $c_{ij} \in E$ such that $a_i = \sum\limits_j c_{ij}$, $x_i = \sum\limits_j c_{ij}$, $b_j = \sum\limits_i c_{ij}$ and $y_j = \sum\limits_i c_{ij}$. An easy argument shows that all c_{ij} belong to V, which gives $a \in V$, a contradiction. $\square$

PROPOSITION 3.5. (1) *An ideal P of a pseudo-effect algebra E is prime if and only if, for all $I, J \in \mathcal{I}(E)$ with $I \cap J \subseteq P$, we have $I \subseteq P$ or $J \subseteq P$.*

(2) *If P is prime, then $I \cap J = P$ implies $I = P$ or $J = P$. If E satisfies* (RDP)*, then an ideal P is prime if and only if, for all $I, J \in \mathcal{I}(E)$ with $I \cap J = P$, we have $I = P$ or $J = P$.*

Proof. (1) Let P be prime and let $I \cap J \subseteq P$. If $I \not\subseteq P$ and $J \not\subseteq P$, there are $a \in I \setminus P$ and $b \in J \setminus P$. Then $I_0(a) \cap I_0(b) \subseteq I \cap J \subseteq P$, which gives $a \in P$ or $b \in P$, a contradiction.

The converse implication is evident.

(2) Let P be prime, and let $I \cap J = P$. Then by (1), $I \subseteq P$ or $J \subseteq P$. In the first case, we have $I \subseteq P = I \cap J \subseteq I$, i.e., $I = P$. The second case is analogous.

For the converse implication, assume now that P is a normal ideal of E satisfying (RDP) and the condition, and let $I \cap J \subseteq P$. By Proposition 3.2, $P = P \vee (I \cap J) = (P \vee I) \cap (P \vee J)$ which gives $P = P \vee I$ or $P = P \vee J$, so that, $I \subseteq P$ or $J \subseteq P$, which by (1) yields that P is prime. $\square$

PROPOSITION 3.6. *Let A be an ideal of a pseudo-effect algebra E with (RDP) and let $a \notin A$. Then every ideal I of E, which is maximal with respect to the property $I \supseteq A$ and $a \notin I$, is prime.*

Moreover, every ideal J of E is the intersection of all prime ideals containing J.

Proof. Due to Zorn's lemma, such an ideal exists. Using the proof of Proposition 3.4, we have I is prime.

Let $\mathcal{P}(J) = \{P \in \mathcal{P}(E) \colon P \supseteq J\}$. Then $E \in \mathcal{P}(J)$, and let $P_0 = \bigcap \mathcal{P}(J)$; then $P_0 \supseteq J$. Assume $a \in P_0 \setminus J$. According to the first part of the present proposition, there is a prime ideal P of E containing J and not containing the element a, which contradicts the choice of $\mathcal{P}(J)$. Hence, $P_0 = J$. $\square$

4. Quotient pseudo-effect algebras

An ideal I of a pseudo-effect algebra E is said to be the *Riesz ideal* if for $x \in I$, $a, b \in E$ and $x \le a + b$, there exist $a_1, b_1 \in I$ such that $x = a_1 + b_1$ and $a_1 \le a$ and $b_1 \le b$.

For example, if E is with (RDP), then any ideal of E is Riesz.

Let P be an ideal of a pseudo-effect algebra E. For $a, b \in E$, we write $a \sim_P b$ iff there are two elements $e, f \in P$ such that $a \setminus e = b \setminus f$. We recall that $a \sim_P b$ iff $e' \mathbin{/} a = b \setminus f$ for some $e', f \in P$ iff $e' \mathbin{/} a = f' \mathbin{/} b$ for some $e', f' \in P$.

PROPOSITION 4.1. *Let P be a normal Riesz ideal of a pseudo-effect algebra E. Then $\sim_P$ is an equivalence on E such that $\big(E/P; +, [0]_P, [1]_P\big)$ is a pseudo-effect algebra, where $[a]_P := \{b \in E \colon b \sim_P a\}$, $E/P := \{[a]_P \colon a \in E\}$, and $[a]_P + [b]_P = [c]_P$ if and only if there are $a_1 \in [a]_P$, $b_1 \in [b]_P$ and $c_1 \in [c]_P$ such that $a_1 + b_1 = c_1$.*

Moreover, $[a]_P \le [b_i]_P$ for $i = 1, \dots, n$ if and only if there exists $a_1 \in [a]_P$ such that $a_1 \le b_i$, $i = 1, \dots, n$.

Moreover, if P is a normal ideal of E satisfying (RDP), or (RDP_1), or (RDP_2), then so satisfies E/P.

Proof. (1) To prove that E/P is a pseudo-effect algebra, see [DvVe 3, Prop 3.6].

(2) Assume now $[a]_P \le [b]_P$. There exists $c \in E$ such that $[a]_P + [c]_P = [b]_P$. Then there are elements $u, v, e, f \in P$ and $a_0 \in [a]_P, b_1 \in [b]_P$ such that $a_0 + c = b_1$, $a_0 \setminus u = a \setminus v$, $b_1 \setminus e = b \setminus f$. Since $a_0 = (a \setminus v) + u$, $b_1 = (b \setminus f) + e$, we have $(a \setminus v) + u + c = (b \setminus f) + e$. Since P is a Riesz ideal, there are $e_1, e_2, e_3 \in P$ with $e_1 \le a \setminus v$, $e_2 \le u$, $e_3 \le c$ and $e = e_1 + e_2 + e_3$, which yields $\big((a \setminus v) \setminus e_1\big) + e_1 + (u \setminus e_2) + e_2 + (c \setminus e_3) + e_3 = (b \setminus f) + e_1 + e_2 + e_3$. Hence,

$\big(a \setminus (e_1+v)\big) + e_1 + e_2 \le (b \setminus f) + e_1 + e_2$, i.e., $\big(a \setminus (e_1+v)\big) \le b \setminus f \le b$. Set $a_1 = a \setminus (e_1+v)$, then $a_1 \in [a]_P$ and $a_1 \le b$.

