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PSEUDO MV-ALGEBRAS AND MINIMAL CLANS

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Dedicated to Professor Ján Jakubík on the occasion of his 80th birthday

ABSTRACT. A categorical isomorphism between the categories of pseudo MV-algebras and positive unital minimal clans is proved.

1. Basic definitions and results

Pseudo MV-algebras were introduced as a generalization of MV-algebras in [6], and an equivalent structure was introduced in [8]. For basic properties see [2].

Definition 1.1. A *pseudo MV-algebra* is an algebra $(M; \oplus, -, \sim, 0, 1)$ of type $(2,1,1,0,0)$ such that the following axioms hold for all $x, y, z \in M$ with an additional binary operation $\odot$ defined via

$$y \odot x := (x^- \oplus y^-)^\sim, \tag{1}$$

(A1) $(x \oplus y) \oplus z = x \oplus (y \oplus z)$,

(A2) $x \oplus 0 = 0 \oplus x = x$,

(A3) $x \oplus 1 = 1 \oplus x = 1$,

(A4) $1^\sim = 0$,

(A5) $(x^- \oplus y^-)^\sim = (x^\sim \oplus y^\sim)^-$,

(A6) $x \oplus (x^\sim \odot y) = y \oplus (y^\sim \odot x) = (x \odot y^-) \oplus y = (y \odot x^\sim) \oplus x$,

(A7) $x \odot (x^- \oplus y) = (x \oplus y^\sim) \odot y$,

(A8) $(x^-)^\sim = x$.

Define $x \le y$ iff $x^- \oplus y = 1$. Then $\le$ is a partial order such that M is a distributive lattice with $x \vee y = x \oplus x^\sim \odot y$ (see [6]). Observe that 0 is the smallest, 1 is the greatest element, and $x \wedge y = (x \wedge y)^{-\sim} = (x^- \vee y^-)^\sim = (x^- \oplus x \odot y^-)^\sim = (x^\sim \oplus y) \odot x$.

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EXAMPLE. Let (G,u) be a lattice ordered group with strong unit u. Define $M := [0,u]$, $x \oplus y := (x+y) \wedge u$, $x^{-} := u-x$, $x^{\sim} := -x+u$, $x \odot y := (x-u+y) \vee 0$. Then $\big([0,u]; \oplus, -, \sim, 0, u\big)$ is a pseudo MV-algebra. Notice that every pseudo MV-algebra can be expressed in this form [1].

Define a partial binary operation $+$ on M as follows: $x+y$ is defined iff $x \leq y^{-}$ and in this case

$$x + y := x \oplus y\,. \tag{2}$$

Throughout the paper, we will need the following results.

LEMMA 1.2. [2, Prop. 6.4.2] *Let M be a pseudo MV-algebra. The following holds for all $x, y \in M$:*

(i) $x + 0 = 0 + x = x$,

(ii) $x + x^{\sim} = 1 = x^{-} + x$,

(iii) $x^{-} + y = 1 \Rightarrow y = x$,

(iv) $x + y = 1 \Rightarrow y = x^{\sim}, x = y^{-}$.

LEMMA 1.3. *Let $x+y$ and $(x+y)+z$ be defined in M. Then $y+z$ and $x+(y+z)$ are defined in M, and*

$$(x+y)+z = x+(y+z)\,.$$

LEMMA 1.4. [2, Prop. 6.4.4] *Let M be a pseudo MV-algebra.*

(i) *If $x+y=b$, then $b = x^{\sim} \odot y$,*

(ii) $x + y_1 = x + y_2 \Rightarrow y_1 = y_2$,

(iii) $x_1 + y = x_2 + y \Rightarrow x_1 = x_2$,

(iv) *if $y = a + x$, then $a = y \odot x^{-}$.*

LEMMA 1.5. *Let M be a pseudo MV-algebra.*

(i) $x \leq y^{-}$ *iff* $y \leq x^{\sim}$ *iff* $y \odot x = 0$,

(ii) $x \leq y$ *iff there is $z \in M$ such that $x+z$ exists and $x+z=y$.*

P r o o f. (i) By definition, $x \leq y^{-}$ iff $x^{-} \oplus y^{-} = 1$, which is equivalent with $(x^{-} \oplus y^{-})^{\sim} = 1^{\sim}$, which entails the desired result. (ii) If $x \leq y$, then $y = x \vee y = x \oplus x^{\sim} \odot y = x + x^{\sim} \odot y$ (indeed, $(x^{\sim} \odot y)^{-} \oplus x^{\sim} = (y^{-} \oplus x^{\sim -})^{\sim -} \oplus x^{\sim}$ $= (y^{-} \oplus x) \oplus x^{\sim} = 1$, hence $x^{\sim} \odot y \leq x^{\sim}$), so put $z = x^{\sim} \odot y$. Uniqueness follows by (i) of Lemma 1.4. If $y = x+z$, then $y = x \oplus z \Rightarrow x^{-} \oplus y = x^{-} \oplus x \oplus z = 1 \Rightarrow$ $x \leq y$. $\square$

