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BIAS CORRECTION FOR THE SHAPE PARAMETER OF THE SKEW NORMAL DISTRIBUTION

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ABSTRACT. In the paper the shape parameter α of the multivariate skew normal distribution is estimated by the method of moments and the main term of the bias of the estimator is derived using matrix derivative technique. A stochastic representation of a skew normal distribution which defines a new simpler simulation algorithm is found.

1. Skew normal distribution

The multivariate skew normal distribution has been introduced by Azzalini & Dalla Valle (1996), the distribution with applications is studied in Azzalini & Capitanio (1999). Following Gupta & Kollo (2000) we shall use slightly different parameterization of the distribution.

Definition 1. A continuous random p-vector Z has a *multivariate skew normal distribution* with parameters α and Σ, $Z \sim SN_p(\alpha, \Sigma)$, if its density function is given by

$$f_Z(x) = 2 f_X(x, \Sigma)\Phi(\alpha' x), \qquad x \in \mathbb{R}^p, \tag{1}$$

where $\alpha \in \mathbb{R}^p$, $\Sigma > 0$, $f_X(x, \Sigma)$ is the density function of the normal distribution $N_p(0, \Sigma)$ and $\Phi(\cdot)$ is the distribution function of $N(0,1)$.

In Definition 1 the parameter Σ is the dispersion matrix of $N_p(0,\Sigma)$, while Azzalini *et al.* use the correlation matrix Ω instead of Σ. From Azzalini & Dalla Valle (1996) we get straightforwardly the moment generating function of the skew normal distribution:

$$M_Z(t) = 2e^{\frac{1}{2}t'\Sigma t}\,\Phi\left(\frac{\alpha'\Sigma t}{(1+\alpha'\Sigma\alpha)^{1/2}}\right). \tag{2}$$

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Differentiating (2) twice we get the first two moments of Z:

$$EZ = \sqrt{\frac{2}{\pi}}\,\frac{\Sigma\alpha}{(1+\alpha'\Sigma\alpha)^{1/2}}\,,$$
$$m_2(Z) = \Sigma\,,$$
$$DZ = \Sigma - EZEZ'.$$

When differentiating $M_Z(t)$, the matrix derivative and its properties are used (see Magnus & Neudecker (1999), for example).

The method of moments gives us the following parameter estimators (Gupta & Kollo (2000)) through the sample mean $\bar{Z}$ and the sample dispersion matrix S:

$$\hat{\Sigma} = S + \bar{Z}\bar{Z}'; \tag{3}$$
$$\hat{\alpha} = \frac{(S+\bar{Z}\bar{Z}')^{-1}\bar{Z}}{\sqrt{\frac{2}{\pi} - \bar{Z}'(S+\bar{Z}\bar{Z}')^{-1}\bar{Z}}}\,. \tag{4}$$

2. Stochastic representation

The following statement gives us a stochastic representation of $Z \sim SN_p(\alpha,\Sigma)$ which forms a basis for simulation. Correspondingly, simulation of the skew normal distribution is based on the simulation of $X \sim N_p(0,\Sigma)$ and independent $W \sim N(0,1)$ variables.

Proposition 1. *Let $X \sim N_p(0,\Sigma)$ and $W \sim N(0,1)$. Defining*

$$Z = \begin{cases} X & \text{if } \alpha'X > W\,, \\ -X & \text{otherwise}\,, \end{cases} \tag{5}$$

then $Z \sim SN_p(\alpha,\Sigma)$, where $\alpha \in \mathbb{R}^p$.

Remark. The rule (5) is different from the one presented in Azzalini & Dalla Valle (1996). The equality (5) is easier to use for simulation, and it gives better understanding of the impact of α on the support of Z: random vector Z tends to be on one side of the plane $\alpha'X = 0$.

