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ON THE POWER OF THE MAXIMUM LIKELIHOOD TESTS FOR MULTIDIMENSIONAL STUDENT'S t-DISTRIBUTED DATA

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ABSTRACT. The idea how to widen the linear Gauss-Markoff model was based on the changing of the assumption about normality and independence of errors. In 1976, Zellner proposed the using of the regression model with the joint multidimensional Student's t-distribution as a distribution of the random error. It begins the searching on distributions with fatter tails than normal ones. In 1990, Anderson and Fang applying elliptical distributions discovered that the maximum likelihood test gives, when zero hypothesis is true, F-statistic, like in the normal case. Both models have a significant difference; the power of the test. The authors of this paper are interested in the comparison of both tests (specifically: the form of statistic, the power of the test).

1. Introduction

This paper deals with an application of the t distribution to the linear Gauss-Markoff model with observations arranged in a block design. Earlier, in [5], the simpler model for the randomized complete block design was investigated. Sutradhar (in [6]) calculated the power of the maximum likelihood test for a regression model, but here a more general model with the singular matrix is considered. The matrix of a block design $\mathbf{A}$ (see Chapter 3) is singular so not every element of the vector of parameters can be estimated.

The using of some t distribution instead of multinormal was for the first time expressed by Zellner in [8]. Later, Anderson and Fang (in [1]), Fang and Zhang (in [2]) have developed so-called generalized multivariate analysis (i.e., the theory similar to the classical multivariate analysis based on a multinormal distribution) in accordance with elliptically contoured distributions.

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2. Notation

In this paper the following notation is used: let $\mathbb{R}$ be the set of real numbers, $\mathbf{0}_n$ be the $(n \times 1)$ vector of zeroes, $\mathbf{1}_n$ denotes the $(n \times 1)$ vector of ones, $\mathbf{C}^-$ denotes a general inversion of a matrix $\mathbf{C}$, $\Gamma(n)$ is the Gamma function, $\beta(m,n)$ denotes the Beta function, $(a)_n = a(a+1)\dots(a+n-1)$ and ${}_2F_1(a,b,c;x)$ is the Gauss hypergeometric series.

3. Model of block design

Let us consider the Gauss-Markoff model $(\mathbf{y}, \mathbf{A}\theta, \lambda^2\mathbf{\Sigma})$ of the form $\mathbf{y} = \mathbf{A}\boldsymbol{\theta} + \boldsymbol{\varepsilon}$, with $\mathbf{A}$ as a matrix of scalars and a vector of parameters $\boldsymbol{\theta}$ consisting of three parts: μ—the global mean, $\boldsymbol{\alpha}$—the vector of k treatment effects $\alpha_1, \dots, \alpha_k$ and $\boldsymbol{\beta}$—the vector of the block effects $\beta_1, \dots, \beta_b$. The symbol $\boldsymbol{\varepsilon}$ denotes the vector of random errors $\boldsymbol{\varepsilon} = [\varepsilon_1, \dots, \varepsilon_n]'$. Hence we get $\boldsymbol{\theta} = [\mu, \boldsymbol{\alpha}, \boldsymbol{\beta}]'$ and $\mathbf{A} = \left[\mathbf{1}_n \vdots \mathbf{\Delta}' \vdots \mathbf{D}'\right]$. The element d_{ij} of the $(n \times b)$ matrix $\mathbf{D}'$ is equal either to 1 or to 0, according whether the ith observation comes or does not come from the jth block. Similarly, the $(n \times k)$ matrix $\mathbf{\Delta}'$ is the treatment-observations incidence matrix, i.e., the element Δ_{ij} of $\mathbf{\Delta}$ is equal either to 1 or to 0, according whether the ith observation comes or does not come from the jth treatment.

Assume now that the vector of the random errors has the expected value $E(\boldsymbol{\varepsilon}) = \mathbf{0}_n$ and the variance-covariance matrix $\mathrm{Cov}(\boldsymbol{\varepsilon}) = \lambda^2\mathbf{\Sigma}$, with a positive scalar λ^2 and a known, symmetric, positive definite matrix $\mathbf{\Sigma}$ is given.