Lemma 1.6. [2, Prop. 6.4.8] *Let M be a pseudo MV-algebra.*

(a) $x \odot (x \wedge y)^- = x \odot y^-$,

(b) $x \odot (x \wedge y)^- + y = x \odot y^- + y = x \vee y$,

(c) $(x \wedge y)^\sim \odot y = x^\sim \odot y$, $x \vee y = x + (x \wedge y)^\sim \odot y$,

(d) $(x \vee y) \odot y^- = x \odot (x \wedge y)^- = x \odot y^-$,

(e) $y^\sim \odot (x \vee y) = y^\sim \odot x = (x \wedge y)^\sim \odot x$.

Lemma 1.7. [2, Prop. 6.4.10] *Let M be a pseudo MV-algebra.*

(i) $a + b_1 \le a + b_2 \Rightarrow b_1 \le b_2$,

(ii) $a_1 + b \le a_2 + b \Rightarrow a_1 \le a_2$,

(iii) $a + b \in M$, $a_1 \le a, b_1 \le b \Rightarrow a_1 + b_1 \in M$.

Minimal clans were introduced as a common abstraction of Boolean rings and lattice ordered groups [9].

Definition 1.8. A *minimal clan* is a nonempty partially ordered set E with a subset $S \subset E \times E$ and a partial binary operation $+\colon S \to E$ satisfying the following axioms:

(MC-1) There is $0 \in E$ such that $(0, x) \in S$, $(x, 0) \in S$ and $0 + x = x + 0 = x$ for all $x \in E$.

(MC-2) For all $x, y, z \in E$, $(x, y) \in S$, $(x+y, z) \in S \Leftrightarrow (y, z) \in S$, $(x, y+z) \in S$, and $(x + y) + z = x + (y + z)$.

(MC-3) For all $x, y \in E$, if $u + x + v = u + y + v$ for some (u, v), then $x = y$.

(MC-4) $(u, x) \in S$, $(u + x, v) \in S$, $(u, y) \in S$, $(u + y, v) \in S$ $u + x + v \le u + y + v \Leftrightarrow x \le y$.

(MC-5) $x \vee y := \sup(x, y)$, $x \wedge y := \inf(x, y)$ exist for all $x, y \in E$.

(MC-6) For all $x, y \in E$ there are $u, v \in E$, $0 \le u, 0 \le v$, such that $u + x = x \vee y = x + v$ and $u + x \wedge y = y = x \wedge y + v$.

A minimal clan E is *positive* if $0 \le x$ for all $x \in E$, *commutative* if $(x, y) \in S$ iff $(y, x) \in S$ and in this case $x + y = y + x$. An element $1 \in E$ will be called a *unit* if for all $x \in E$ there are uniquely defined elements $x^\sim$ and x^- such that $x + x^\sim = 1 = x^- + x$. Observe that E has a unit iff it has a greatest element 1. If E has a unit, we will call it *unital.*

2. Connections between minimal clans and pseudo MV-algebras

Theorem 2.1. *Let $(M; \oplus, -, ^\sim, 0, 1)$ be a pseudo MV-algebra. Then there is a subset $S \subset M \times M$ and a binary operation $+$ such that $(M, +, \le)$, where*

$\leq$ is the original partial order on M, is a positive minimal clan with the unit $1 \in M$.

P r o o f . Put $S := \{(x,y) \in M \times M : x \leq y^-\}$ and the operation $+ : S \to M$ defined by $x + y = x \oplus y$. We have to prove that axioms (MC-1)–(MC-6) are satisfied.

(MC-1): follows from Lemma 1.2 (i).

(MC-2): follows by Lemma 1.3.

(MC-3): follows by Lemma 1.4 (ii) and (iii).

(MC-4): follows by Lemma 1.7 (i) and (ii).