(3) Suppose now $[a]_P \le [b_i]_P$ for $i = 1,\dots,n$. Using the previous proof of (2), we have that there exists $a_1 \in [a]_P$ such that $a_1 \le b_i$ for $i = 1,\dots,n$.

(4) Let E satisfy (RDP_1) and let P be a normal ideal of E. Then it is a Riesz ideal, and let $[a_1] + [a_2]_P = [b_1]_P + [b_2]_P$. Without loss of generality, we can assume that $a_1 + a_2$ and $b_1 + b_2$ are defined in E. There are $e, f \in P$ such that $(a_1 + a_2) \setminus e = (b_1 + b_2) \setminus f$, so that $a_1 + a_2 = \big((b_1+b_2) \setminus f\big) + e$. By (RDP) holding in E, there are $c_{11}, c_{12}, c_{21}, c_{22}$ in E such that

$$\begin{aligned} a_1 &= c_{11} + c_{12}\,, \quad (b_1+b_2) \setminus f = c_{11} + c_{21}\,,\\ a_2 &= c_{21} + c_{22}\,, \quad e = c_{12} + c_{22}\,. \end{aligned}$$

This gives $c_{11} + c_{21} + f = b_1 + b_2$. Again due to (RDP_1), there are d_{11}, d_{12}, d_{21}, $d_{22}, d_{31}, d_{31} \in E$ with d_{12} **com** $(d_{21} + d_{31})$ such that

$$\begin{aligned} c_{11} &= d_{11} + d_{12}\,, \quad b_1 = d_{11} + d_{21} + d_{31}\,,\\ c_{21} &= d_{21} + d_{22}\,, \quad b_2 = d_{12} + d_{22} + d_{32}\,,\\ f &= d_{31} + d_{32}\,. \end{aligned}$$

It is clear that $c_{12}, c_{22}, d_{31}, d_{32} \in P$, which gives $[a_1]_P = [c_{11}]_P = [d_{11}]_P + [d_{12}]_P$, $[a_2]_P = [c_{21}]_P = [d_{21}]_P + [d_{22}]_P$, $[b_1]_P = [d_{11}]_P + [d_{21}]_P$, $[b_2]_P = [d_{12}]_P + [d_{22}]_P$. Assume $[x]_P \le [d_{12}]_P$ and $[y]_P \le [d_{21}]_P$. Then there are $x_1 \in [x]_P$ and $y_1 \in [y]_P$ such that $x_1 \le d_{12}$ and $y_1 \le d_{21}$, i.e., $[d_{12}]_P$ **com** $[d_{21}]_P$ which proves that E/P is with (RDP_1).

In a similar way we proceed if E satisfies (RDP_2) or (RDP). $\square$

We now present some connections among ideals of E and o-ideals of the corresponding unital po-groups (see also [DvVe 3]). An *o-ideal* of a po-group G is any normal convex directed subgroup H of G. A subgroup H of an ℓ-group G is an o-ideal of G iff H is an ℓ-ideal of G. If (G,u) is a unital po-group, we denote by $\mathcal{OI}(G,u)$ the set of all o-ideals of (G,u).

For example, if $f\colon (G,u) \to (H,v)$ is a homomorphism of unital po-groups, then the kernel of f, $\mathrm{Ker}(f)$, defined by

$$\mathrm{Ker}(f) := \big\{g \in G\colon\, f(g) = 0\big\}$$

is a normal convex subgroup of G which is not necessary an o-ideal.

Indeed, take $\big(\mathbb{Z}^2, (1,1)\big)$ and for a state s on it defined by $s(x,y) = (x+y)/2$, we have $\mathrm{Ker}(s) = \big\{(n,-n)\colon\, n \in \mathbb{Z}\big\}$, which is a normal convex subgroup of $\mathbb{Z}^2$ but not an ℓ-subgroup.

Therefore, we define for any homomorphism $f\colon (G,u) \to (H,v)$, the set

$$\mathrm{K}(f) := \{x - y\colon x, y \in G^+ \cap \mathrm{Ker}(f)\}. \tag{4.1}$$

Then $\mathrm{K}(f)$ is an o-ideal of (G,u). Indeed, it is sufficient to verify that if $x, y \in \mathrm{Ker}(f) \cap G^+$, and if $-y + x = x + y_0$, then $-y_0 \in \mathrm{Ker}(f) \cap G^+$. In fact, $y_0 = -x - y + x = -(y+x) + x$. Since $y + x \geq x$, we have $-(y+x) \leq -x$, i.e., $y_0 \leq 0$. It is clear that $y_0 \in \mathrm{Ker}(f)$. Let now $x_1, y_1, x_2, y_2 \in \mathrm{Ker}(f) \cap G^+$. Then $x_1 - y_1 + x_2 - y_2 = x_1 + x_2 - y_1' - y_2 \in \mathrm{K}(f)$, where $y_1' \in \mathrm{Ker}(f) \cap G^+$.

In general, it is possible to show that if H is a convex subgroup of a po-group G, then the set

$$H_0 := \{x - y\colon x, y \in H \cap G^+\} \tag{4.2}$$

is a directed convex subgroup of G, and it is the largest directed convex subgroup of G that is contained in H. Moreover, if H is a convex normal subgroup of G, then H_0 is an o-ideal of G.