4. Additional assumptions

Moreover, we assume that the vector of random errors has the n-dimensional Student's t-distribution with ν degrees of freedom, with the expected value $E(\boldsymbol{\varepsilon}) = \mathbf{0}_n$ and the variance-covariance matrix $\mathrm{Cov}(\boldsymbol{\varepsilon}) = \left(\nu\sigma^2/(\nu-2)\right)\mathbf{\Sigma}$, with a positive scalar σ^2 and $\nu > 2$ (degrees of freedom).

The pdf of $\boldsymbol{\varepsilon}$ is given (for every $\mathbf{x} \in \mathbb{R}^n$) by the formula

$$L(\mathbf{y}, \mu, \boldsymbol{\alpha}, \boldsymbol{\beta}) = \frac{\Gamma(\frac{\nu+n}{2})\left(\det(\mathbf{\Sigma})\right)^{-\frac{1}{2}}}{\Gamma(\frac{\nu}{2})(\nu\pi)^{\frac{n}{2}}(\sigma^2)^{\frac{1}{2}}} \tag{1}$$
$$\times\left[1 + \frac{1}{\nu\sigma^2}\left(\mathbf{y} - \mu\mathbf{1}_n - \mathbf{\Delta}'\boldsymbol{\alpha} - \mathbf{D}'\boldsymbol{\beta}\right)'\mathbf{\Sigma}^{-1}\left(\mathbf{y} - \mu\mathbf{1}_n - \mathbf{\Delta}'\boldsymbol{\alpha} - \mathbf{D}'\boldsymbol{\beta}\right)\right]^{-\frac{\nu+n}{2}}.$$

5. Tested hypothesis

We are interested in testing the hypothesis about lack of the treatment influence on the value of the response taking the likelihood ratio approach. Thus we are testing the linear hypothesis

$$H_0\colon \mathbf{H}\boldsymbol{\alpha} = \mathbf{0} \quad \text{vs.} \quad H_1\colon \mathbf{H}\boldsymbol{\alpha} \neq \mathbf{0} \tag{2}$$

using the test statistic

$$\tau = \frac{\sup\limits_{\boldsymbol{\theta}\in\Theta_0} L(\boldsymbol{\theta}, y_1, \dots, y_n)}{\sup\limits_{\boldsymbol{\theta}\in\Theta} L(\boldsymbol{\theta}, y_1, \dots, y_n)} \tag{3}$$

based on the likelihood function $L(\boldsymbol{\theta}, \mathbf{y})$. The function L is defined on $\Theta = \{(\mu, \boldsymbol{\alpha}, \boldsymbol{\beta})\colon \mu \in \mathbb{R}\,,\ \boldsymbol{\alpha} \in \mathbb{R}^k,\ \boldsymbol{\beta} \in \mathbb{R}^b\}$ and the null hypothesis determines its subspace $\Theta_0 = \{(\mu, \boldsymbol{\alpha}, \boldsymbol{\beta})\colon \mu \in \mathbb{R},\ \mathbf{H}\boldsymbol{\alpha} = \mathbf{0},\ \boldsymbol{\beta} \in \mathbb{R}^b\}$.

We substitute the parameters by its ML estimators and obtain

$$\tau = \left\{ \frac{1 + \frac{1}{\nu\sigma^2}\mathbf{y}'\mathbf{P}\mathbf{y}}{1 + \frac{1}{\nu\sigma^2}\mathbf{y}'\mathbf{Q}\mathbf{y}} \right\}^{-\frac{\nu+n}{2}}, \tag{4}$$

where matrices $\mathbf{P}$ and $\mathbf{Q}$ of the form

$$\mathbf{Q} = \boldsymbol{\Sigma}^{-1} - \boldsymbol{\Sigma}^{-1}\mathbf{A}(\mathbf{A}'\boldsymbol{\Sigma}^{-1}\mathbf{A})^{-}\mathbf{A}'\boldsymbol{\Sigma}^{-1}, \tag{5}$$
$$\mathbf{P} = \mathbf{Q} + \boldsymbol{\Sigma}^{-1}\mathbf{A}(\mathbf{A}'\boldsymbol{\Sigma}^{-1}\mathbf{A})^{-}\mathbf{A}'\boldsymbol{\Sigma}^{-1} - \boldsymbol{\Sigma}^{-1}\mathbf{A}(\mathbf{A}'\boldsymbol{\Sigma}^{-1}\mathbf{A} - \mathbf{W}'\mathbf{W})^{-}\mathbf{A}'\boldsymbol{\Sigma}^{-1}$$