(MC-5): has been proved in [6].

(MC-6): By definition, $x \vee y = x \oplus x^\sim \odot y = x + x^\sim \odot y$, so put $v = x^\sim \odot y$.

By (A6), $x \oplus x^\sim \odot y = y \odot x^- + x$, so put $u = y \odot x^-$.

From $x \oplus x^\sim \odot y = x + x^\sim \odot y$, and $x \wedge y \leq x$, $x^\sim \odot y \leq x^\sim$ we get, by Lemma 1.7 (i), that $x \wedge y + v \in M$. Moreover, there is $z \in M$ such that $x \wedge y + z = y$, and by Lemma 1.4 (i), then $z = (x \wedge y)^\sim \odot y = x^\sim \odot y = v$ by Lemma 1.6 (c). So $x \wedge y + v = y$.

Similarly we prove that $u + x \wedge y \in M$, and $u + x \wedge y = y \odot x^- + x \wedge y = y \odot (x \wedge y)^- + x \wedge y$ by Lemma 1.6 (d), so that $u + x \wedge y = y \vee (x \wedge y) = y$. This concludes the proof. $\square$

To prove the next theorem, it will be convenient to introduce the notion of a pseudo effect algebra, and use the results of [3] and [4]. The following definition was introduced in [3].

Definition 2.2. A structure $(A; +, 0, 1)$, where $+$ is a partial binary operation and 0 and 1 are constants, is called a *pseudo effect algebra* if, for all $a, b, c \in A$, the following hold.

- (E1) $a + b$ and $(a + b) + c$ exist if and only if $b + c$ and $a + (b + c)$ exist, and in this case, $(a + b) + c = a + (b + c)$.
- (E2) There exists exactly one $d \in A$ and exactly one $e \in A$ such that $a + d = e + a = 1$.
- (E3) If $a + b$ exists, there are elements d, e in A such that $a + b = d + a = b + c$.
- (E4) If $1 + a$ exists, then $a = 0$.

In view of (E2), we may define two unary operations $^\sim$ and $^-$ by requiring for any $a \in A$, $a + a^\sim = a^- + a = 1$, and consider a pseudo effect algebra as a structure $(A; +, ^\sim, ^-, 0, 1)$ of type $(2, 1, 1, 0, 0)$. A partial order in a pseudo effect algebra A is defined as follows:

$$a \leq b \quad \text{iff} \quad a + c = b \quad \text{for some } c \in A.$$

Observe that owing to (E3),

$$a \leq b \quad \text{iff} \quad d + a = b \quad \text{for some } d \in A\,.$$

We will make use of the following basic properties of pseudo effect algebras.

Lemma 2.3. [Prop. 1.4] *Let $(A; +, 0, 1)$ be a pseudo effect algebra. For all $a, b, c \in A$ the following hold.*

(i) $a + 0 = 0 + a = a$.

(ii) $a + b = 0$ *implies* $a = 0$, $b = 0$.

(iii) $0^{\sim} = 0^{-} = 1$, $1^{\sim} = 1^{-} = 0$.

(iv) $a^{\sim -} = a^{-\sim} = a$.

(v) $a + b = c$ *iff* $a = (b + c^{\sim})^{-}$ *iff* $b = (c^{-} + a)^{\sim}$.

Lemma 2.4. [Lemma 1.6]

(i) $\leq$ *is a partial order on* A.

(ii) $a \leq b$ *iff* $b^{-} \leq a^{-}$ *iff* $b^{\sim} \leq a^{\sim}$.

(iii) *If* $a + b$ *exists, and* $a_1 \leq a, b_1 \leq b$, *then* $a_1 + b_1$ *exists.*

(iv) $a + b$ *exists iff* $a \leq b^{-}$ *iff* $b \leq a^{\sim}$.

(v) *Suppose* $b + c$ *exist. Then* $a \leq b$ *iff* $a + c$ *exists and* $a + c \leq b + c$. *Suppose* $c + b$ *exists. Then* $a \leq b$ *iff* $c + a$ *exists and* $c + a \leq c + b$.

Let $(A; +, 0, 1)$ be a pseudo effect algebra. Let $/$ and $\backslash$ be partial binary operations such that for $a, b \in A$, $b\backslash a$ is defined iff a/b is defined iff $a \leq b$, in which case we require

$$(b\backslash a) + a = a + (a/b)\,. \tag{3}$$

We will say that a lattice ordered pseudo effect algebra A satisfies condition (MV) if the following hold:

(MV) $a\backslash(a \wedge b) = (a \vee b)\backslash b$, or equivalently, $(a \wedge b)/a = b/(a \vee b)$

(see [3, Prop. 8.15]).