If H is an o-ideal of G, then G/H is a po-group, where $x/H \leq y/H$ iff there exists $h \in H$ such that $x \leq h + y$; we write also $a \sim_H b$ iff $a - b \in H$. Moreover, if (G,u) is a unital po-group, so is $(G/H, u/H)$.

THEOREM 4.2. *Let $E = \Gamma(G,u)$, where (G,u) is a unital po-group satisfying* (RDP_1). *Let I be an ideal of E. Set*

$$\phi(I) = \{x \in G\colon \exists\, x_i, y_j \in I,\ x = x_1 + \cdots + x_n - y_1 - \cdots - y_m\}. \tag{4.3}$$

Then $\phi(I)$ is an o-ideal of (G,u) if and only if I is a normal ideal of E. In such a case

$$(E/I, u/I) = \Gamma\big(G/\phi(I),\ u/\phi(I)\big).$$

In addition, if K is an o-ideal of (G,u), then its restriction to E, denoted by $\psi(K)$, gives a normal ideal of E, i.e.,

$$\psi(K) := K \cap E \in \mathcal{I}(E), \quad K \in \mathcal{I}(G,u).$$

Moreover, both mappings, ϕ and ψ, are mutually bijective and preserving the set-theoretical inclusion.

Proof. (1) Let I be a normal ideal of E. We assert that $\phi(I)$ is an ideal of (G,u). Assume that (H,v) is a unital po-group guaranteed by Theorem 2.2 such that $\Gamma(H,v) = E/I$. Let f be a unique homomorphism from Theorem 2.3 such that $f\colon (G,u) \to (H,v)$ and $\Gamma(f) = h$, where $h\colon E \to E/I$ is the canonical homomorphism of the pseudo-effect algebras. Then $\phi(I) \subseteq \mathrm{K}(f)$, where $\mathrm{K}(f)$ is defined by (4.1). If $x \in \mathrm{K}(f)$ and $x \geq 0$, then $x = x_1 + \cdots + x_n$, where $x_i \in E$ with $f(x_i) = 0$ for any i, i.e., $f(x) = 0$ and $x \in \phi(I)$. If now $x \in \mathrm{K}(f)$, then $x = x_1 - x_2$, where $x_1, x_2 \in \mathrm{Ker}(f) \cap G^+$ which gives $x \in \phi(I)$. Consequently, we have proved that $\phi(I) = \mathrm{K}(f)$, and $\phi(I)$ is an o-ideal of (G,u).

Conversely, let $\phi(I)$ be an o-ideal of (G,u) and let $x \in I$ and $y \in E$ be given such that $x+y \in E$. Then $x+y = y+x_0$ for some $x_0 \in E$. Due to the normality of $\phi(I)$, $x_0 = x_1 + \dots + x_n - y_m - \dots - y_1$, where $x_i, y_j \in I$. Hence, $x_0 + y_1 + \dots + y_m = x_1 + \dots + x_n$. Due to (RDP_1) in (G,u), there exists a system $\{c_{ij}\}$ of elements in E such that $x_0 = c_{01} + \dots + c_{0n}$, $y_j = \sum\limits_i c_{ij}$ and $x_i = \sum\limits_j c_{ij}$. Then each $c_{ij} \in I$, i.e., $x_0 \in I$. In a similar way we prove that if $x \in I$ and $y \in E$ are such that $y + x \in E$, then $(y+x) \setminus y \in I$.

(2) Assume now that I is a normal ideal of E, then so is $\phi(I)$ in (G,u) by (1), and if $a, b \in E$, then $a \sim_I b$ iff $a \sim_{\phi(I)} b$. Indeed, one direction is clear. Assume thus $a \sim_{\phi(I)} b$, i.e., $a - b =: x_1$, $b - a = x_2 \in \phi(I)$. Since $\phi(I)$ is directed, there is $x \in \phi(I)$, $x \geq 0$, such that $a - b \leq x$ and $b - a \leq x$. Hence, $a \leq x + b$ which gives $x_1, \dots, x_n \leq x$, $(x_i \in \phi(I))$ and $b_1 \leq b$ such that $a = x_1 + \dots + x_n + b_1$. Then $a \setminus e = b_1 \leq b$, where $e = x_1 + \dots + x_n \in I$. In a similar way, $b \leq x + a$, i.e., $b = f + a_1$, where $f \in I$ and $a_1 \leq a$, i.e., $b \setminus f = a_1 \leq a$. Hence $[a]_I \leq [b]_I$ and $[b]_I \leq [a]_I$, i.e., $a \sim_I b$.

Summarizing that with (H,v), we see that $(H,v) \cong \big(G/\phi(I), u/\phi(I)\big)$.

The rest follows from [DvVe 3, Prop 5.6]. □

If $A \subseteq E$ and P is an ideal of E, then we set $A/P := \{a/P \colon a \in A\}$, where $a/P := [a]_P$ $(a \in E)$.

PROPOSITION 4.3. *Let E be a pseudo-effect algebra with* (RDP) *and let P be a proper normal ideal of E.*

(i) *If I is an ideal of E, so is I/P in E/P. Moreover, if I is a proper ideal of E containing P, then I/P is a proper ideal of E/P.*

(ii) *If M is an ideal of E/P, then*

$$\kappa(M) := \{x \in E \colon x/P \in M\} \tag{4.4}$$

is an ideal of E, and $\kappa(M)/P = M$. If M is a proper ideal of E so is $\kappa(M)$ in E.

