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# LATTICES OF ANTICHAINS OF A ROOT SYSTEM

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*Dedicated to Professor Ján Jakubík on the occasion of his 80th birthday*

ABSTRACT. Except for very special cases the lattice of antichains $A(I)$ of a root system $I$ is not projectable. However, it enjoys a related property named $r$-projectability. Here we point at some more properties which are satisfied by $A(I)$, and which, when satisfied in a lattice, imply that this lattice can be well embedded in a lattice of antichains.

After introducing the needed notions, we prove (3.12) that a distributive lattice, with a least element, which is $r$-projectable, branching and separated, can be embedded in the lattice of antichains of a root system, in such a way that this embedding commutes with relative orthogonals, respects components, and separates points in the root system. (This result can be used when studying some classes of lattice ordered groups which enjoy some of the properties of full Hahn powers on a root system.)

## 1. Root systems

A partially ordered set is a *root-system* $(I, \leq)$ if no two incomparable elements have a lower bound (an equivalent criterion is that for all $i \in I$, $\{j : i \leq j\}$ is totally ordered). Classical examples are the set of values of a lattice-ordered group, and the real spectrum of a commutative ring.

### 1.1 Root-systems

In this section $I$ and $J$ will stand for root systems.

We use here the notation $i \perp j$ to say that two elements $i$ and $j$ of a root system are incomparable; then $i^{\perp}$ denotes the subset of those elements which are incomparable with $i$, we call it the *orthogonal* of $i$. We say that a root

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system is *branching* if two comparable elements have different orthogonals: $(\forall\, i,$ $j \in I)(j < i \Rightarrow \exists\, k < i;\; j \perp k)$: ($I$ is branching if for all $i, j \in I$, such that $j < i$, there exists $k < i$ such that $j \perp k$).

To each root system $I$ we can associate a branching quotient $\hat{I}$, saying that two points are equivalent if they have the same orthogonal.

We say that a point is a branching point if it is the join of two incomparable elements. (In the root system $I_3 = \{i, j, k\}$ with $j < i$, $k < i$ and $k \perp j$, $i$ is a branching point). Remark that there are branching root systems without any branching points, as it can be seen in the following example: We consider first the set of finite sequences of reals whose terms are required to be rational except the last one. We equip this set with the partial order: $x \leq y$ if and only if $\mathrm{length}(x) \geq \mathrm{length}(y) = n$ & $m < n \Rightarrow x_m = y_m$ & $x_n \leq y_n$. (Here $\mathrm{length}(x)$ is the number of terms of the finite sequence $x$.) This defines a branching root system in which the branching points are the sequences with only rational terms. The subset of sequences with an irrational last term is a branching root system without any branching point.

### 1.2 Antichains

An *antichain* of $I$ is a subset of pairwise incomparable elements of $I$.

We denote by $A(I)$ the set of antichains of $(I, \leq)$, and by $A_f(I)$ the subset of finite antichains. The finiteness condition can be changed into another one bounding the infinite cardinality of antichains, giving rise to $A_\alpha(I)$. All of them are partially ordered by:

$a \leq b$ iff $(\forall\, t \in a\; \exists\, t' \in b;\; t \leq t')$ as defined in [12].

$A(I)$ is *laterally-complete* (i.e., any family of pairwise orthogonal elements has a least upper bound).

Without further mention, *all the lattices considered below will be distributive with least element.*

In such a lattice, we say that $c$ is a *component* of $a$ if, for some $b$, $a = b \vee c$ and $b \perp c$ and $c \neq 0 \neq b$. Then we write $a = b \oplus c$ and $c \subseteq' a$. The relation $\subseteq'$ so defined is a partial order. If $L$ is a subset of some $A(I)$ then $\subseteq'$ coincides with $\subseteq$. An element is *irreducible* if it has no component.

A lattice $L$ is *separated* ([10]) if the join of two elements $a$ and $b$ is an orthogonal join of a component of $a$ and a component of $b$:

$(\forall\, a, b \in L)\; (\exists\, a_1, a_2, b_1, b_2)\; (a = a_1 \oplus a_2$ & $b = b_1 \oplus b_2$ & $a \vee b = a_1 \oplus b_2)$.

Remark that this implies $a \wedge b = a_2 \oplus b_1$.

It is easy to verify that the lattices $A(I)$ and $A_f(I)$ are separated.

