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ON TWO TYPES OF PARTIAL SUMMATIONS

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ABSTRACT. Let $\{P_j^*\}_{j=0}^{\infty}$, $\{P_j\}_{j=0}^{\infty}$ be discrete distributions on the nonnegative integers. Generalized partial summations, namely

$$P_x = \sum_{j=x}^{\infty} g(j) P_j^* , \quad x = 0, 1, 2, \dots$$

and

$$P_x = h(x) \sum_{j=x}^{\infty} P_j^* , \quad x = 0, 1, 2, \dots ,$$

are analyzed. Relations between the probability generating functions of the parent and descendant distributions are investigated. For a wide class of discrete distributions we can find the functions $g(x)$ and $h(x)$ such that the distribution remains unaltered under the above mentioned summations.

I. Introduction

Let $\{P_j^*\}_{j=0}^{\infty}$ (parent) and $\{P_j\}_{j=0}^{\infty}$ (descendant) be discrete probability distributions defined on the set of the nonnegative integers. So far three types of partial summations have been analyzed (in the following, a, b and c are proper constants). The summation

$$P_x = a \sum_{j=x}^{\infty} P_j^* , \qquad x = 0, 1, 2, \dots \tag{1}$$

is mentioned in [3] and is generalized in [5]. Due to [11] the parent and descendant distributions are identical if and only if they are geometric distributions.

The second type,

$$P_x = b \sum_{j=x}^{\infty} \frac{P_j^*}{j} , \qquad x = 0, 1, 2, \dots , \tag{2}$$

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can be found in [1] and is generalized in [7]. In [9] there are many pairs of parent and descendant distributions from the generalized hypergeometric family (see [8]). According to [12] the Yule distribution remains unaltered under this summation.

The third type yields

$$P_x = \frac{c}{x} \sum_{j=x}^{\infty} P_j^* , \qquad x = 0, 1, 2, \dots \tag{3}$$

The summation is introduced in [10]. Its generalization is given in [4]. The Salvia–Bolinger distribution (see [8]) is invariant with respect to this summation (proved in [10]).

II. Invariant distributions

Two types of partial summations will be considered. They are

$$P_x = \sum_{j=x}^{\infty} g(j) P_j^* , \qquad x = 0, 1, 2, \dots \tag{4}$$

and

$$P_x = h(x) \sum_{j=x}^{\infty} P_j^* , \qquad x = 0, 1, 2, \dots \tag{5}$$

It is easy to see that the summations (1), (2) and (3) are special cases of the summations (4) or (5). For a wide class of discrete distributions, the functions $g(x)$ and $h(x)$ can be found such that the distribution is invariant (i.e., the parent and descendant distributions are identical) under the summations (4) and (5), respectively.

THEOREM II.1. *Let $\{P_j^*\}_{j=0}^{\infty}$ be a discrete probability distribution defined on the set of the nonnegative integers. Let $P_{x+1}^* = f(x+1) P_x^*$, $x = 0, 1, 2, \dots$ The distribution is invariant (i.e., $P_x^* = P_x$, $x = 0, 1, 2, \dots$) with respect to the summation (4) if and only if*

$$g(x) = 1 - f(x+1) , \qquad x = 0, 1, 2, \dots$$

P r o o f. Let $P_x^* = P_x$, $x = 0, 1, 2, \dots$ For $x = 0, 1, 2, \dots$ it holds

$$P_x^* = \sum_{j=x}^{\infty} g(j) P_j^* \implies P_x^* - P_{x+1}^* = g(x) P_x^* \implies$$

$$g(x) = 1 - \frac{P_{x+1}^*}{P_x^*} = 1 - f(x+1) .$$

Let $g(x) = 1 - f(x+1)$. Then

$$P_x = \sum_{j=x}^{\infty} g(j) P_j^* = \sum_{j=x}^{\infty} \bigl(1 - f(j+1)\bigr) P_j^* = \sum_{j=x}^{\infty} P_j^* - \sum_{j=x}^{\infty} P_{j+1}^* = P_x^* .$$

□

The function $g(x)$ for the distributions belonging to the Katz family (see [8]) is derived in [6].

Throughout the paper let us suppose that a product with the lower limit greater than the upper one is equal to 1.