Proof. For the distribution function of Z we have

$$F_Z(z) = P(Z \le z) = P(X \le z \mid \alpha'X - W > 0) = 2P(X \le z\,,\ W < \alpha'X)\,,$$

since $P(\alpha' X - W > 0) = \frac{1}{2}$. Using total probability formula with $t \le z$, $t \in \mathbb{R}^p$, we get

$$F_Z(z) = 2 \int\limits_{-\infty}^{z} P(X \le z,\, W < \alpha' X \mid X = t) f_X(t, \Sigma)\, dt\,,$$
$$= 2 \int\limits_{-\infty}^{z} \Phi(\alpha' t) f_X(t, \Sigma)\, dt\,.$$

Thus, the density function of Z is (1), what completes the proof. □

3. Bias correction for $\hat{\alpha}$

It is well-known (e.g., Azzalini & Capitanio (1999)) that estimator $\hat{\alpha}$ in (4) is a biased estimator for α. In the following we are going to find the main term of the bias. This will be derived from the Taylor expansion of $\hat{\alpha}$ at the point α. Let us rewrite (4) in a shorter form:

$$\hat{\alpha} = \frac{\hat{M}_2^{-1} \bar{Z}}{\sqrt{\frac{2}{\pi} - \bar{Z}' \hat{M}_2^{-1} \bar{Z}}}\,, \tag{6}$$

where $\hat{M}_2$ denotes the classical estimator of the matrix of the second order moments of Z. Denote

$$Y = \begin{pmatrix} \bar{Z} \\ \operatorname{vec} \hat{M}_2 \end{pmatrix}, \quad Y_0 = \begin{pmatrix} EZ \\ \operatorname{vec} M_2 \end{pmatrix}.$$

In the next statement we shall present two first derivatives of $\hat{\alpha}$.

Proposition 2. *Let statistic $\hat{\alpha}$ be given by (6). Then in the notations given above, the first order derivative of $\hat{\alpha}$ is the following partitioned matrix*

$$\left.\frac{d\hat{\alpha}}{dY}\right|_{Y=Y_0} = \left(\sqrt{\frac{2}{\pi}} (1 + \alpha' \Sigma \alpha)^{\frac{1}{2}} (\Sigma^{-1} + \alpha\alpha') \,\middle|\, -\alpha' \otimes \Sigma^{-1} - \frac{1}{2} \alpha\alpha' \otimes \alpha' \right) \tag{7}$$

and the second order derivative is

$$\left.\frac{d^2 \hat{\alpha}}{dY^2}\right|_{Y=Y_0} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \tag{8}$$

where

$$A_{11} = \frac{\pi}{2}(1+\alpha'\Sigma\alpha)\big(\operatorname{vec}\Sigma^{-1}\cdot\alpha' + \alpha\otimes\Sigma^{-1} + 3\alpha\alpha'\otimes\alpha + \Sigma^{-1}\otimes\alpha\big)\,, \tag{9}$$

$$\begin{aligned} A_{12} = -\sqrt{\frac{\pi}{2}(1+\alpha'\Sigma\alpha)}\bigg(&\frac{1}{2}\operatorname{vec}\ \Sigma^{-1}(\alpha'\otimes\alpha') + \Sigma^{-1}\otimes\Sigma^{-1} + \alpha\alpha'\otimes\Sigma^{-1} \\ &+ \Sigma^{-1}\otimes\alpha\alpha' + \frac{3}{2}\alpha\alpha'\otimes\alpha\alpha'\bigg)\,, \end{aligned} \tag{10}$$

$$\begin{aligned} A_{21} = -\sqrt{\frac{\pi}{2}(1+\alpha'\Sigma\alpha)}\bigg(&\alpha\alpha'\otimes\operatorname{vec}\ \Sigma^{-1} + \Sigma^{-1}\otimes\operatorname{vec}\ \Sigma^{-1} \\ &+ \frac{1}{2}(\alpha\otimes\alpha\otimes I_p)(\Sigma^{-1}+3\alpha\alpha') + \frac{1}{2}(I_{p^2}\otimes\alpha)(K_{p,p}+I_{p^2})(\alpha\otimes\Sigma^{-1})\bigg)\,, \end{aligned} \tag{11}$$

$$\begin{aligned} A_{22} = &\frac{1}{2}\,\alpha\alpha'\otimes\operatorname{vec}\ \Sigma^{-1}\cdot\alpha' + \Sigma^{-1}\otimes\operatorname{vec}\ \Sigma^{-1}\cdot\alpha' + \big(\alpha\otimes I_{p^2}\big)(\Sigma^{-1}\otimes\Sigma^{-1}) \\ &+ \frac{1}{2}(\alpha\otimes\alpha\otimes I_p)\bigg(\alpha'\otimes\Sigma^{-1} + \frac{3}{2}\,\alpha\alpha'\otimes\alpha'\bigg) \\ &+ \frac{1}{2}(I_{p^2}\otimes\alpha)(K_{p,p}+I_{p^2})(\Sigma^{-1}\otimes\alpha\alpha')\,, \end{aligned} \tag{12}$$

where $K_{p,p}$ is the commutation matrix.