## 2. $r$-projectable lattices

It is well known that the projectability property for an $\ell$-group $G$ is in fact a property of the lattice $G^+$: The usual definition says that $G$ is projectable if every principal polar is a cardinal summand, and an equivalent property is that $G^+$ is a locally pseudocomplemented lattice, i.e., for each positive $g \in G$ the positive interval bounded by $g$ is pseudocomplemented.

In [11] we introduced the notion of an $r$-projectable $\ell$-group as a property of its lattice of positive elements:

**Definition 2.1.** A lattice $L$ is called *$r$-projectable* if it has a least element 0 and satisfies the following condition:

$$(\forall\, a, b \neq 0)(\exists\, a', a'')\bigl(a' \perp b \ \&\ a = a' \oplus a'' \&$$
$$(\forall\, c \neq 0,\ d \neq 0)(a'' = c \oplus d \Rightarrow (c \notin b^{\perp} \ \&\ d \notin b^{\perp}))\bigr)\,.$$

The last condition says that there is a component of $a$ which is orthogonal to $b$ and, in some sense, maximal with this property.

The elements $a'$ and $a''$ whose existence is asserted in (1) are uniquely determined. We denote them respectively by $a^{\perp b}$ the $b$-orthogonal of $a$ and $a_b$ the $b$-projection of $a$. $a^{\perp b}$ is the largest component of $a$ orthogonal to $b$.

The lattices $A(I)$ and $A_f(I)$ of antichains of a root system $I$, are $r$-projectable. In these two cases

$$a^{\perp b} = \{t \in a \colon t \perp b\}, \quad \text{and} \quad a_b = \bigl\{t \in a \colon (\exists\, t' \in b;\ t' \leq t \ \text{or}\ t \leq t')\bigr\}\,.$$

Each laterally-complete sublattice of $A(I)$ containing all the singletons $\{i\}$, $i \in I$, is $r$-projectable.

### 2.1 The rooted case

In [11] we defined an element $a \in L$ to be *sup-irreducible* if it is not the join of two incomparable elements; in a lattice which is separated, the sup-irreducible elements are exactly the irreducible ones. In $A(I)$ the sup-irreducible elements are the singletons. Remark that this way we can recover $I$ in $A(I)$ by a first order formula. In [9] D. Glushankof studied the relations between $(I, \leq)$ and $\bigl(A(I), \leq\bigr)$.

We defined a lattice $L$ to be *rooted* if, above each $a \in L$ there is a sup-irreducible element, and the subset $I(L)$ of sup-irreducible elements is a root system. (One can prove directly that if $L$ is rooted and $r$-projectable, and $I(L)$ is branching, then $L$ can be embedded in $A\bigl(I(L)\bigr)$. This is a particular case of our 3.12.)

Let $R$ be the relation given by $R(a, b)$ iff

$$b^{\perp} \subseteq a^{\perp} \ \&\ (\forall\, b', b'')$$
$$\bigl((b = b' \oplus b'') \Rightarrow (\exists\, a', a'';\ a = a' \oplus a'' \ \&\ a' \in b'^{\perp\perp} \ \&\ a'' \in b''^{\perp\perp})\bigr)\,. \quad (2)$$

Roughly speaking, the second part of the formula says that for each orthogonal decomposition of $b$ there is a compatible orthogonal decomposition of $a$. The relation $R$ is reflexive and transitive. Remark that if $L$ is $r$-projectable and rooted, then $R$ is the order relation in $L$.

In a more general context we say that $L$ is *branching* if $R$ coincides with the order.

## 3. The general case, without any assumption on $I(L)$

In this section, $L$ will stand for an $r$-projectable distributive lattice with a least element $0$, which is supposed to be separated and branching, without any assumption on $I(L)$ (which can be empty for example). We construct a special embedding of $L$ in a lattice of antichains.

### 3.1 Branching and separated $r$-projectable lattices

Before developing properties of the lattices which are $r$-projectable, separated and branching, we give an example of a lattice which is $r$-projectable and branching but is not separated:

Let $I$ be the disjoint union of a family of copies of $I_3$ indexed by $\mathbb{Z}$ and $L = \langle I, a, b\rangle$ the sublattice of $A(I)$ generated by the points of $I$, $a$ and $b$, where $a = \{i_m, j_{m'}, k_{m'},\ m < 0, m' \geq 0\}$ and $b = \{j_m, k_m, i_{m'},\ m < 0, m' \geq 0\}$. $L$ is $r$-projectable and branching but is not separated.