Theorem II.2. *Let $\{P_j^*\}_{j=0}^{\infty}$ be a distribution with the same properties as in the Theorem II.1. $\{P_j^*\}_{j=0}^{\infty}$ is invariant with respect to the summation* (5) *if and only if*

$$h(x) = \left(1 + \sum_{j=1}^{\infty} \prod_{k=1}^{j} f(x+k)\right)^{-1} .$$

Proof. Let $P_x^* = P_x$, $x = 0,1,2,\dots$ For $x = 0,1,2,\dots$ it holds

$$P_x^* = h(x) \sum_{j=x}^{\infty} P_j^* \implies$$

$$h(x) = \frac{P_x^*}{\sum\limits_{j=x}^{\infty} P_j^*} = \frac{P_0^* \prod\limits_{j=1}^{x} f(j)}{P_0^* \sum\limits_{j=x}^{\infty} \prod\limits_{k=1}^{j} f(k)} = \left(1 + \sum_{j=1}^{\infty} \prod_{k=1}^{j} f(x+k)\right)^{-1} .$$

Let

$$h(x) = \left(1 + \sum_{j=1}^{\infty} \prod_{k=1}^{j} f(x+k)\right)^{-1} .$$

Then

$$\begin{aligned} h(x) \sum_{j=x}^{\infty} P_j^* &= \left(1 + \sum_{j=1}^{\infty} \prod_{k=1}^{j} f(x+k)\right)^{-1} \sum_{j=x}^{\infty} P_0^* \prod_{k=1}^{j} f(k) \\ &= P_0^* \frac{1 + f(x+1) + f(x+1)f(x+2) + \dots}{1 + f(x+1) + f(x+1)f(x+2) + \dots} \prod_{j=1}^{x} f(j) = P_x^* . \end{aligned}$$

□

In the following ${}_mF_n(a_1,\dots,a_m;\, b_1,\dots,b_n;\, z)$ is a hypergeometric function, see, e.g., [2]. Let us suppose that

$$f(x+1)=\frac{P^*_{x+1}}{P^*_x}=C\,\frac{\prod\limits_{i=1}^{m}(x+a_i)}{\prod\limits_{j=1}^{n}(x+b_j)}\,.$$

We obtain

$$\frac{1}{h(x)}=1+\frac{\prod\limits_{i=1}^{m}(x+a_i)}{\prod\limits_{j=1}^{n}(x+b_j)}\,C+\frac{\prod\limits_{i=1}^{m}\big[(x+a_i)(x+a_i+1)\big]}{\prod\limits_{j=1}^{n}\big[(x+b_j)(x+b_j+1)\big]}\,C^2+\cdots \implies$$

$$h(x)=\big({}_{m+1}F_n(x+a_1,\dots,x+a_m,1;\, x+b_1,\dots,x+b_n;\, C)\big)^{-1}.$$

In addition, if $m=1$, $n=1$, $C=1$ and $b_1-a_1>1$ (i.e., $f(x+1)=\frac{x+a_1}{x+b_1}$, $b_1-a_1>1$) then

$$\begin{aligned} h(x)&=\big({}_2F_1(x+a_1,1;x+b_1;1)\big)^{-1}\\ &=\frac{\Gamma(x+b_1-1)\Gamma(b_1-a_1)}{\Gamma(x+b_1)\Gamma(b_1-a_1-1)}=\frac{b_1-a_1-1}{x+b_1-1}\,. \end{aligned}$$

Special cases of the functions $g(x)$ or $h(x)$ are in [6] (for the Katz family of distributions), [10], [11] and [12] (for the Salvia–Bolinger, geometric and Yule distribution, respectively). All the distributions and distribution families from the following table can be found in [8].

EXAMPLES.