(iii)
$$\mathcal{N}(E/P) = \big\{N/P \colon N \in \mathcal{N}(E) \text{ and } P \subseteq N\big\}.$$

P r o o f. (i) $0/P \in I/P$. Let $x/P \leq y/P$, where $y \in I$. There exists $x_1 \in [x]_P$ such that $x_1 \leq y$, which gives $x_1 \in I$, and $x_1/P = x/P \leq y/P$. Assume $x/P + y/P$ is defined in E/P for some $x, y \in I$. There are $x_1 \in [x]_P$, $y_1 \in [y]_P$ and $e, f, u, v \in P$ such that $x_1 \setminus e = x \setminus f \in I$, $y_1 \setminus u = y \setminus v \in I$, $x_1 + y_1 \in E$. Then $x/P + y/P = x_1/P + y_1/P = (x_1 + y_1)/P = \big((x \setminus f) + e + (y \setminus v) + u\big)/P = \big((x \setminus f) + (y \setminus v)\big)/P$ and $(x \setminus f) + (y \setminus v) \in I$.

Let now $I \supseteq P$ and $1/P = x/P$, where $x \in I$. There are $e, f \in P$ such that $1 \setminus e = x \setminus f$, i.e., $x \,/\, 1 = f \,/\, e \in P \subseteq I$ which gives a contradiction.

(ii) We have $\kappa(M) \supseteq P$. If $x \leq y \in \kappa(M)$, then $x/P \leq y/P \in M$, so that $x \in \kappa(M)$. Let now $x, y \in \kappa(M)$ and $x+y \in E$. Then $(x+y)/P = x/P + y/P \in M$, i.e., $x + y \in \kappa(M)$.

Finally, assume that M is a proper ideal of E/P. Then $1/P \notin M$, hence, $1 \notin \kappa(M)$.

(iii) It follows from (ii). □

A pseudo-effect algebra E is said to be (i) *simple* if it has only trivial ideals, and (ii) *local* if it has only one maximal ideal.

Proposition 4.4. *A proper normal ideal I of a pseudo-effect algebra E with* (RDP) *is maximal if and only if E/I is simple.*

Proof. Let I be maximal and let J be a proper ideal of E/I. Define $\kappa(J)$ by (4.4). Then $\kappa(J)$ is an ideal of E containing I. The maximality of I entails $J = \{0/I\}$.

Conversely, let E/I be simple and let I_1 be a proper ideal of E containing I. The set $\{a/I \colon a \in I_1\}$ is an ideal of E/I. Therefore, this set coincides with zero-ideal $\{0/I\}$ of E/I, i.e., $I_1 = I$, which proves that I is maximal. □

We recall that a poset $(E; \leq)$ is an *antilattice* if only comparable elements of E have a supremum or an infimum. It is clear that any linearly ordered poset is an antilattice. Let E be a pseudo-effect algebra. Then E is an antilattice iff $a \wedge b = 0$ implies $a = 0$ or $b = 0$. The same is true if G is a directed po-group.

Proposition 4.5. *A pseudo-effect algebra E with* (RDP) *is an antilattice if and only if the null ideal $\{0\}$ is prime.*

Proof. Let $\{0\}$ be a prime ideal and let $a \wedge b = 0$. Then $I_0(a) \cap I_0(b) = \{0\}$ which proves $a = 0$ or $b = 0$, i.e., E is an antilattice.

Conversely, let E be an antilattice, and let $I_0(a) \cap I_0(b) = \{0\}$. Then $a \wedge b = 0$ which yields $a = 0$ or $b = 0$, i.e., $\{0\}$ is prime. □

Proposition 4.6. *A proper normal ideal P of a pseudo-effect algebra E with* (RDP) *is prime if and only if E/P is an antilattice.*

Proof. Let P be prime and assume $a/P \wedge b/P = 0/P$. We assert that $I_0(a) \cap I_0(b) \subseteq P$. Take $x \in I_0(a) \cap I_0(b)$. Then $x = a_1 + \cdots + a_m = b_1 + \cdots + b_n$, where $a_i \leq a$ and $b_j \leq b$ for all i and all j. (RDP) implies that there is a system $\{c_{ij}\}$ of elements of E such that $a_i = \sum_j c_{ij}$ and $b_j = \sum_i c_{ij}$. Since $c_{ij} \leq a, b$, we have $c_{ij}/P = 0/P$, i.e., $c_{ij} \in P$, which yields $a_i \in P$ and $x \in P$. Since P is prime, then $a \in P$ or $b \in P$, i.e., $a/P = 0/P$ or $b/P = 0/P$, which proves that E/P is an antilattice.

Conversely, let E/P be an antilattice, and let $I_0(a) \cap I_0(b) \subseteq P$, where $a, b \in E$. We claim $I_0(a/P) \cap I_0(b/P) = \{0/P\}$. Indeed, let $x/P = a_1/P + \cdots + a_m/P = b_1/P + \cdots + b_n/P$, where $a_i/P \le a/P$ and $b_j/P \le b/P$. By Proposition 4.1, there are $a_i' \in [a_i]_P$ and $b_j' \in [b_j]_P$ such that $a_i' \le a$ and $b_j' \le b$. Hence, there is a system $\{c_{ij}\}$ of elements in E such that $a_i'/P = \sum\limits_j c_{ij}/P$ and $b_j'/P = \sum\limits_i c_{ij}/P$. Following Proposition 4.1, for each i and each j, there is an element $c_{ij}' \in [c_{ij}]_P$ such that $c_{ij}' \le a_i', b_j'$, i.e., $c_{ij}' \le a_i, b_j$. Then $c_{ij}' \in P$ which proves $x \in P$, i.e., $I_0(a/P) \cap I_0(b/P) = \{0/P\}$. Since E/P is an antilattice, by Proposition 4.5, $\{0/P\}$ is a prime ideal of E/P, consequently $a/P = 0/P$ or $b/P = 0/P$, i.e., $a \in P$ or $b \in P$ proving that P is prime. □

COROLLARY 4.7. *Every simple pseudo-effect algebra E with* (RDP) *is an antilattice.*

Proof. Since $E \cong E/\{0\}$, by Proposition 4.4, $\{0\}$ is maximal, and by Proposition 3.4, $\{0\}$ is prime which in view of Proposition 4.6 yields that $E \cong E/\{0\}$ is an antilattice. □

We note that the converse statement to Corollary 4.7 is not true, in general. There exists an antilattice commutative pseudo-effect algebra with (RDP) which is not simple, see the following example.