*From now on, the considered lattices are r-projectable, separated and branching.*

We look at the relation "$a$ is a component of $b$" ($a \subseteq' b$), which is a partial order. When $c$ and $d$ have a meet for this order, we denote it by $c \cap' d$. The following properties are easily verified:

1. $a_b \subseteq' a$, $a^{\perp b} \subseteq' a$, $a \perp b$ implies $a \cap' b = 0$.
2. If $a \leq b$, $a \subseteq' c$ and $b \subseteq' c$ then $a \subseteq' b$.
3. $b^{\perp a} = \sup\{c \colon c \subseteq' b \ \&\ c \perp a\}$ and $b_a = \big\{c \colon c \subseteq' b \ \&\ \forall c' \neq 0\ (c' \subseteq' c \Rightarrow c' \notin a^{\perp})\big\}$.
4. If $a \subseteq' b$ then $c_a \subseteq' c_b$.
5. If $c \subseteq' a$ and $d \subseteq' a$, then $c \cap' d = c_d = d_c = c \wedge d$.
6. If $a \leq b \leq c$ and $d \subseteq' a$ and $d \subseteq' c$, then $d \subseteq' b$.

**Lemma 3.1.** *If $a, b \in L$ ($r$-projectable and branching), $a \leq b$ and $a^{\perp} = b^{\perp}$ then $a$ has a proper component whose components are not components of $b$.*

P r o o f. We give it as an example of how making use of the branching hypothesis. There is an orthogonal decomposition of $a$: $a = u \oplus v$ such that for all proper orthogonal decompositions of $b$: $b = u' \oplus v'$ either $u' \notin u^{\perp\perp}$ or $v' \notin v^{\perp\perp}$. Consider $b_u$, $b_v$ and $e = b_u \wedge b_v$ which is a common component of $b_u$ and $b_v$.

If $e = 0$ then $b = b_u \oplus b_v \oplus b'$, but $b'$ is orthogonal to both $u$ and $v$, hence, owing to $a^\perp = b^\perp$, $b' = 0$. Therefore by hypothesis $b_u \notin u^{\perp\perp}$, there is $w$ such that $w \perp u$, $w \wedge b_u \neq 0$, $(w \wedge b_u) \perp u$. We have $b_u \perp v$ so $(w \wedge b_u)$ is orthogonal to $v$, hence to $a$, and by the second hypothesis it is orthogonal to $b$, contradiction.

If $e \neq 0$, then $u_e$ is a component of $a$ smaller than $b$. Let $c$ be a component of $u_e$, if it is a component of $b$, by the last one of the previous list of properties, it will be a component of $e$. But it is a component of $u$, hence orthogonal to $v$, contradicting the fact that it has to be a component of $b_v$. ☐

**Lemma 3.2.** *Suppose $d^\perp = (c \oplus c')^\perp$ and $d = d_c = d'_c$ then $c \leq d$ and $c' \leq d$.*

P r o o f . Easy, using the fact that $L$ is separated. ☐

### 3.2 Filters

We *denote* by $\mathcal{F}$ the set of filters of the ordered set $L' = (L, \subseteq', 0)$: $F \in \mathcal{F}$ iff:

1. $F \in \mathcal{P}(L)$, $0 \notin F$;
2. if $a \in F$ and $a \subseteq' b$ then $b \in F$;
3. $(\forall\, a, b \in F)$ $(\exists\, a' \in F)$ $(a' \subseteq' a$ & $a' \subseteq' b)$.

We *denote* by $\mathcal{M}(L)$ the set of maximal filters of $L'$. (By Zorn's lemma, each filter is contained in a maximal one.) One can prove that:

**Lemma 3.3.** *$F$ is maximal if and only if it is $\subseteq'$-prime $(\forall\, a, b \in L)\big((a \perp b$ & $a \oplus b \in F) \Rightarrow (a \in F$ or $b \in F)\big)$.*

For $a \in L$ we *define* $I(a)$ to be the subset of those elements $b \in L$ such that their $a$-projection $b_a$ is less than $a$, and none of the components of $b_a$ is a component of $a$ $(c \subset' a \Rightarrow \neg c \subset' b_a)$.