Proof of the Proposition 2 is given in Appendix. Now we shall state the main result of the paper.

Theorem 1. *Let $Z \sim SN_p(\alpha,\Sigma)$ with the shape parameter $\alpha \in \mathbb{R}^p$ and $\Sigma > 0\colon p\times p$. In the notations given above, the main term of the bias b of $\hat\alpha$ is of the form:*

$$b = \frac{1}{2n}(\operatorname{vec}' B\otimes I_p)\operatorname{vec}\frac{d^2\hat\alpha}{dY^2}\Big|_{Y=Y_0}\,,$$

where the second order derivative is given with equalities (8)–(12), and blocks of $B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22}\end{pmatrix}$ are presented through parameters α and Σ:

$$B_{11} = \Sigma - \frac{2}{\pi}\frac{\Sigma\alpha\alpha'\Sigma}{1+\alpha'\Sigma\alpha}\,, \tag{13}$$

$$B_{21} = B_{12}' = \sqrt{\frac{2}{\pi}}\,(1+\alpha'\Sigma\alpha)^{-\frac{1}{2}}\bigg(\Sigma\otimes\Sigma\alpha + \Sigma\alpha\otimes\Sigma - \frac{\Sigma\alpha\alpha'\Sigma\otimes\Sigma\alpha}{1+\alpha'\Sigma\alpha}\bigg)\,, \tag{14}$$

$$B_{22} = \big(I_{p^2}+K_{p,p}\big)(\Sigma\otimes\Sigma)\,. \tag{15}$$

P r o o f. By K o l l o (1991) the Taylor expansion of $\hat\alpha$ is of the form

$$\hat\alpha = \alpha + \frac{d\hat\alpha}{dY}\Big|_{Y=Y_0}(Y-Y_0) + \frac{1}{2}\big((Y-Y_0)'\otimes I_p\big)\frac{d^2\hat\alpha}{dY^2}\Big|_{Y=Y_0}(Y-Y_0) + \cdots$$

The main term of the expected bias from here is

$$E(\hat{\alpha}-\alpha)=\frac{1}{2}E\Big\{\big((Y-Y_0)'\otimes I_p\big)\frac{d^2\hat{\alpha}}{dY^2}\Big|_{Y=Y_0}(Y-Y_0)\Big\}. \tag{16}$$

Let us find the expression on right hand side of (16).

$$E\Big\{\big((Y-Y_0)'\otimes I_p\big)\frac{d^2\hat{\alpha}}{dY^2}\Big|_{Y=Y_0}(Y-Y_0)\Big\}$$
$$=\Big(\big[\operatorname{vec}E\{(Y-Y_0)(Y-Y_0)'\}\big]'\otimes I_p\Big)\operatorname{vec}\frac{d^2\hat{\alpha}}{dY^2}\Big|_{Y=Y_0}.$$

The second order derivative $\frac{d^2\hat{\alpha}}{dY^2}\Big|_{Y=Y_0}$ is given by (8)–(12) in Proposition 2.

Denoting $B=nE\{(Y-Y_0)(Y-Y_0)'\}$ on the right hand side of the last equality we get by Fang, Kollo & Parring (2000):

$$B=\begin{pmatrix} DZ & M_3'(Z)-EZ\operatorname{vec}'\Sigma \\ M_3(Z)-\operatorname{vec}\Sigma\cdot EZ' & M_4(Z)-\operatorname{vec}\Sigma\cdot\operatorname{vec}'\Sigma\end{pmatrix}, \tag{17}$$

where

$$M_3(Z)=E(Z\otimes Z'\otimes Z); \tag{18}$$
$$M_4(Z)=E(ZZ'\otimes ZZ'). \tag{19}$$