Theorem 2.5. [3, Theorem 8.7]. *Let $(A; +, {}^{\sim}, {}^{-}, 0, 1)$ be a lattice pseudo effect algebra. Then we may define $a \oplus b = (a \wedge b^{-}) + b$, and $(A; \oplus, {}^{\sim}, {}^{-}, 0, 1)$ is a pseudo MV-algebra if and only if condition* (MV) *is satisfied.*

The following theorem is a special case of § 5 in [5].

Theorem 2.6. *Let $(E; \leq, +)$ be a positive minimal clan with the greatest element* 1. *Then $(E; +, 0, 1)$ is a pseudo effect algebra with the original partial order. Moreover, it is lattice ordered and satisfies condition* (MV). *Consequently,* *it can be endowed with a structure of a pseudo MV-algebra, in which the operation $\oplus$ extends the partial operation $+$, and the order and lattice operations coincide with the original ones.*

Proof. We prove that properties (E1)–(E4) of the definition of pseudo effect algebras are satisfied.

(E1) follows by (MC-2). We have $a \leq b$ in the minimal clan iff $a \vee b = b$. Applying (MC-6), and taking into account that the $+$ operation is the same, we get that $a \leq b$ also in E as a pseudo effect algebra. Conversely, if $a \leq b$ in the pseudo effect algebra, the there are elements c, d in E such that $a+c = d+a = b$, which entails, by positivity of E as a minimal clan and (MC-4), that $a \leq b$.

(E2) Since 1 is the greatest element, we have $a \leq 1$ for all $a \in E$. Therefore by (MC-6), there are $d, e \in E$ such that $a + d = a \vee 1 = 1 = e + a$. Uniqueness of d and e follows by (MC-3).

(E3) If $a + b$ exists, then owing to positivity of E, we have by (MC-4) that $a \vee (a + b) = a + b = b \vee (a + b)$ and applying (MC-6), we obtain the existence of d, e such that $a + b = d + a = b + e$.

(E4) Assume that $a + 1$ or $1 + a$ is defined. From $0 \leq a$ for all $a \in E$ we get by (MC-4) and (MC-1) that $1 = 0 + 1 \leq a + 1$ and $1 = 1 + 0 \leq 1 + a$. Since 1 is the greatest element of E, we obtain $a = 0$.

E is lattice ordered by (MC-5). Condition (MV) follows from (MC-6). $\square$

Now we will introduce homomorphisms of minimal clans and homomorphisms of pseudo MV-algebras.

DEFINITION 2.7. Let $(E; +, 0, \leq)$ be a minimal clan. A mapping $\phi\colon E_1 \to E_2$ is a *homomorphism* (of minimal clans) if

(i) $(x, y) \in S_1$ implies $\bigl(\phi(x), \phi(y)\bigr) \in S_2$ and $\phi(x + y) = \phi(x) + \phi(y)$;

(ii) for all $x, y \in E_1$, $\phi(x \vee y) = \phi(x) \vee \phi(y)$, $\phi(x \wedge y) = \phi(x) \wedge \phi(y)$.

Let $(E; +, 0, 1, \leq)$ b be u unital minimal clan. A homomorphism of unital minimal clans is a homomorphism of minimal clans satisfying the condition

(iii) $\phi(1) = 1$.

Observe that $\phi(0) = 0$, and $a \leq b$ implies $\phi(a) \leq \phi(b)$ for all $a, b \in E_1$.

DEFINITION 2.8. Let $(M, \oplus, -, \sim, 0, 1)$ be a pseudo MV-algebra. A mapping $\psi\colon M_1 \to M_2$ is a *homomorphism* (of pseudo MV-algebras) if

(i) for all $a, b \in M_1$, $\psi(a \oplus b) = \psi(a) \oplus \psi(b)$;

(ii) $\psi(1) = 1$;

(iii) for all $a \in M_1$, $\psi(a^-) = \psi(a)^-$, $\psi(a^\sim) = \psi(a)^\sim$.

It is easily seen that $\psi(0) = 0$, $a \le b \Rightarrow \psi(a) \le \psi(b)$, $\psi(a \odot b) = \psi(a) \odot \psi(b)$, $\psi(a \vee b) = \psi(a) \vee \psi(b)$, $\psi(a \wedge b) = \psi(a) \wedge \psi(b)$.