It is clear that: $a \notin I(a)$; $a^\perp \subseteq I(a)$; $b \in I(a)$ if and only if $b_a \in I(a)$; and if $b \geq a_b$ then $b \notin I(a)$. It is also easy to verify that if $a \neq 0$ then $I(a)$ is an $\leq$-ideal.

**Lemma 3.4.**

1. *If $a \neq 0$ and $a \subseteq' b$ then: $I(b) \subseteq I(a)$.*
2. *If $b \perp c$ then: $b \oplus c \in I(a)$ iff $\big(b \in I(a)$ and $c \in I(a)\big)$.*
3. *If $0 \neq a \leq b$ and $b = b_a$ then $I(a) \subseteq I(b)$.*

Proof. 1) First $a \subseteq' b$ implies that for each $c$, $c_a \subseteq' c_b$ so $c_a \leq c_b$ and $(c_a)_a = c_a \leq (c_b)_a$. Let $c \in I(b)$: we have $c_b \leq b$ so $(c_b)_a \leq b_a$, hence $c_a \leq a$. In order to prove that $c \in I(a)$ we have to prove that there is no common component of $a$ and $c_a$. Suppose $d \subseteq' a$ and $d \subseteq' c_a$, this implies $d \subseteq' b$ and $d \subseteq' c_b$ which is impossible since $c \in I(b)$.

2) $I(a)$ is an $\leq$-ideal so $b \oplus c \in I(a)$ implies $b \in I(a)$ and $c \in I(a)$. Conversely if $b \in I(a)$ and $c \in I(a)$ then $b \vee c = b \oplus c \in I(a)$.

3) Let $d \in I(a)$, we have $d = d_a \oplus d^{\perp a}$, and by 2) $d_a \in I(a)$ and $d^{\perp a} \in I(b)$. Since $d \in I(a)$ we have $d_a \leq a$ so $d_a = (d_a)_b \leq b$ and no component of $d_a$ can be a component of $b$; hence $d_a \in I(b)$. So we can restrict ourselves to the case $d \perp a$. Consider $d \in I(a)$, $d \perp a$, and suppose $d \notin I(b)$. Then:

If $d_b \leq b$, and there were a common component of $d_b$ and $b$, this would contradict $b_a = b$ and $d \perp a$.

If not, since $L$ is separated, there are $0 \neq b' \subseteq' b$ and $0 \neq d' \subseteq' d_b$ such that $b' \leq d'$ and since $d \perp a$, $d' \wedge a = 0$, $b' \wedge a = 0$ which contradicts $b_a = b$. □

We *define* $I(F) = \{c \in L \colon (\exists\, a \in F;\ c \in I(a))\}$. It is an $\leq$-ideal.

**Lemma 3.5.** *1. If $c$ belongs to $F$ then it does not belong to $I(F)$.*

*2. If $a \perp b$ and $b \in F$ then $a \in I(F)$.*

**Lemma 3.6.** *If a filter $F$ is $\subseteq'$-prime, then the $\leq$-ideal $I(F)$ is prime.*

Proof. We shall prove that if $x \wedge y \in I(F)$ then $x \in I(F)$ or $y \in I(F)$. Suppose $x \wedge y \in I(F)$, then there is some $a \in F$ such that $x \wedge y \in I(a)$, i.e., $(x \wedge y)_a \leq a$ and no component of $x \wedge y$ is a component of $a$. We can add the hypothesis $x_a = x$ and $y_a = y$, because if $x_a \in F$ ($y_a \in F$) then $x \in F$ ($y \in F$). Since $L$ is separated there are $x', x'', y', y'' \in L$ such that $x = x' \oplus x''$, $y = y' \oplus y''$, $x' \leq y''$, $x'' \leq y'$ and $x \wedge y = x' \oplus y'$.

If $a_{x'} \in F$ let $x_{(a_{x'})} = x'_a \oplus v$, then $x = x'_a \oplus v \oplus w$ with $w \perp a_{x'}$ so if $x'_a \oplus v \in I(F)$ then $x \in I(F)$. We have $x'_a \leq a$, let $a^* = a_v$, then $a_{x'_a} = a^* \oplus a'$ for some $a'$, and $x'_a = x'_{a^*} \oplus x'_{a'}$. Applying 3.4 to $x'_{a^*}$ and $v$ we obtain that $x'_{a^*} \leq a^*$ and $v \leq a^*$ and no component of $x'_{a^*} \oplus v$ is a component of $a^*$. Hence $x'_{a^*} \oplus v \in I(a)$. We also have $x'_{a'} \leq a$ and $x'_{a'}$ is a component of $x'$, and no component of $x'_{a'}$ is contained in $a$, so $x'_{a'}$ is in $I(a)$, hence in $I(F)$.