Expression for the third moment (18) can be obtained from Gupta & Kollo (2000), which gives us

$$M_3(Z)-\operatorname{vec}\Sigma\cdot EZ'=\sqrt{\frac{2}{\pi}}(1+\alpha'\Sigma\alpha)^{-\frac{1}{2}}\left(\Sigma\otimes\Sigma\alpha+\Sigma\alpha\otimes\Sigma-\frac{\Sigma\alpha\alpha'\Sigma\otimes\Sigma\alpha}{1+\alpha'\Sigma\alpha}\right).$$

The fourth moment (19) equals to the fourth moment of the normal vector $N(0,\Sigma)$ which in turn equals to the fourth central moment of $N(0,\Sigma)$. Therefore the expression of $M_4(Z)$ we get from Traat (1984)

$$M_4(Z)=\big(I_{p^2}+K_{p,p}\big)(\Sigma\otimes\Sigma)+\operatorname{vec}\Sigma\cdot\operatorname{vec}'\Sigma.$$

As a result, the matrix of moments (17) equals:

$$E\{(Y-Y_0)(Y-Y_0)'\}=\frac{1}{n}\begin{pmatrix}B_{11} & B_{12}\\ B_{21} & B_{22}\end{pmatrix},$$

where the blocks B_{ij} are defined by (13)–(15).

The proof is completed. □

Appendix: Proof of the Proposition 2

Proof. We are going to differentiate twice $\hat{\alpha}$ as a function of $\bar{Z}$ and $\hat{M}_2$. For necessary properties of the matrix derivative one can consult Magnus & Neudecker (1999) or Kollo (1991), for instance.

The first derivative consists of two blocks:

$$\frac{d\hat{\alpha}}{dY} = \begin{pmatrix} \frac{d\hat{\alpha}}{d\bar{Z}} & \frac{d\hat{\alpha}}{d\operatorname{vec}\hat{M}_2} \end{pmatrix}.$$

Let us find the blocks separately.

$$\begin{aligned}\frac{d\hat{\alpha}}{d\bar{Z}} &= \frac{d}{d\bar{Z}}\left[(\hat{M}_2^{-1}\bar{Z})\left(\frac{2}{\pi} - \bar{Z}'\hat{M}_2^{-1}\bar{Z}\right)^{-\frac{1}{2}}\right] \\ &= \hat{M}_2^{-1}\left(\frac{2}{\pi} - \bar{Z}'\hat{M}_2^{-1}\bar{Z}\right)^{-\frac{1}{2}} + \hat{M}_2^{-1}\bar{Z}\left(\frac{2}{\pi} - \bar{Z}'\hat{M}_2^{-1}\bar{Z}\right)^{-\frac{3}{2}}\bar{Z}'\hat{M}_2^{-1}.\end{aligned}$$

At the point Y_0 we get:

$$\left.\frac{d\hat{\alpha}}{d\bar{Z}}\right|_{Y=Y_0} = \sqrt{\frac{2}{\pi}}(1+\alpha'\Sigma\alpha)^{\frac{1}{2}}(\Sigma^{-1}+\alpha\alpha').$$

The second block is:

$$\begin{aligned}\frac{d\hat{\alpha}}{\operatorname{dvec}\hat{M}_2} &= (\bar{Z}'\otimes I_p)(-\hat{M}_2^{-1}\otimes\hat{M}_2^{-1})\left(\frac{2}{\pi} - \bar{Z}'\hat{M}_2^{-1}\bar{Z}\right)^{-\frac{1}{2}} \\ &\quad - \frac{1}{2}\hat{M}_2^{-1}\bar{Z}\left(\frac{2}{\pi} - \bar{Z}'\hat{M}_2^{-1}\bar{Z}\right)^{-\frac{3}{2}}(\bar{Z}\otimes\bar{Z})'(\hat{M}_2^{-1}\otimes\hat{M}_2^{-1}) \\ &= -\left(\frac{2}{\pi} - \bar{Z}'\hat{M}_2^{-1}\bar{Z}\right)^{-\frac{1}{2}}\left(\bar{Z}'\hat{M}_2^{-1}\otimes\hat{M}_2^{-1} + \frac{1}{2}\hat{M}_2^{-1}\bar{Z}\left(\frac{2}{\pi} - \bar{Z}'\hat{M}_2^{-1}\bar{Z}\right)^{-1}(\bar{Z}'\hat{M}_2^{-1}\otimes\bar{Z}'\hat{M}_2^{-1})\right).\end{aligned}$$