EXAMPLE 4.8. Let (G, u) be a unital po-group, where $G = \mathbb{R}^2$ and $u = (1, 1)$, and $G^+ := \{(0,0)\} \cup \{(x, y) \in \mathbb{R}^2 \colon x > 0,\ y \ge 0\}$. Then (G, u) is with (RDP), and $I := \{(x, 0) \in \mathbb{R}^2 \colon 0 \le x < 1\}$ is a unique maximal ideal of $E = \Gamma(G, u)$. In addition, E is an antilattice and $\mathcal{I}(E) = \{\{0\}, I, E\}$.

A non-zero element a of a pseudo-effect algebra E is said to be an *atom* of E if $b \le a$ for some $b \in E$ implies $b = 0$ or $b = a$. E is said to be *atomic* if E possesses at least one atom.

PROPOSITION 4.9. *Every simple pseudo-effect algebra E with* (RDP) *is an antilattice.*

Every atomic simple pseudo-effect algebra with (RDP) *is commutative. If a is an atom of E, for any $g \in E$, $g = na$ for some $n \in \mathbb{N}$, and E is linear.*

Moreover, if a simple E satisfies (RDP_2)*, then it is a commutative linearly ordered lattice.*

Proof. If E is simple, then the null ideal of E is a value of any non-zero element of E. Therefore, $\{0\}$ is prime, and by Proposition 4.5, E is an antilattice.

Since $a > 0$, the simplicity of E yields $I_0(a) = E$. By (3.3) and Proposition 3.1, $g = a_1 + \cdots + a_n$, where $a_i \le a$. Since a is an atom, $a_i \in \{0, a\}$, which proves the assertion in question.

If now E satisfies (RDP_2), then it can be converted into a pseudo MV-algebra, and the commutativity for simple pseudo MV-algebras was proved in [DvPu, Thm 6.4.23]; the linear structure follows from the first part. □

We recall that it can happen that if E is an atomless simple pseudo-effect algebra with (RDP_0), then it is not necessary commutative.

EXAMPLE 4.10. Let G be a free group generated by the countably many elements $g_0, g_1, \dots$, and let h be a group homomorphism from $G \to \mathbb{R}$ such that $h(g_i) = 1/2^i$, $i \ge 0$. We define the positive conus $G^+ := \{g \in G \colon g = 0$ or $v(g) > 0\}$. Then by [DvVe 1, Ex. 3.6], (G, g_0) satisfies (RDP_0) but not (RDP). The pseudo-effect algebra $E = \Gamma(G, g_0)$ is an atomless simple pseudo-effect algebra which is not commutative.

PROPOSITION 4.11. *Let E be a pseudo-effect algebra with* (RDP). *Then $a \wedge b \in E$ if and only if $a \vee b \in E$. Moreover, if I is an ideal of E, $a, b \in I$ and $a \vee b$ is defined in E, then $a \vee b \in I$.*

Proof. Let $a \wedge b \in E$. Due to (RDP), there are $a_1, b_1, c \in E$ such that $a = a_1 + c$, $b = b_1 + c$, and $a_1 + b_1 + c = b_1 + a_1 + c \in E$. Since $c \le a \wedge b$, then $a_1 \ge a \setminus (a \wedge b)$, $b_1 \ge b \setminus (a \wedge b)$, which yield $x_0 = \big(a \setminus (a \wedge b)\big) + \big(b \setminus (a \wedge b)\big) + a \wedge b \in E$. According to (2.3) and (2.4), we have $a_0 + b_0 = a_0 \vee b_0$, where $a_0 = a \setminus (a \wedge b)$ and $b_0 = b \setminus (a \wedge b)$. We state $x_0 = a \vee b$. Assume $x \ge a, b$; then $x \ge a_0 + (a \wedge b)$ and $x \ge b_0 + (a \wedge b)$, which gives $x \setminus (a \wedge b) \ge a_0 \vee b_0$, i.e., $x \ge x_0 = a_0 \vee b_0 + (a \wedge b) = a_0 + b_0 + (a \wedge b) = a \vee b$.

Assume now $a \vee b \in E$. Then $(a \vee b)^- = a^- \wedge b^- \in E$ which gives by the above $a^- \vee b^- \in E$. Hence, $a \wedge b = (a^- \vee b^-)^\sim \in E$.