For the Kemp-Dacey-hypergeometric family $(a_i, b_j, \theta\colon\, i=1,\dots,k;\, j=1,\dots,r)$ it holds

$$f(x+1)=\frac{\theta\prod\limits_{i=1}^{k}(x+a_i)}{(x+1)\prod\limits_{j=1}^{r}(x+b_j)}\,,$$

$$g(x)=1-\frac{\theta\prod\limits_{i=1}^{k}(x+a_i)}{(x+1)\prod\limits_{j=1}^{r}(x+b_j)}\,,$$

and

$$h(x)=\big({}_{k+1}F_{r+1}(x+a_1,\dots,x+a_k\,,1;\; x+b_1,\dots,x+b_r,\,x+1\,,:\;\theta)\big)^{-1}.$$

distribution (*parameters*)	$f(x+1)$	$g(x)$	$h(x)$
geometric (p)	$1-p$	p	p
Poisson (λ)	$\frac{\lambda}{x+1}$	$\frac{x-\lambda+1}{x+1}$	$\frac{1}{{}_1F_1(1;x+1;\lambda)}$
negative binomial (k,p)	$\frac{(x+k)(1-p)}{x+1}$	$\frac{1-k+p(k+x)}{x+1}$	$\frac{1}{{}_2F_1(x+k,1;x+1;1-p)}$
1-shifted logarithmic (θ)	$\frac{\theta(x+1)}{x+2}$	$\frac{x(1-\theta)-\theta+2}{x+2}$	$\frac{1}{{}_2F_1(x+1,1;x+2;\theta)}$
2-shifted Flory	$\frac{x+2}{2(x+1)}$	$\frac{x}{2(x+1)}$	$\frac{1}{{}_2F_1(x+2,1;x+1;\frac{1}{2})}$
Salvia-Bolinger (α)	$\frac{x-\alpha+1}{x+2}$	$\frac{\alpha+1}{x+2}$	$\frac{\alpha}{x+1}$
Yule (b)	$\frac{x+1}{x+b+2}$	$\frac{b+1}{x+b+2}$	$\frac{b}{x+b+1}$
family (*parameters*)	$f(x+1)$	$g(x)$	$h(x)$
power series $\left(a_x,\theta,f(\cdot)\right)$	$\frac{\theta a_{x+1}}{a_x}$	$\frac{a_x-\theta a_{x+1}}{a_x}$	$\frac{a_x}{\sum_{j=0}^{\infty} a_{x+j}\theta^j}$
Katz (α,β), $\beta\neq 0$	$\frac{\alpha+\beta x}{x+1}$	$\frac{1-\alpha+(1-\beta)x}{x+1}$	$\frac{1}{{}_2F_1(x+\frac{\alpha}{\beta},1;x+1;\beta)}$
Irwin $\left(a_x,\theta,\Psi(\cdot)\right)$	$\frac{a_{x+1}}{(x+\theta+1)a_x}$	$-\frac{(x+\theta+1)a_x-a_{x+1}}{(x+\theta+1)a_x}$	$\frac{a_x}{\sum_{j=0}^{\infty}\frac{a_{x+j}}{(x+\theta+1)^{(j)}}}$

III. Generalized partial summations

In the previous section the parent and descendant distributions were identical and the problem of finding proper functions $g(x)$ and $h(x)$ was solved. Now these functions and the parent distribution will be given and we shall study properties of the descendant distribution (in general different from its parent).

Let $f(x)$ be such a function that

$$P^*_{x+1} = f(x+1)P^*_x\,, \qquad x = 0,1,2,\dots \tag{6}$$

for a discrete distribution $\{P^*_j\}_{j=0}^{\infty}$. Let us consider the discrete distribution $\{P_x\}_{x=0}^{\infty}$ given by the summation

$$P_x = c_1\sum_{j=x}^{\infty} g(j)P^*_j\,, \qquad x = 0,1,2,\dots\,,$$

where c_1 is a proper constant, $g(x) = 1 - f(x+1)$ and $\{P^*_j\}_{j=0}^{\infty}$ is a distribution satisfying the conditions: (a) its mean μ^* is finite and (b) the sequence $\left\{\sum_{j=x}^{\infty} g(j)P^*_j\right\}_{x=0}^{\infty}$ does not change a sign.

Lemma III.1.

$$c_1 = \frac{1}{1 + \mu^* - \sum\limits_{x=0}^{\infty} (x+1) f(x+1) P_x^*}\,.$$

Proof.

$$1 = \sum_{x=0}^{\infty} P_x = c_1 \sum_{x=0}^{\infty} \sum_{j=x}^{\infty} g(j) P_j^* = c_1 \sum_{x=0}^{\infty} \sum_{j=x}^{\infty} \big(1 - f(j+1)\big) P_j^* \implies$$

$$c_1 = \frac{1}{1 + \mu^* - \sum\limits_{x=0}^{\infty} (x+1) f(x+1) P_x^*}\,.$$

□

It is easy to see that if $f(x+1) = \frac{P_{x+1}^*}{P_x^*}$ (i.e., the parent and descendant distributions are identical) then $c_1 = 1$.