If $a_{x'} \notin F$, then $b = a^{\perp a_{x'}} \in F$. In order to prove that $y \in I(F)$ it suffices to prove that $y_b \in I(F)$, we have $y_b = y'_b \oplus u$, $b = b_u \oplus b'$, $y'_b = y'_{b_u} \oplus y_{b'}$ and we conclude by the same argument as in the previous case. □

**Lemma 3.7.** *Let $F, F' \in \mathcal{M}(L)$, $a \in F$, $b \in F'$ such that $a \leq b$ and $I(F') \subseteq I(F)$ then:*

1. *There is $d \in F'$ such that $I(a) \subseteq I(d)$,*
2. *$I(F) = I(F')$.*

Proof. Remark first that if $c \in F$ then $b_c \in F'$. Indeed if $b_c \notin F'$, then $b^{\perp c} \in F'$, $b_c \in I(F') \subseteq I(F)$ and $b \in I(F')$. There would be $d \in F$ such that $b \in I(d)$, i.e., $b_d \leq d$ and no proper component of $b_d$ is a component of $d$. We can choose $d \subseteq' a$, then because $a \leq b$, we would have $d \leq b_d$ so $d = b_d$, contradiction. 1) There is $c$ such that $c \in F$ and $c \subseteq' a$. We have $c \leq a \leq b$ so $c \leq b_c$ and by 3.4.3, $I(b) \subseteq I(b_c)$, and because $c \subseteq' a$, $I(a) \subseteq I(c) \subseteq I(b_c)$. By the remark above, $b_c \in F'$ which proves 1). 2) is a clear consequence of 1). □

**Lemma 3.8.** *Let $F, F' \in \mathcal{M}(L)$, $a \in F$, $b \in F'$ with $a \perp a'$, then:*

1. *$I(a)$ is not included in $I(a')$,*
2. *$I(F)$ is not included in $I(F')$.*

Proof. 1) is clear because $a' \in I(a)$ and $a' \notin I(a')$.

2) $a' \in I(a)$ so $a' \in I(F)$, and $a' \in F'$ so $a' \notin I(F')$. □

**Lemma 3.9.** *Let $F, F' \in \mathcal{M}(L)$, the following are equivalent:*

1. *$I(F) \subseteq I(F')$,*
2. *$(\forall\, a \in F)\ \ (\exists\, a' \in F')\big(I(a) \subseteq I(a')\big)$.*

Proof. 1) $\Rightarrow$ 2) using definitions. 1) $\Rightarrow$ 2): Take $b \in F'$, $b = b_a \oplus b^{\perp a}$ with $b^{\perp a} \in I(F) \subseteq I(F')$, so $b^{\perp a} \notin F'$, which implies $b_a \in F'$. Therefore we can suppose $b = b_a$. Because $L$ is separated there are $a', a'', b', b''$ such that $a = a' \oplus a''$, $b = b' \oplus b''$, $a' \leq b'$ and $b'' \leq a''$. By 3.8, neither $a' \in F$ and $b'' \in F'$, nor $a'' \in F$ and $b' \in F'$ can happen. If $a'' \in F$ and $b' \in F''$ then by 3.7 $I(F) = I(F')$. If $a' \in F$ and $b' \in F'$ then by 3.4 $I(a') \subseteq I(b')$ then $I(a) \subseteq I(b' \oplus a'')$. □