Let I be an ideal of E. Then by the first part of the present proof, $a \wedge b \in E$ and $a \vee b = \big(a \setminus (a \wedge b)\big) + \big(b \setminus (a \wedge b)\big) + a \wedge b$ which proves $a \vee b \in I$. □

PROPOSITION 4.12. *Let E be a pseudo-effect algebra with* (RDP) *and let P be a proper normal ideal of E. If $a \wedge b \in E$, then $[a \wedge b]_P = [a]_P \wedge [b]_P$ and $[a \vee b]_P = [a]_P \vee [b]_P$. Moreover, if E is a lattice, so is E/P.*

Proof. It is clear that $[a \wedge b]_P \le [a]_P, [b]_P$. Assume $[x]_P \le [a]_P$ and $[x]_P \le [b]_P$. According to Proposition 4.1, there are $x_1, x_2 \in [x]_P$ such that $x_1 \le a$ and $x_2 \le b$. Since $x_1 \sim_P x_2$, there are $e, f \in P$ with $x_1 \setminus e = x_2 \setminus f$. Hence, for $x_0 = x_1 \setminus e$, we have $x_0 \in [x]_P$, and $x_0 \le x_1$, $x_0 \le x_2$. Consequently, $x_0 \le a, b$ which yields $x_0 \le a \wedge b$, i.e., $[x]_P = [x_0]_P \le [a \wedge b]_P$.

The second equality follows from the first part of the present proof and from Proposition 4.11. □

5. Antilattice po-groups and prime subgroups

In the present Section, we give new properties of ideals and their interrelations with corresponding notions in unital po-groups. In particular, we will characterize antilattice po-groups via prime subgroups.

Proposition 5.1. *Let $E = \Gamma(G,u)$, where (G,u) is a unital po-group satisfying* (RDP_1)*. Let I be an ideal of E. Set*

$$\begin{aligned}\delta(I) &= \{x \in G\colon x = x_1 - y_1 + \dots + x_n - y_n\,,\ x_i, y_i \in I\}\,,\\ \delta_c(I) &= \{h \in G\colon h = x + p_1 = y - p_2\,,\ x,y \in \delta(I)\,,\ p_1,p_2 \in G^+\}\,,\\ \delta_0(I) &= \{h_1 - h_2\colon h_1,h_2 \in \delta_c(I) \cap G^+\}.\end{aligned}$$

Then $\delta(I)$ and $\delta_c(I)$ are the subgroup and the convex subgroup, respectively, of G generated by I, and $\delta_0(I)$ is the largest directed convex subgroup of G that is contained in $\delta_c(I)$.

Let I and J be two ideals of E. Then $I \subseteq J$ if and only if $\delta(I) \subseteq \delta(J)$ if and only if $\delta_c(I) \subseteq \delta_c(J)$ if and only if $\delta_0(I) \subseteq \delta_0(J)$.

Proof. It is evident that $\delta(I)$ is a subgroup of (G,u) generated by I. By [Fuc, p. 18 (d)], $\delta_c(I)$ is a convex subgroup generated by I. The assertion on $\delta_0(I)$ follows from (4.2). Let $\delta(I) \subseteq \delta(J)$ and $a \in I$. Then $a = x_1 - y_1 + \dots + x_n - y_n = x_1 + \dots + y_1' + \dots + y_n'$, where $x_i, y_i \in J$. Then $a + y_1'' + \dots + y_n'' = x_1 + \dots + x_n$, where $y_i'' \geq 0$. Applying (RDP_1) in G, we have a system of positive elements $\{c_{ij}\}$ in G such that $a = c_{01} + \dots + c_{0n}$, $y_i'' = \sum\limits_j c_{ij}$ and $x_j = \sum\limits_i c_{ij}$. Then each $c_{ij} \in J$, i.e., $a \in J$.

Assume now $\delta_c(I) \subseteq \delta_c(J)$ and let $a \in J$. Then $a = x_1 - y_1 + \dots + x_n - y_n - p$, where $x_i, y_j \in J$ and $p \in G^+$. Then $a + p = x_1 + \dots + x_n + y_1' + \dots + y_n'$ which gives $a + p + y_1'' + \dots + y_n'' = x_1 + \dots + x_n$ $(y_j'' \geq 0)$. Applying again (RDP_1) to this equality, we obtain $a \in J$.

Finally, let $\delta_0(I) \subseteq \delta_0(J)$ and let $a \in I$. Then $a \in \delta_c(J) \cap G^+$, so that $a \in J$ as above. ☐

Proposition 5.2. *Let $E = \Gamma(G,u)$, where (G,u) is a unital po-group satisfying* (RDP_1)*. Let K be a convex subgroup of (G,u). Then*

$$\gamma(K) := K \cap E\,,$$

is an ideal of E, and $\delta_c\big(\gamma(K)\big) \subseteq K$. If K is directed, then $\delta_0\big(\gamma(K)\big) = K$, and $\gamma\big(\delta_0(I)\big) = I$ for any ideal I of E. Moreover, if K_1 and K_2 are two directed convex subgroups of (G,u), then $\gamma(K_1) \subseteq \gamma(K_2)$ if and only if $K_1 \subseteq K_2$.

Proof. It is clear that $\delta_0\big(\gamma(K)\big) \subseteq K$. Let now K be directed and let $g \in K$, $g \geq 0$. Then $g = x_1 + \dots + x_n$, where each $x_i \in E$, which proves

$x \in \delta_0(\gamma(K))$. If now $x \in K$, then $x = x_1 - x_2$, where $x_1, x_2 \in K$ and $x_1, x_2 \geq 0$. Hence, $x \in \delta_0(\gamma(K))$.

Let I be an ideal of E. Then $\gamma(\delta_0(I)) \supseteq I$. Assume $x \in \gamma(\delta_0(I))$, then $x \in E$ and $x = x_1 - y_1 + \cdots + x_n - y_n - p$, where $x_i, y_j \in I$ and $p \in G^+$. This gives $x + p = x_1 + \cdots + y_1' + \cdots + y_n'$ and $x + p + y_1'' + \cdots + y_n'' = x_1 + \cdots + x_n$. Applying (RDP_1), we have $x \in I$.