At the point Y_0 we get:

$$\left.\frac{d\hat{\alpha}}{\operatorname{dvec}\hat{M}_2}\right|_{Y=Y_0} = -\alpha'\otimes\Sigma^{-1} - \frac{1}{2}\alpha\alpha'\otimes\alpha'.$$

Denote $A = \frac{d\hat{\alpha}}{d\bar{Z}}$, $B = \frac{d\hat{\alpha}}{\operatorname{dvec}\hat{M}_2}$.

Let us write out the second order derivative in patterned form:

$$\begin{aligned}\frac{d^2\hat{\alpha}}{dY^2} &= \begin{pmatrix} \frac{d(\frac{d\hat{\alpha}}{dY})}{d\bar{Z}} & \frac{d(\frac{d\hat{\alpha}}{dY})}{\operatorname{dvec}\hat{M}_2} \end{pmatrix} = \begin{pmatrix} \frac{d(A\vdots B)}{d\bar{Z}} & \frac{d(A\vdots B)}{\operatorname{dvec}\hat{M}_2} \end{pmatrix} \\ &= \begin{pmatrix} \frac{dA}{d\bar{Z}} & \frac{dA}{\operatorname{dvec}\hat{M}_2} \\ \frac{dB}{d\bar{Z}} & \frac{dB}{\operatorname{dvec}\hat{M}_2} \end{pmatrix}.\end{aligned}$$

We have to find the derivatives in 4 blocks. The first block is:

$$\frac{dA}{d\bar{Z}} = \frac{d}{d\bar{Z}}\left(\left(\frac{2}{\pi} - \bar{Z}'\hat{M}_2^{-1}\bar{Z}\right)^{-\frac{1}{2}} \otimes \hat{M}_2^{-1} + \hat{M}_2^{-1}\bar{Z}\left(\frac{2}{\pi} - \bar{Z}'\hat{M}_2^{-1}\bar{Z}\right)^{-\frac{3}{2}}\bar{Z}'\hat{M}_2^{-1}\right)$$

$$= \left(\frac{2}{\pi} - \bar{Z}'\hat{M}_2^{-1}\bar{Z}\right)^{-\frac{3}{2}}\Big[\mathrm{vec}\,\hat{M}_2^{-1}\cdot\bar{Z}'\hat{M}_2^{-1} + \hat{M}_2^{-1}\bar{Z}\otimes\hat{M}_2^{-1}$$

$$+ (I_p \otimes \hat{M}_2^{-1}\bar{Z})\left(3\hat{M}_2^{-1}\bar{Z}\bar{Z}'\hat{M}_2^{-1}\left(\frac{2}{\pi} - \bar{Z}'\hat{M}_2^{-1}\bar{Z}\right)^{-1} + \hat{M}_2^{-1}\right)\Big].$$

At the point Y_0 we get:

$$\frac{dA}{d\bar{Z}}\Big|_{Y=Y_0} = \frac{\pi}{2}(1+\alpha'\Sigma\alpha)(\mathrm{vec}\ \Sigma^{-1}\cdot\alpha' + \alpha\otimes\Sigma^{-1} + 3\alpha\alpha'\otimes\alpha + \Sigma^{-1}\otimes\alpha) = A_{11}\,.$$

The second block is:

$$\frac{dA}{d\mathrm{vec}\,\hat{M}_2} = \frac{d}{d\hat{M}_2}\left(\left(\frac{2}{\pi} - \bar{Z}'\hat{M}_2^{-1}\bar{Z}\right)^{-\frac{1}{2}} \otimes \hat{M}_2^{-1} + \hat{M}_2^{-1}\bar{Z}\left(\frac{2}{\pi} - \bar{Z}'\hat{M}_2^{-1}\bar{Z}\right)^{-\frac{3}{2}}\bar{Z}'\hat{M}_2^{-1}\right)$$