satisfy the conditions: $(\mathbf{P}-\mathbf{Q})'\boldsymbol{\Sigma}\mathbf{Q} = \mathbf{0}$, $(\mathbf{P}-\mathbf{Q})'\boldsymbol{\Sigma}(\mathbf{P}-\mathbf{Q}) = \mathbf{P}-\mathbf{Q}$, $\mathbf{Q}'\boldsymbol{\Sigma}\mathbf{Q} = \mathbf{Q}$, $\mathbf{P}'\boldsymbol{\Sigma}\mathbf{P} = \mathbf{P}$ and $\mathbf{P}'\boldsymbol{\Sigma}\mathbf{Q} = \mathbf{Q} = \mathbf{Q}'\boldsymbol{\Sigma}\mathbf{P}$. Additionally, we denote $\mathbf{W} = [\mathbf{0}\,\mathbf{H}\,\mathbf{0}]$ and $n_1 = \operatorname{rank}(\mathbf{P}-\mathbf{Q})$, $n_2 = \operatorname{rank}(\mathbf{Q})$. Hence

$$\tau^{-\frac{2}{\nu+n}} - 1 = \frac{y'(\mathbf{P}-\mathbf{Q})y}{\nu\sigma^2 + y'\mathbf{Q}y} \ (= U,\ \text{say}).$$

The likelihood ratio criterion requires the rejection of H_0 for small values of τ (i.e., for large values of U). Therefore the rejection region at the significant level α is $R = \{U\colon U > u_\alpha\}$ and u_α is determined by the condition $\sup\limits_{\Theta_0} \Pr(U > u_\alpha \mid H_0) = \alpha$.

6. Distribution of U

The random variable U has the pdf of the form

$$f(x) = \frac{\pi^{\frac{n}{2}-1}\nu^{\frac{\nu}{2}}(n_1-3)!!\,(n_2-3)!!\,(\nu\sigma^2)^{\frac{n_1+n_2}{2}-3}x^{\frac{n_1}{2}-1}}{2^{\frac{n_2}{2}-1}\Gamma\left(\frac{\nu}{2}\right)\Gamma\left(\frac{n_2}{2}\right)\Gamma\left(\frac{n_1-1}{2}\right)\Gamma\left(\frac{n_2-1}{2}\right)\Gamma\left(\frac{\nu+n}{2}\right)(\nu+\rho+x\nu\sigma^2)^{\frac{\nu+n}{2}}}$$
$$\times\sum_{j=1}^{\infty}\frac{(2j-1)!!\,(2j-2)!!\,\beta\left(\frac{n_2}{2},\frac{\nu+n-n_1-n_2}{2}+j\right)}{j!(2j+n_1-2)!!\,\beta\left(\frac{\nu+n}{2},2j\right)}\left(\frac{2\nu\sigma^2\rho x}{(\nu+\rho+\nu\sigma^2x)^2}\right)^j \tag{6}$$
$$\times\ {}_2F_1\left(\frac{\nu+n}{2}+2j,\ \frac{n_2}{2},\frac{\nu+n-n_1}{2}+j+1,\ 1-\frac{1+x}{\frac{\nu+\rho}{\nu\sigma^2}+x}\right)$$

for every $x>0$. (See [4].)

The noncentrality parameter $\rho = \mathbf{w}'(\mathbf{P}-\mathbf{Q})\mathbf{w}$, where $\mathbf{w} = E(\mathbf{y}) = \mathbf{A}(\mathbf{A}'\mathbf{\Sigma}^{-1}\mathbf{A}-\mathbf{W}'\mathbf{W})^{-}(\mathbf{A}'\mathbf{\Sigma}^{-1}-\mathbf{W})\mathbf{y}$.