**Remark.** We can have $I(F) = I(F')$ with $F \neq F'$, $F, F' \in \mathcal{M}(L)$: Consider $I = \{i_n : n \in \mathbb{N}\} \cup \{j_n : n \in \mathbb{N}\} \cup \{k_n : n \in \mathbb{N}\} \cup \{m_n : n \in \mathbb{N}\}$ endowed with the partial order $i_n \leq k_n$ and $j_n \leq k_n$ for each $n$. It is a root-system. Let $L$ be the sublattice of $A(I)$ generated by $A_f(I)$. For each $h \in \mathbb{N}$ consider the antichains $a_h = \{i_n, j_n, m_n : n \geq h\}$ and $b_h = \{k_n, m_n : n \geq h\}$. Given $a \in L$, it contains almost every $m_n$ and, either every $k_n$, or, almost every $i_n$ and almost every $j_n$. Let $F$ (resp. $F'$) be the filter generated by $\{a_h : h \in \mathbb{N}\}$, (resp. $\{b_h : h \in \mathbb{N}\}$). These two maximal filters are different but $I(F) = I(F')$. For each $h$, $a_h \leq b_h$ so $I(F') \subseteq I(F)$. If $c \in I(F)$ avoids $a_h$ then it contains only a finite number of $m_k$ so it avoids some $b_{h'}$; hence $I(F) \subseteq I(F')$. Besides if $L$ is closed by meets for $\subseteq'$, then $F = F'$ iff $I(F) = I(F')$.

**Lemma 3.10.** *Let $F, F', F'' \in \mathcal{M}(L)$, $c' \in F'$, $c'' \in F''$, if $c' \perp c''$ then $I(F') \nsubseteq I(F)$ or $I(F) \nsubseteq I(F')$.*

Proof. Suppose that $I(F) \subseteq I(F')$ and $I(F) \subseteq I(F'')$ then for each $a \in F$ there are $a' \in F'$ and $a'' \in F''$ such that $I(a)$ is contained in both $I(a')$ and $I(a'')$, and we can choose $a' \perp a''$. Since $L$ is separated, there are $a_1$ and $a_2$ such that $a = a^{\perp a' \oplus a''} \oplus a_1 \oplus a_2$ with $a_1 \le a' \oplus a''$ and $(a' \oplus a'')_{a_2} \le a_2$.

If $a^{\perp a' \oplus a''} \in F$ then $I(F) \nsubseteq I(F')$, contradiction.

If $a_1 \in F$: We have $a_1 \le a' \oplus a''$ then, since $L$ is branching $R(a_1, a' \oplus a'')$ so there are $a_1'$ and $a_1''$ such that $a_1 = a_1' \oplus a_1''$ with $a_1' \in (a')^{\perp\perp}$ and $a_1'' \in (a'')^{\perp\perp}$. Suppose $a_1' \in F$, since $a_1' \perp a''$, by 3.8.2 $I(F) \nsubseteq I(F'')$.

If $a_2 \in F$ then $(a' \oplus a'')_{a_2} \le a_2 = (a_2)_{a' \oplus a''}$ and $a' \in F'$. If $(a')^{\perp a_2} \in F'$, then by 3.5 it belongs to $I(F)$ and not to $I(F')$. If not, $(a')_{a_2} \in F'$, $a_2 \in F$ and $(a')_{a_2} \le a_2$, so by 3.7, if $I(F)$ is contained in $I(F')$, then they are equal and the assertion $I(F) \subseteq I(F'')$ would imply $I(F') \subseteq I(F'')$ which is false because $a'' \in I(F')$ and $a'' \notin I(F'')$. □

### 3.3 The root system

Let us denote by $\mathcal{I}(L)$ the set $\{I(F) \colon F \in \mathcal{M}(L)\}$ with the order given by set theoretical inclusion.

**Lemma 3.11.** *If (as assumed) $L$ is $r$-projectable, separated and branching, then $\mathcal{I}(L)$ is a root-system.*

Proof. Let $I, I', I'' \in \mathcal{I}(L)$, there are $F, F', F'' \in \mathcal{M}(L)$ such that $I = I(F)$, $I' = I(F')$ and $I'' = I(F'')$. Suppose $I \subseteq I'$ and $I \subseteq I''$ and $I' \nsubseteq I''$, we shall prove that $I'' \subseteq I'$: $\forall c \in F''\ \exists b \in F'\big(I(c) \subseteq I(b)\big)$. Since $I' \nsubseteq I''$ there is a $b \in F'$ such that for all $c \in F''$ $I(b) \nsubseteq I(c)$. Let $a \in F$, we have by previous lemmas $b_a \in F'$ and $b_a \in F''$ and $a_b \in F$ and $a_c \in F$. So there is a common component $a' \in F$ of $a_b$ and $a_c$. Now $b_{a'} \in F'$ and $c_{a'} \in F''$ and $a'$ is less or equal to both $c_{a'}$ and $b_{a'}$ so $a' \le c_{a'} \wedge b_{a'}$. Writing $c_{a'} \wedge b_{a'} = c_1 \oplus b_1$ with $c_1$ component of $c_{a'}$ and $b_1$ component of $b_{a'}$ and $c_1 \le b_{c_1}$ and $b_1 \le c_{b_1}$. Either $c_1 \in F''$ and $b_{c_1} \in F'$, or $b_1 \in F'$ and $c_{b_1} \in F''$. Suppose the second option, then $I(b) \subseteq I(b_1) \subseteq I(c_{b_1})$, contradiction, so $I(c) \subseteq I(c_1) \subseteq I(b_{c_1})$ and $I'' \subseteq I'$. □