Let $G^*(t) = \sum\limits_{x=0}^{\infty} P_x^* t^x$, $G(t) = \sum\limits_{x=0}^{\infty} P_x t^x$ be the probability generating functions of the distributions $\{P_j^*\}_{j=0}^{\infty}$ and $\{P_j\}_{j=0}^{\infty}$, respectively.

Theorem III.2.

$$G(t) = \frac{c_1}{1-t}\left(1 - tG^*(t) - \sum_{x=0}^{\infty} P_x^* f(x+1)\big(1 - t^{x+1}\big)\right).$$

Proof.

$$\begin{aligned}
G(t) &= \sum_{x=0}^{\infty} P_x t^x = c_1 \sum_{x=0}^{\infty} t^x \sum_{j=x}^{\infty} g(j) P_j^* = c_1 \sum_{x=0}^{\infty} t^x \sum_{j=x}^{\infty} \big(1 - f(j+1)\big) P_x^* \\
&= c_1 \sum_{x=0}^{\infty} P_x^* \big(1 - f(x+1)\big) \sum_{j=0}^{x} t^j \\
&= \frac{c_1}{1-t} \sum_{x=0}^{\infty} P_x^* \big(1 - f(x+1)\big)\big(1 - t^{x+1}\big) \\
&= \frac{c_1}{1-t}\left(1 - tG^*(t) - \sum_{x=0}^{\infty} P_x^* f(x+1)\big(1 - t^{x+1}\big)\right).
\end{aligned}$$

□

Special cases of the Theorem III.2 can be found in [5], [6], [7] and [9].

Let us consider the discrete distribution $\{Q_x\}_{x=0}^{\infty}$ given by the summation

$$Q_x = c_2 h(x) \sum_{j=x}^{\infty} P_j^* , \qquad x = 0, 1, 2, \dots ,$$

where c_2 is a proper constant, $h(x) = \left(1 + \sum\limits_{j=1}^{\infty} \prod\limits_{k=1}^{j} f(x+k)\right)^{-1}$ ($f(x)$ being the function from (6)) and $\left\{P_j^*\right\}_{j=0}^{\infty}$ is a distribution with a finite mean. Obviously

$$c_2 = \frac{1}{\sum\limits_{x=0}^{\infty} h(x) \sum\limits_{j=x}^{\infty} P_j^*}$$

and

$$H(t) = c_2 \sum_{x=0}^{\infty} \left(\sum_{j=x}^{\infty} P_j^* \right) h(x) t^x ,$$

$H(t) = \sum\limits_{x=0}^{\infty} Q_x t^x$ being the probability generating function of the distribution $\{Q_j\}_{j=0}^{\infty}$. One can find special cases of the relationship between the probability generating functions of the parent and descendant distributions in [4] and [10].

If $h(x) = \frac{a}{b+x}$ then

$$\begin{aligned}
H(t) &= c_2 \sum_{x=0}^{\infty} \left(\sum_{j=x}^{\infty} P_j^* \right) h(x) t^x = c_2 \sum_{x=0}^{\infty} P_x^* \sum_{j=0}^{x} \frac{a}{b+j} t^j \\
&= \frac{ac_2}{t^b} \sum_{x=0}^{\infty} \sum_{j=0}^{x} P_j^* \frac{t^{b+j}}{b+j} = \frac{ac_2}{t^b} \sum_{x=0}^{\infty} \sum_{j=0}^{x} \int_0^t P_j^* z^{b+j-1} dz \\
&= \frac{ac_2}{t^b} \int_0^t z^{b-1} \sum_{x=0}^{\infty} P_x^* \sum_{j=0}^{x} z^j dz = \frac{ac_2}{t^b} \int_0^t \frac{z^{b-1}}{1-z} \sum_{x=0}^{\infty} P_x^* \left(1 - z^{x+1}\right) dz \\
&= \frac{ac_2}{t^b} \int_0^t \frac{z^{b-1}}{1-z} \big(1 - zG^*(z)\big) dz ,
\end{aligned}$$

see [4] and [10].

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