The contour plot on Fig. 1 (on the left) shows the dependence on pdf on the argument x (on the right) and the noncentrality parameter ρ (below) for the chosen value of n, n_1, n_2, ν, and σ^2.

7. Distribution of U when H_0 is true

If H_0 is true, then the joint distribution of quadratic forms $y'(\mathbf{P}-\mathbf{Q})y$ and $y'\mathbf{Q}y$ is 2-dimensional Gamma(n_1, n_2, $n-n_1-n_2$) and the pdf of U under H_0 is given below. (See [4].)

$$f_0(u) = \begin{cases} \int_0^g h_2(u,x)\,dx + \int_g^\infty h_1(u,x)\,dx & \text{for } u<1\,, \\ \int_0^\infty h_2(u,x)\,dx & \text{for } u\geqslant 1 \end{cases} \tag{7}$$

with $g=\frac{u}{1-u}\nu\sigma^2$, where

$$h_1(u,x) = \frac{u^{\frac{n_1+n_2}{2}+1}x^{\frac{n-n_1-n_2}{2}}(\nu\sigma^2+x)^{\frac{n_1+n_2}{2}}e^{-x}}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)\Gamma\left(\frac{n-n_1-n_2}{2}\right)}\,I(n_1,\,n_2,\,n-n_1-n_2,\,u,\,x)\,,$$
$$h_2(u,x) = \frac{u^{\frac{n_2}{2}}x^{\frac{n-n_2}{2}+1}(\nu\sigma^2+x)^{\frac{n_2}{2}}e^{-u(\nu\sigma^2+x)}}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)\Gamma\left(\frac{n-n_1-n_2}{2}\right)}\,I(n_1,\,n-n_1-n_2\,,n_2,\,u,\,x)$$

and

$$I(a,b,c,u,x) = \int_0^1 z^{\frac{a}{2}}(1-z)^{\frac{b}{2}}\left(1-\frac{u(\nu\sigma^2+x)}{x}z\right)^{\frac{c}{2}}e^z\,dz\,.$$

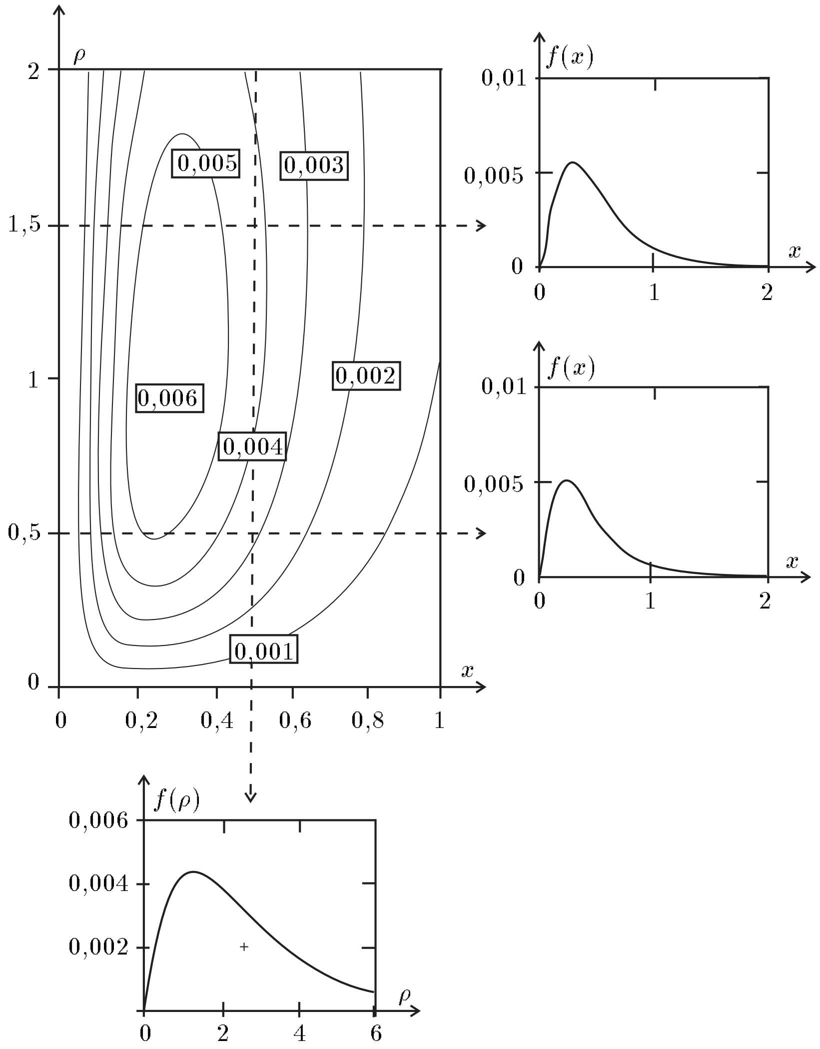
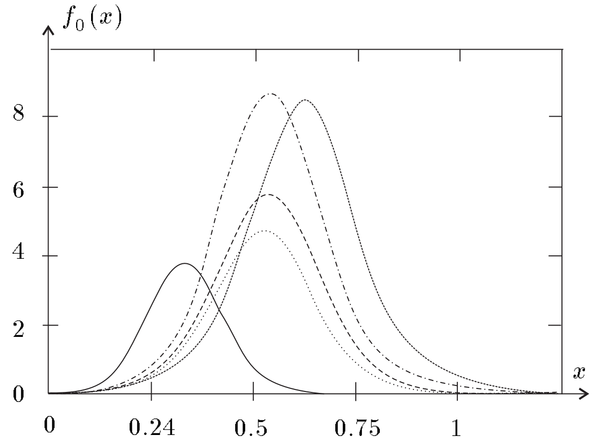