$$= -\left(\frac{2}{\pi} - \bar{Z}'\hat{M}_2^{-1}\bar{Z}\right)^{-\frac{3}{2}}\Big[\frac{1}{2}\mathrm{vec}\,\hat{M}_2^{-1}(\bar{Z}'\hat{M}_2^{-1}\otimes\bar{Z}'\hat{M}_2^{-1}) + \left(\frac{2}{\pi} - \bar{Z}'\hat{M}_2^{-1}\bar{Z}\right)(\hat{M}_2^{-1}\otimes\hat{M}_2^{-1})$$

$$+ \hat{M}_2^{-1}\bar{Z}\bar{Z}'\hat{M}_2^{-1}\otimes\hat{M}_2^{-1} + \hat{M}_2^{-1}\otimes\hat{M}_2^{-1}\bar{Z}\bar{Z}'\hat{M}_2^{-1}$$

$$+ \frac{3}{2}\left(\frac{2}{\pi} - \bar{Z}'\hat{M}_2^{-1}\bar{Z}\right)^{-1}(\hat{M}_2^{-1}\bar{Z}\bar{Z}'\hat{M}_2^{-1}\otimes\hat{M}_2^{-1}\bar{Z}\bar{Z}'\hat{M}_2^{-1})\Big].$$

At the point Y_0 we have:

$$\frac{dA}{d\mathrm{vec}\,\hat{M}_2}\Big|_{Y=Y_0} = -\sqrt{\frac{\pi}{2}(1+\alpha'\Sigma\alpha)}\Big(\frac{1}{2}\mathrm{vec}\ \Sigma^{-1}(\alpha'\otimes\alpha') + \Sigma^{-1}\otimes\Sigma^{-1} + \alpha\alpha'\otimes\Sigma^{-1}$$

$$+ \Sigma^{-1}\otimes\alpha\alpha' + \frac{3}{2}\alpha\alpha'\otimes\alpha\alpha'\Big) = A_{12}\,.$$

The third block is:

$$\frac{dB}{d\bar{Z}} = \frac{d}{d\bar{Z}}\Big(-\left(\frac{2}{\pi} - \bar{Z}'\hat{M}_2^{-1}\bar{Z}\right)^{-\frac{1}{2}}(\bar{Z}'\hat{M}_2^{-1}\otimes\hat{M}_2^{-1})$$

$$- \frac{1}{2}\hat{M}_2^{-1}\bar{Z}\left(\frac{2}{\pi} - \bar{Z}'\hat{M}_2^{-1}\bar{Z}\right)^{-\frac{3}{2}}(\bar{Z}'\hat{M}_2^{-1}\otimes\bar{Z}'\hat{M}_2^{-1})\Big)$$

$$= -\left(\frac{2}{\pi} - \bar{Z}'\hat{M}_2^{-1}\bar{Z}\right)^{-\frac{3}{2}}\Big(\hat{M}_2^{-1}\bar{Z}\bar{Z}'\hat{M}_2^{-1}\otimes\mathrm{vec}\,\hat{M}_2^{-1} + \left(\frac{2}{\pi} - \bar{Z}'\hat{M}_2^{-1}\bar{Z}\right)(\hat{M}_2^{-1}\otimes\mathrm{vec}\,\hat{M}_2^{-1})$$

$$+ \frac{1}{2}(\hat{M}_2^{-1}\bar{Z}\otimes\hat{M}_2^{-1}\bar{Z}\otimes I_p)\left(\hat{M}_2^{-1} + 3\left(\frac{2}{\pi} - \bar{Z}'\hat{M}_2^{-1}\bar{Z}\right)^{-1}\hat{M}_2^{-1}\bar{Z}\bar{Z}'\hat{M}_2^{-1}\right)$$

$$+ \frac{1}{2}(I_{p^2}\otimes\hat{M}_2^{-1}\bar{Z})(K_{p,p} + I_{p^2})(\hat{M}_2^{-1}\bar{Z}\otimes\hat{M}_2^{-1})\Big).$$