LEMMA 2.9. *A map $\psi\colon M_1 \to M_2$ between two pseudo MV-algebras is a homomorphism if and only if*

(j) $\psi(a+b) = \psi(a) + \psi(b)$ *whenever* $a \le b^-$;

(jj) $\psi(1) = 1$;

(jjj) $\psi(a \wedge b) = \psi(a) \wedge \psi(b)$.

Proof. If ψ is a homomorphism, then (j), (jj) and (jjj) are clearly satisfied. Let ψ have properties (j), (jj) and (jjj). Condition (jj) corresponds to condition (ii) in Definition 2.8. Condition (iii) of Definition 2.8 follows from (j) and (jj) by Lemma 1.2 (iv).

It suffices to prove that for all $a, b \in M_1$, $\psi(a \oplus b) = \psi(a) \oplus \psi(b)$.

By Lemma 1.6 (e),

$$y^\sim \odot x^\sim = (x^\sim \wedge y^\sim) \odot x^\sim ,$$

which entails

$$x \oplus y = x \oplus x^\sim \wedge y = x + x^\sim \wedge y ,$$

since $x^\sim \wedge y \le x^\sim$ (see Lemma 2.4 (iv)). Applying (j), (iii) and (jjj), we get the desired result. □

From this we easily obtain the following

COROLLARY 2.10. *To every homomorphism $\phi\colon E \to E'$ of positive unital minimal clans there is a homomorphism ϕ_* of the corresponding pseudo MV-algebras, and every homomorphism $\psi\colon M_1 \to M_2$ of pseudo MV-algebras equals to ϕ_* for some homomorphism ϕ of the corresponding minimal clans. Moreover, $\phi_{1*} = \phi_{2*}$ implies $\phi_1 = \phi_2$.*

3. Categorical isomorphism

Let MC_1^+ denote the category of positive unital minimal clans with the homomorphisms of minimal clans as morphisms, and let PMV denote the category of pseudo MV-algebras with the homomorphisms of pseudo MV-algebras as morphisms. According to Theorem 2.6, there is an object mapping $\Gamma\colon \mathrm{MC}_1^+ \to \mathrm{PMV}$, and, by Corollary 2.10, to every morphism μ of positive unital minimal clans, there is a morphism $\Gamma(\mu) := \mu_*$ of pseudo MV-algebras. The proof of the following lemma is straightforward.

Lemma 3.1. (i) *If $j\colon E \to E$ is the identity mapping on E, then $\Gamma(j)$ is the identity mapping on $\Gamma(E)$.*

(ii) *Given the diagram*

$$E \xrightarrow{\alpha} E' \xrightarrow{\beta} E'',$$

where α and β are morphisms of positive unital minimal clans, then

$$\Gamma(\beta \circ \alpha) = \Gamma(\beta) \circ \Gamma(\alpha).$$

Using [7], we obtain the following result.

Theorem 3.2. *The map $\Gamma\colon \mathrm{MC}_1^+ \to \mathrm{PMV}$ is a functor from the category of positive unital minimal clans to the category of pseudo MV-algebras.*

Similarly, if we define the map $\Delta\colon \mathrm{PMV} \to \mathrm{MC}_1^+$ according Theorem 2.1 between the objects, and assign to every mapping ψ a mapping $\Delta(\psi)$ such that $\psi = \Delta(\psi)_*$, by Corollary 2.10 we prove that Δ is a functor.

Theorem 3.3. *There is a categorical isomorphism between the category MC_1^+ of positive unital minimal clans and the category PMV of pseudo MV-algebras.*

Proof. By Theorems 2.6, and 2.1, a set A can be endowed with the structure of a positive unital minimal clan if and only if A can be endowed with a structure of a pseudo MV-algebra. Moreover, if we consider A as an object in the category MC_1^+, and endow it with the structure of a pseudo MV-algebra to obtain $\Gamma(A)$, and then endow $\Gamma(A)$ with the structure of a positive unital minimal clan to obtain $\Delta \circ \Gamma(A)$, we come back to the original object A. A similar statement holds about the morphisms by Corollary 2.10. So $\Delta \circ \Gamma = \mathrm{id}_{\mathrm{MC}_1^+}$, and similarly, $\Gamma \circ \Delta = \mathrm{id}_{\mathrm{PMV}}$. □

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