**Theorem 3.12.** *Let $L$ be a lattice (distributive with a least element) which is $r$-projectable, branching and separated, then the map $\rho$ from $L$ into the lattice of antichains of the root-system $A\big(\mathcal{I}(L)\big)$ defined by $\rho(a) = \{I(F) \colon F \in \mathcal{M}(L)$ and $a \in F\}$ is an embedding of lattices which satisfies:*

1. *$\rho$ commutes with the relative projections, the relative orthogonals and respects the component relation: (i.e., $\rho(a^{\perp b}) = \big(\rho(a)\big)^{\perp \rho(b)}$ and $a \subseteq' b \Leftrightarrow \rho(a) \subseteq \rho(b)$).*

2. $\rho(L)$ *separates points of* $\mathcal{I}(L)$, *i.e.:*
$\forall I(F),\ I(F') \in \mathcal{I}(L)$,
$\big(I(F) \neq I(F') \Rightarrow \exists\, a \in F\big(I(F) \in \rho(a) \,\&\, I(F') \notin \rho(a)\big)\big)$.

Proof. Write $\mathcal{I}$ for $\mathcal{I}(L)$.

- For each $a \in L$, $\rho(a)$ is an antichain of $\mathcal{I}$. If $a$ belongs to two different $F$ and $F'$, then $I(F) \nsubseteq I(F')$. There are $a_1, a_2 \in F$ such that $a = a_1 \oplus a_2$ and $a_1, a_2 \notin F'$ (take $b$ such that $b \in F$ and $b \notin F'$, $a_1$ a common component of $a$ and $b$, and $a_2 = a^{\perp a_1}$). Then $a_2 \in I(a_1)$, $a_2 \in I(F)$, $a_2 \in F'$ and $a_2 \notin I(F')$.
- $\rho$ is injective: Suppose $a \neq a'$, then using the hypothesis on $L$, we can write $a = a^{\perp a'} \oplus a_1 \oplus a_2$ with $a_1 \leq a'$ and $a'_{a_2} \leq a_2$. If $a^{\perp a'} \neq 0$ it contains an $F \in \mathcal{M}(L)$ such that $a \in F$, $a' \in I(F')$ and $a \notin I(F)$. If $a^{\perp a'} = 0$ then one of $a_1$ and $a_2$ is $\neq 0$, suppose $a_1$, and $a_1 < a'_{a_1}$, there is a nonzero component $d$ of $a_1$ such that $d \leq a'$ and $d$ has no common component with $a'$. Then $a'_d$ contains some $F \in \mathcal{M}(L)$ such that $a' \notin I(F)$ and $a_1 \in I(F)$. The last case is similar.
- $a \leq b$ iff $\rho(a) \leq \rho(b)$, i.e.,
$\big(\forall\, F; I(F) \in \rho(a)\big)\big(\exists\, F'; I(F') \in \rho(b)\big)\ \big(I(F) \subseteq I(F')\big)$:

  Suppose $a \leq b$. Remark first that if $a$ is a component of $b$, then $\rho(a)$ is a subset (a component) of $\rho(b)$. So we can suppose without loss of generality that $a^{\perp} \neq b^{\perp}$. Given F such that $a \in F$, consider the filter $F'$ generated by $\{b_c \colon c \in F\}$ and $F''$ a maximal filter containing $F'$. We have $I(F) \subseteq I(F'')$ because for each $d \in F$ there is a component of $a$: $a' \in F$ such that $d \in I(a')$, i.e., $d_{a'} \leq a'$ and there is no common component of $d_{a'}$ and $a'$, so $d_{b_{a'}} \leq b_{a'}$ and there is no common component of $d_{b_{a'}}$ and $b_{a'}$.