At the point Y_0 we have:

$$\frac{dB}{d\bar{Z}}\Big|_{Y=Y_0} = -\sqrt{\frac{\pi}{2}(1+\alpha'\Sigma\alpha)}\Big(\alpha\alpha'\otimes\mathrm{vec}\ \Sigma^{-1} + \Sigma^{-1}\otimes\mathrm{vec}\ \Sigma^{-1}$$

$$+ \frac{1}{2}(\alpha\otimes\alpha\otimes I_p)(\Sigma^{-1} + 3\alpha\alpha') + \frac{1}{2}(I_{p^2}\otimes\alpha)(K_{p,p} + I_{p^2})(\alpha\otimes\Sigma^{-1})\Big)$$

$$= A_{21}\,.$$

The 4th block is:

$$\frac{dB}{\mathrm{dvec}\,\hat{M}_2} = \frac{d}{\mathrm{dvec}\,\hat{M}_2}\Bigg(-\Big(\frac{2}{\pi} - \bar{Z}'\hat{M}_2^{-1}\bar{Z}\Big)^{-\frac{1}{2}}(\bar{Z}'\hat{M}_2^{-1}\otimes\hat{M}_2^{-1})$$

$$-\frac{1}{2}\hat{M}_2^{-1}\bar{Z}\Big(\frac{2}{\pi} - \bar{Z}'\hat{M}_2^{-1}\bar{Z}\Big)^{-\frac{3}{2}}(\bar{Z}'\hat{M}_2^{-1}\otimes\bar{Z}'\hat{M}_2^{-1})\Bigg)$$

$$= \Big(\frac{2}{\pi} - \bar{Z}'\hat{M}_2^{-1}\bar{Z}\Big)^{-\frac{3}{2}}\Bigg[\frac{1}{2}(\hat{M}_2^{-1}\bar{Z}\bar{Z}'\hat{M}_2^{-1}\otimes \mathrm{vec}\,\hat{M}_2^{-1}\cdot\bar{Z}'\hat{M}_2^{-1})$$

$$+\Big(\frac{2}{\pi} - \bar{Z}'\hat{M}_2^{-1}\bar{Z}\Big)\Big(\hat{M}_2^{-1}\otimes\mathrm{vec}\,\hat{M}_2^{-1}\cdot\bar{Z}'\hat{M}_2^{-1} + (\hat{M}_2^{-1}\bar{Z}\otimes I_{p^2})(\hat{M}_2^{-1}\otimes\hat{M}_2^{-1})\Big)$$

$$+\frac{1}{2}(\hat{M}_2^{-1}\bar{Z}\otimes\hat{M}_2^{-1}\bar{Z}\otimes I_p)\Bigg(\bar{Z}'\hat{M}_2^{-1}\otimes\hat{M}_2^{-1}+\frac{3}{2}\Big(\frac{2}{\pi}-\bar{Z}'\hat{M}_2^{-1}\bar{Z}\Big)^{-1}(\hat{M}_2^{-1}\bar{Z}\bar{Z}'\hat{M}_2^{-1}\otimes\bar{Z}'\hat{M}_2^{-1})\Bigg)$$

$$+\frac{1}{2}(I_{p^2}\otimes\hat{M}_2^{-1}\bar{Z})(K_{p,p}+I_{p^2})(\hat{M}_2^{-1}\otimes\hat{M}_2^{-1}\bar{Z}\bar{Z}'\hat{M}_2^{-1})\Bigg].$$

At the point Y_0 we have:

$$\frac{dB}{\mathrm{dvec}\,\hat{M}_2}\Big|_{Y=Y_0} = \frac{1}{2}\alpha\alpha'\otimes\mathrm{vec}\ \Sigma^{-1}\cdot\alpha' + \Sigma^{-1}\otimes\mathrm{vec}\ \Sigma^{-1}\cdot\alpha' + (\alpha\otimes I_{p^2})(\Sigma^{-1}\otimes\Sigma^{-1})$$

$$+\frac{1}{2}(\alpha\otimes\alpha\otimes I_p)\Big(\alpha'\otimes\Sigma^{-1}+\frac{3}{2}\alpha\alpha'\otimes\alpha'\Big)$$

$$+\frac{1}{2}(I_{p^2}\otimes\alpha)(K_{p,p}+I_{p^2})(\Sigma^{-1}\otimes\alpha\alpha')$$

$$= A_{22}\,.$$

The obtained equalities complete the proof. □

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