FIGURE 1. The dependence of the pdf of U on the argument x and the noncentrality parameter ρ for $n = 14$, $n_1 = 4$, $n_2 = 6$, $\nu = 4$, $\sigma^2 = 1$.

On Fig. 2 the pdf of U under H_0 is shown for three models (for $\nu\sigma^2 = 5, 7$, or 15) with the same number of n on components n_1 (rank($\mathbf{P} - \mathbf{Q}$)), n_2 (rank($\mathbf{Q}$)) and the remain. The changing of the product $\nu\sigma^2$ denotes the using another model (what is visible as the translation of the plot).

8. Power of the test

The power of the test, denoted by M, being equal to the integral



Values of parameters

	n	n_1	n_2	$n-n_1-n_2$	$\nu\sigma^2$
	9	4	2	3	15
	9	2	4	3	7
	9	4	3	2	7
	9	4	2	3	7
	9	2	3	4	5

FIGURE 2. The plot of the pdf of U when H_0 is true.

$$M(\rho) = \int_{u_\alpha}^{\infty} f(u)\,du\,,$$

depends on the noncentrality parameter ρ according to the formulae

$$M(\rho) = 1 - a\frac{\pi^{\frac{n}{2}-1}\nu^{\frac{\nu}{2}}(\nu\sigma^2)^{\frac{n_1+n_2}{2}}}{\Gamma\left(\frac{\nu}{2}\right)\Gamma\left(\frac{n_1-1}{2}\right)\Gamma\left(\frac{n_2-1}{2}\right)(\nu+\rho)^{\frac{\nu+n}{2}}}\sum_{j=0}^{\infty} b(j,\rho,u_\alpha)\,,$$

$$a = \begin{cases} \dfrac{(n_2-3)!!\,\pi}{2^{\frac{n_2}{2}-1}(\frac{n_2}{2}-1)!} & \text{for even } (n_2-2)\,, \\[2ex] \dfrac{1+\sum\limits_{k=0}^{\frac{n_2-5}{2}}\dfrac{\Gamma\left(\frac{n_2-1}{2}\right)(n_2-2k-6)!!}{\Gamma\left(\frac{n_2-3}{2}-k\right)(n_2-4)!!}4^{k+1}}{(n_2-2)2^{\frac{n_2}{2}-3}} & \text{for odd } (n_2-2)\,, \end{cases}$$