The last statement follows from Proposition 5.1. ☐

Remark 5.3. We recall that if I is a normal ideal of E, where E satisfies the conditions of Theorem 4.2, then $\phi(I)$ defined by (4.3) and $\delta(I)$ and $\delta_0(I)$ coincide.

PROPOSITION 5.4. *Let $E = \Gamma(G, u)$, where (G, u) is a unital po-group satisfying (RDP_1). Then every maximal ideal of E is normal if and only if every maximal directed convex subgroup of (G, u) is an o-ideal of (G, u).*

Proof. Let I be a maximal ideal of E, then it is normal and $\phi(I) = \delta_0(I)$ is an o-ideal of (G, u) by Theorem 4.2. Let now K be a maximal directed convex subgroup of (G, u). Then $I = \gamma(K)$ is an ideal of E. Assume that J is any proper ideal of E containing I, i.e., $I \subseteq J$. Then by Proposition 5.2, $K = \delta_0(\gamma(K)) \subseteq \delta_0(J)$. The maximality of K implies $K = \delta_0(J)$, and therefore, $I = J$ which proves that I is maximal in E. By Theorem 4.2, K is an o-ideal of (G, u).

Conversely, let every maximal directed convex subgroup of (G, u) be an o-ideal of (G, u). Assume that I is a maximal ideal of E. Then $\delta_0(I)$ is a directed convex subgroup of (G, u). Assume that K is a proper directed subgroup of (G, u) containing $\delta_0(I)$. By Proposition 5.2, $\gamma(K) \subseteq \gamma(\delta_0(I)) = I$. The maximality of I yields $\gamma(K) = I$, i.e., $K = \delta_0(I)$ which proves $\delta_0(I)$ is a maximal directed convex subgroup of (G, u), consequently, it is an o-ideal. Applying Theorem 4.2, we have I is a normal ideal of E. ☐

Analogously to pseudo-effect algebras and Proposition 3.5, we say that a directed convex subgroup P of a po-group G is a *prime subgroup* of G if, for all directed convex subgroups I and J of G, $I \cap J \subseteq P$ implies $I \subseteq P$ or $J \subseteq P$. We denote by $\mathcal{P}(G, u)$ the set of all prime subgroups of a unital po-group (G, u).

PROPOSITION 5.5. *Let $E = \Gamma(G, u)$, where (G, u) is a unital po-group satisfying (RDP_1). If K is a prime subgroup of (G, u), then $\gamma(K) := K \cap E$ is a prime ideal of E, and if P is a prime ideal of E, then $\delta_0(P)$ is a prime subgroup of (G, u), where δ_0 is defined in Proposition 5.1. In addition, both mappings, γ and δ_0 are mutually bijective and preserving the set-theoretical inclusion.*

Moreover, the mappings γ and δ_0 restricted to normal prime ideals and prime o-ideals are mutually bijective.

Proof. Let K be a prime subgroup of (G,u), and assume $I\cap J\subseteq\gamma(K)=K\cap E$. By Proposition 5.1 and Proposition 5.2, $\delta_0(I)\cap\delta_0(J)\subseteq\delta_0(\gamma(K))=K$, so that $\delta_0(I)\subseteq K$ or $\delta_0(J)\subseteq K$ which gives by Proposition 5.2 $I=\gamma(\delta_0(I))\subseteq\gamma(K)$ or $J=\gamma(\delta_0(J))\subseteq\gamma(K)$.

Conversely, let P be a prime ideal of E and assume that, for two directed convex subgroups K_1 and K_2 of (G,u), we have $K_1\cap K_2\subseteq\delta_0(P)$. Hence $\gamma(K_1)\cap\gamma(K_2)\subseteq\gamma(\delta_0(P))=P$ which gives $\gamma(K_1)\subseteq P$ or $\gamma(K_2)\subseteq P$, i.e., $K_1=\delta_0(\gamma(K_1))\subseteq\delta_0(P)$ or $K_2=\delta_0(\gamma(K_2))\subseteq\delta_0(P)$ which proves that $\delta_0(P)$ is prime. □

For $a\in G$, let $G_0(a)$ be a directed convex subgroup of G generated by a. If G satisfies (RDP), then by [Fuc, p. 18], for $a\geq 0$

$$G_0(A)=\{x\in G\colon x=na+p=ma-q\,,\ n,m\in\mathbb{Z}\,,\ p,q\in G^+\}\,.$$

Proposition 5.6. *For a directed convex subgroup P of a po-group G the following statements are equivalent:*

(i) *P is prime.*

(ii) *For $a,b\in G$, $G_0(a)\cap G_0(b)\subseteq P$ implies $a\in P$ or $b\in P$.*

(iii) *For $a,b\in G^+$, $G_0(a)\cap G_0(b)\subseteq P$ implies $a\in P$ or $b\in P$.*

Proof. (i) $\Rightarrow$ (ii). It is evident.

(ii) $\Rightarrow$ (i). Assume that $G_0(a)\cap G_0(b)\subseteq P$ implies $a\in P$ or $b\in P$. Let, for two directed convex subgroups I and J of G, us have $I\cap J\subseteq P$. If $I\nsubseteq P$ and $J\nsubseteq P$, there are two elements $a\in I\setminus P$ and $b\in J\setminus P$. Then $G_0(a)\cap G_0(b)\subseteq I\cap J\subseteq P$ which gives $a\in P$ or $b\in P$, a contradiction.

(ii) $\Rightarrow$ (iii). It is evident.