  Conversely: $b = b^{\perp a} \oplus b_1 \oplus b_2$ with $b_1 \leq a$ and $a_{b_2} \leq b_2$. By hypothesis for each $F$ such that $a \in F$ there is $F'$ such that $b \in F'$ and $I(F) \subseteq I(F')$. If $b^{\perp a} \neq 0$, for all $F'$ if $b \in F'$ then $b^{\perp a} \in F'$ and $a \notin F'$ so $I(F) \nsubseteq I(F')$ contradiction. If $b_1 \in F'$, since $a \in F$ and $b_1 \leq a$ we would have $I(F) = I(F')$. We can conclude that $b_1 = a_{b_1}$ so we proved that $a \leq b$.
- $\rho(a \vee b) = \rho(a) \vee \rho(b)$: Suppose $\alpha \in A\big(\mathcal{I}(L)\big)$ greater than both $\rho(a)$ and $\rho(b)$, we shall prove that $\big(\forall\, F; I(F) \in \rho(a \vee b)\big)\big(\exists\, F'; \big(I(F') \in \alpha \,\&\, I(F) \subseteq I(F')\big)\big)$. So suppose $F \in \rho(a \vee b)$. We have $a \vee b = a^{\perp b} \oplus a_1 \oplus b_1$. One and only one of these components is in $F$. If $a^{\perp b}$ or $a_1$ is in $F$, then $a \in F$, $I(F) \in \rho(a)$ so there is $F'$ such that $I(F) \subseteq I(F') \in \alpha$. If not $I(F) \in \rho(b)$ so there is $F'$ such that $I(F) \subseteq I(F') \in \alpha$.

- $\rho(a \wedge b) = \rho(a) \wedge \rho(b)$: Suppose $\alpha$ less or equal than both $\rho(a)$ and $\rho(b)$, we shall prove that $\alpha \leq \rho(a \wedge b)$. There are $F$, $F'$, $F''$ such that $I(F'') \in \alpha$, $I(F) \in \rho(a)$, $I(F') \in \rho(b)$ and $I(F'')$ is contained in both $I(F)$ and $I(F')$. We shall prove that there is $F'''$ such that $I(F''') \in \rho(a \wedge b)$ and $I(F'') \subseteq I(F''')$. Since $\mathcal{I}$ is a root system $I(F) \subseteq I(F')$ or $I(F) \subseteq I(F')$. Suppose that $I(F) \subseteq I(F')$, we can write $a \wedge b = a_1 \oplus b_1$ with $a_1$ component of $a$ and $b_1$ of $b$. If $a_1 \in F$ then $a \wedge b \in F$ so we can take $F''' = F$. If $b_1 \in F$ we can take $F''' = F'$.
- $\rho(a^{\perp b}) = \rho(a)^{\perp \rho(b)}$: Using the already proved properties, $a \perp b$ iff $\rho(a) \perp \rho(b)$, $\rho(a \oplus b) = \rho(a) \oplus \rho(b)$ and $a \subseteq' b \Rightarrow \rho(a) \subseteq \rho(b)$. So $\rho(a^{\perp b})$ is contained in $\rho(a)$ and orthogonal to $\rho(b)$. We prove that it is maximal with this property. Suppose $I(F) \in \rho(a)$, $I(F) \perp \rho(b)$ and $I(F) \notin \rho(a)^{\perp}$. We write $a = a^{\perp b} \oplus a_1 \oplus a_2$ with $a_1 \leq b$ and $b_{a_2} \leq a_2$. Therefore $\rho(a) = \rho(a^{\perp b}) \oplus \rho(a_1) \oplus \rho(a_2)$ and $I(F) \in \rho(a_1) \oplus \rho(a_2)$, it is not orthogonal to $\rho(b)$; there is $F'$ such that $b \in F'$ $(I(F') \in \rho(b))$ and $I(F) \subseteq I(F')$ or $I(F') \subseteq I(F)$. Contradiction.
- If $I(F)$ and $I(F')$ are different, then there is some $a \in L$ such that $I(F) \in \rho(a)$ and $I(F') \notin \rho(a)$.

$\square$

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