$$b(j,\rho,u_\alpha)=\frac{(-1)^j\Gamma\left(\frac{\nu+n}{2}+j\right)\beta\left(\frac{n_2}{2},\frac{\nu+n-n_1-n_2}{2}+j\right)c(j)(u_\alpha)^{\frac{n_1+j}{2}}}{\left(\frac{n_1+j}{2}\right)j!\left(\frac{\nu+\rho}{2\sqrt{\rho}}\right)^j}\,d(j,\rho,u_\alpha)\,,$$

$$c(j)=\begin{cases}\dfrac{(n_1-3)!!j!!(j-1)!!\pi}{2^{\frac{nj}{2}}\left(\frac{j}{2}\right)!(n_1+j-2)!!} & \text{for even } (n_1-2) \text{ and even } j\,,\\[2ex] \dfrac{1+\sum_{k=1}^{\frac{n_1-3}{2}}\frac{\Gamma\left(\frac{n_1-1}{2}\right)(n_1+j-2k-4)!!}{\Gamma\left(\frac{n_1-1}{2}-k\right)(n_1+j-4)!!}4^k}{(n_1+j-2)2^{\frac{n_1+j-2}{2}-2}} & \text{for odd } (n_1-2)\,,\\[2ex] 0 & \text{for even } (n_1-2) \text{ and even } j\,,\end{cases}$$

$$d(j,\rho,u_\alpha)=\sum_{q=0}^{\infty}\frac{\beta\left(\frac{\nu+n-n_1}{2}+j,q\right)\left(1-\frac{\nu\sigma^2}{\nu+\rho}\right)^q}{q\beta\left(\frac{\nu+n}{2}+j,q\right)\beta\left(\frac{n_2}{2},q\right)}\,h(j,q,\rho,u_\alpha)\,,$$

and

$$h(j,q,\rho,u_\alpha)=\ {}_2F_1\left(\frac{\nu+n}{2}+j+q,\frac{n_1+j}{2},\frac{n_1+j}{2}+1,-\frac{\nu\sigma^2}{\nu+\rho}u_\alpha\right)$$

for every $\rho>0$ and $n_1\geq 2$ and $n_2\geq 2$. (See [4] for even n_1 and even n_2).

TABLE 1. Comparison of two models.

Distribution of error vector $\boldsymbol{\varepsilon}$	$t_n(\nu,\mathbf{0}_n,\frac{\nu\sigma^2}{\nu-2}\boldsymbol{\Sigma})$	$N_n(\mathbf{0}_n,\sigma^2\boldsymbol{\Sigma})$
Assumptions	$\nu>2$, $\sigma^2>0$, $\boldsymbol{\Sigma}>0$ known	$\sigma^2>0$, $\boldsymbol{\Sigma}>0$ known
Null hypothesis	$H_0:\mathbf{H}\boldsymbol{\alpha}=\mathbf{0}$	$H_0:\mathbf{H}\boldsymbol{\alpha}=\mathbf{0}$
Alternative	$H_1:\mathbf{H}\boldsymbol{\alpha}\neq\mathbf{0}$	$H_1:\mathbf{H}\boldsymbol{\alpha}\neq\mathbf{0}$
Test statistic	$U=\tau^{-\frac{2}{\nu+n}}-1=\frac{y'(\mathbf{P}-\mathbf{Q})y}{\nu\sigma^2+y'\mathbf{Q}y}$	$U_1=\tau^{-\frac{2}{n}}-1=\frac{y'(\mathbf{P}-\mathbf{Q})y}{y'\mathbf{Q}y}$
Ranks of $\mathbf{P}-\mathbf{Q}$ and $\mathbf{Q}$	n_1 and n_2 (respectively)	n_1 and n_2 (respectively)
Distribution of test statistic	See Chapter 6 and Fig. 1	Noncentral $F(n_1,n_2,\delta)$
Distribution of test statistic under H_0	See Chapter 7 and Fig. 2	Central $F(n_1,\ n_2)$
Power of the test	Chapter 8	See [7a], [7b]

9. Conclusion

In this paper we have calculated the power of the maximum likelihood test for the model of block design with t-distributed error vector. The idea was based on the method suggested by Anderson and Fang for elliptical distributions (in [1]). The formulae presented here can be directly applied for computing in statistical packages.

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