(iii) $\Rightarrow$ (ii). As in the implication (ii) $\Rightarrow$ (i) we can find two elements $a\in I\setminus P$ and $b\in J\setminus P$. There are two elements $a_1,a_2\in G_0(a)$ such that $a=a_1-a_2$. Since $-a_2\leq a\leq a_1$, we have $a_1\notin P$ or $a_2\notin P$. In a similar way, $b=b_1-b_2$, where $b_1,b_2\in G_0(b)$, and $b_1\notin P$ or $b_2\notin P$. Say $a_1\notin P$ and $b_1\notin P$, then $G_0(a_1)\cap G_0(b_1)\subseteq I\cap J\subseteq P$ which gives $a_1\in P$ or $b_1\in P$, a contradiction. □

We recall that a po-group G is an antilattice iff $a\wedge b=0$ implies $a=0$ or $b=0$.

Proposition 5.7. *A directed po-group G with* (RDP) *is an antilattice if and only if the null subgroup $\{0\}$ is prime.*

Proof. Let $\{0\}$ be a prime ideal and let $a\wedge b=0$. Then $G_0(a)\cap G_0(b)=\{0\}$ which proves $a=0$ or $b=0$, i.e., G is an antilattice.

Conversely, let G be an antilattice, and let $G_0(a)\cap G_0(b)=\{0\}$. Then $a\wedge b=0$ which yields $a=0$ or $b=0$, i.e., $\{0\}$ is prime. □

PROPOSITION 5.8. *A proper o-ideal P of a unital po-group (G,u) satisfying* (RDP) *is prime if and only if G/P is an antilattice.*

P r o o f . Let P be prime and assume $a/P \wedge b/P = 0/P$. Without loss of generality we can assume $a, b \geq 0$ (indeed, there are $e, f \in P$ such that $0 \leq a+e$ and $0 \leq b+f$ and $e = e_1 - e_2$, $f = f_1 - f_2$, where $e_1, e_2, f_1, f_2 \in P^+$, which gives $e_2 \leq a + e_1$ and $f_2 \leq b + f_1$).

We claim $G_0(a) \cap G_0(b) \subseteq P$. There are two cases.

(i) Let $x \in G_0(a) \cap G_0(b)$ and let $x \geq 0$. Then $x = a_1 + \cdots + a_m = b_1 + \cdots + b_n$, where $a_i \leq a$ and $b_j \leq b$ for all i and all j. (RDP) implies that there is a system $\{c_{ij}\}$ of elements of E such that $a_i = \sum_j c_{ij}$ and $b_j = \sum_i c_{ij}$. Since $c_{ij} \leq a, b$, we have $c_{ij}/P = 0/P$, i.e., $c_{ij} \in P$, which yields $a_i \in P$ and $x \in P$.

(ii) Let $x \in G_0(a) \cap G_0(b)$. Then $x = x_1 - x_2 = -y_1 + y_2$, where $x_1, x_2 \in G_0(a)^+$ and $y_1, y_2 \in G_0(b)^+$. Due to (RDP) holding in (G,u), there are four elements $c_{11}, c_{12}, c_{21}, c_{22} \in G^+$ such that $y_1 = c_{11} + c_{12}$, $x_1 = c_{21} + c_{22}$, $y_2 = c_{11} + c_{21}$, and $x_2 = c_{12} + c_{22}$. By (i), $c_{12}, c_{21} \in P$. Therefore, $x = x_1 - x_2 = c_{21} + c_{22} - c_{22} - c_{12} = c_{21} - c_{12} = -c_{12} - c_{11} + c_{11} + c_{12} \in P$.

Combining (i) and (ii), we have $G_0(a) \cap G_0(b) \subseteq P$ which gives $G_0(a) \subseteq P$ or $G_0(b) \subseteq P$, i.e., $a \in P$ or $b \in P$, and $a/P = 0/P$ or $b/P = 0/P$ proving that G/P is an antilattice.

Conversely, let G/P be an antilattice. According to Proposition 5.6, let for $a, b \in G^+$ we have $G_0(a) \cap G_0(b) \subseteq P$. We claim that $G_0(a/P) \cap G_0(b/P) = \{0/P\}$.

Indeed, let $0/P \leq x/P = a_1/P + \cdots + a_m/P = b_1/P + \cdots + b_n/P$, where $a_i/P \leq a/P$ and $b_j/P \leq b/P$. There are $a'_i \in a_i/P$ and $b'_j \in b_j/P$ such that $a'_i \leq a$ and $b'_j \leq b$. Hence, there is a system $\{c_{ij}\}$ of elements in G^+ such that $a'_i/P = \sum_j c_{ij}/P$ and $b'_j/P = \sum_i c_{ij}/P$. For each i and each j, there is an element $c'_{ij} \in c_{ij}/P$ such that $c'_{ij} \leq a'_i, b'_j$, i.e., $c'_{ij} \leq a_i, b_j$. Then $c'_{ij} \in P$ which proves $x \in P$.

If now $x/P \in G_0(a/P) \cap G_0(b/P)$, then $x/P = x_1/P - x_2/P = -y_1/P + y_2/P$, where $x_1/P,\ x_2/P \in G_0(a/P)^+$ and $y_1/P,\ y_2/P \in G_0(b/P)^+$. As in (ii), we can prove $G_0(a/P) \cap G_0(b/P) = \{0/P\}$. Since G/P is an antilattice, by Proposition 5.7, $\{0/P\}$ is a prime ideal of G/P, consequently $a/P = 0/P$ or $b/P = 0/P$, i.e., $a \in P$ or $b \in P$ proving that P is prime. □

THEOREM 5.9. *Let $E = \Gamma(G,u)$, where (G,u) is a unital po-group with* (RDP_1)*. Then E is an antilattice if and only if G is an antilattice.*

P r o o f . It follows from Proposition 4.6, Proposition 5.5, Proposition 5.8, and Theorem 4.2. □

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